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THE STANDARD DEVIATIONS OF FRATERNAL AND PARENTAL CORRELATION COEFFICIENTS.

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CONTENTS.

INTRODUCTION

No attempts have been made as far as I know to calculate special formulae for the standard deviations of fraternal and parental correlation coefficients. The usual formula for the standard deviation of a correlation coefficient* which is deduced on the supposition that the values of the same variable are mutually uncorrelated is generally used also for this case, although it is only correct for a

* *Vide* Pearson and Filon: *Phil. Trans.* Vol. 191 A, p. 229, 1898

fraternal correlation coefficient calculated from only two siblings of each family and for a parental correlation coefficient when only one offspring value from each family enters into the calculation. When the material of observation, as is usually the case in investigations of inheritance in higher mammals, consists of families of varying size, and correlation tables are used in which the same weight is given to each observed pair of siblings or pair of parent and offspring, without regard to the size of the family, a rational treatment of the probable error is excluded at the outset. With material in hand which makes it possible to examine numerous siblings, it is most reasonable to confine the investigation to a constant number of offspring from each family. In this case the deduction of formulae for the standard deviations of the two correlation coefficients does not present special difficulties, and this problem will be solved here.

We shall suppose that each group of q siblings belongs to the same litter or that from other reasons their order of birth is indifferent. Then each pair of siblings or each pair of parent and offspring ought to take a like part in the calculation, and q siblings give rise to $\frac{1}{2}q(q-1)$ pair of brothers and q pair of parent and offspring which all of them are entered in the calculation.

The fraternal correlation can thus be calculated either from a correlation table which is made symmetrical so that it contains $q(q-1)$ entries from each fraternity or by the formula quoted p. 10 which gives an identical result.

I. Fraternal Correlation

Although this investigation aims especially at fraternal correlation it concerns of course other calculations of correlation in which the material consists of classes of equal size inside which the individuals are mutually correlated, all of them forming like parts. In the following we shall therefore name a group of siblings a *class*.

Suppose we have a material consisting of q individuals from each of n classes inside which the individuals are correlated while individuals from different classes are uncorrelated. We can then consider such a material as one of many possible samples of the same nature and size drawn from a population consisting of classes of individuals correlated as mentioned. It is therefore possible to face the problem of finding the law of errors for the *mean value*, the *standard deviation* of the character concerned and further for the *correlation coefficient* inside a class, supposing that these are all calculated from a sample like the one now considered.

Let the sample be $y_1, y_2, y_3 \ldots y_{nq}$ with mean value y and standard deviation σ. No special notation will be introduced for individuals of the same class, but summation of products is indicated by Σ when all factors of the product belong to the same class, and by S when factors of the same product belong to two or more classes. The summations always extend to all n classes.

(a) *The Mean Value.*

For the sample in hand we have

$$\bar{y} = \frac{1}{nq}\Sigma(y_1).$$

The mean value of $\bar{y}$ for a great number of samples coincides, according to the suppositions, with the mean of the population and this we choose for the zero point of y. The squared standard deviation of $\bar{y}$ is therefore found simply by squaring the expression above, summing for all the samples imagined and taking the mean value of the result. We thus find

$$\sigma_{\bar{y}}^2 = \frac{1}{n^2q^2}\{\overline{\Sigma(y_1^2)} + 2\overline{\Sigma(y_1y_2)} + 2\overline{S(y_1y_2)}\},$$

where a bar above a summation indicates that the mean value has to be taken of the sums for all samples, i.e. for the population. Let the standard deviation of the population be s and the correlation coefficient for individuals of the same class r, we then have

$$\overline{\Sigma(y_1^2)} = nqs^2$$

and

$$\overline{\Sigma(y_1y_2)} = \tfrac{1}{2}nq(q-1)rs^2.$$

As individuals of different classes are uncorrelated $\overline{\Sigma(y_1y_2)}$ is equal to 0, and accordingly we find

$$\sigma_{\bar{y}}^2 = \frac{s^2}{nq}\{1+(q-1)r\}\,^* \text{..........................} (1).$$

This contains s and r for the population, which are, as a rule, only known from the sample in hand. It will be seen in the following, what is the approximation obtained by putting s and r equal to the values found from the sample.

(b) *The Mean Value of σ^2—the* presumptive *Standard Deviation.*

For our sample we find

$$\sigma^2 = \frac{1}{nq}\Sigma(y_1^2) - \bar{y}^2 \text{..............................} (2).$$

By taking the mean of σ^2 for a great number of samples we find from this, remembering that the mean of $\bar{y}^2$ equals $\sigma_{\bar{y}}^2$,

$$\overline{\sigma^2} = s^2\left(1 - \frac{1+(q-1)r}{nq}\right) \text{.........................} (3).$$

When we take the value found for σ^2 as an approximation to $\overline{\sigma^2}$, we find accordingly the *presumptive* value of the standard deviation of the population by the formula

$$_p\sigma^2 = s^2 = \sigma^2\frac{nq}{nq - \{1+(q-1)r\}},$$

which for $r=0$ or $q=1$ takes the form known for uncorrelated observations.

* *Vide Comptes-Rendus des Trav. du Lab. Carlsberg*, Vol. XIV. No. 11, 1921, Copenhagen, p. 22.

For the S.D. of y we find by introducing s in (1)

$$\sigma_y^2 = \sigma^2 \frac{1+(q-1)r}{nq-\{1+(q-1)r\}}.$$

(c) *The Standard Deviation of σ^2*

The S.D. of the σ^2 of our sample is found from $\sigma_{\sigma^2}^2 = \overline{\sigma^4} - (\overline{\sigma^2})^2$, where the latter term is already known. From (2) we find for the calculation of σ^4

$$n^2q^2\sigma^2 = (nq-1)\Sigma(y_1^2) - 2\Sigma(y_1y_2) - 2S(y_1y_2) \quad \ldots\ldots \ldots \quad \ldots\ldots(4),$$

and from this

$$n^4q^4\sigma^4 = (nq-1)^2(\Sigma(y_1^2))^2 + 4(\Sigma(y_1y_2))^2 + 4(S(y_1y_2))^2 +$$

$$-4(nq-1)\Sigma(y_1^2)\Sigma(y_1y_2) - 4(nq-1)\Sigma(y_1^2)S(y_1y_2) + 8\Sigma(y_1y_2)S(y_1y_2) \;\ldots(5).$$

For the calculation of the mean values contained in this equation the six products of product sums must be examined. We find

$$\left.\begin{aligned} (\Sigma(y_1^2))^2 &= \Sigma(y_1^4) + 2\Sigma(y_1^2y_2^2) + 2S(y_1^2y_2^2) \\ (\Sigma(y_1y_2))^2 &= \Sigma(y_1^2y_2^2) + 2\Sigma(y_1^2y_2y_3) + 6\Sigma(y_1y_2y_3y_4) + 2S(y_1 {}_s\, y_2y_3 {}_s\, y_4)^* \\ \Sigma(y_1^2)\Sigma(y_1y_2) &= \Sigma(y_1^3y_2) + \Sigma(y_1^2y_2y_3) + S(y_1^2y_2 {}_s\, y_3) \end{aligned}\right\} \quad (6)$$

When the multiplication of products containing the factor $S(y_1y_2)$ is carried out it is clear that we need not consider such sums of products where the product contains a factor which is uncorrelated with all the other factors of the product, because the mean values of such product sums are 0. In the products $\Sigma(y_1^2)S(y_1y_2)$ and $\Sigma(y_1y_2)S(y_1y_2)$ all the sums of products are of this kind, the factors being distributed either in two classes of which one contains 3 and the other 1 factor or in three classes with respectively 2, 1 and 1 in each.

We therefore find

$$\left.\begin{aligned} (S(y_1y_2))^2 &= S(y_1^2y_2^2) + 2S(y_1^2y_2 {}_s\, y_3) + 4S(y_1 {}_s\, y_2y_3 {}_s\, y_4) + \alpha_1 \\ \Sigma(y_1^2)S(y_1y_2) &= \alpha_2 \\ \Sigma(y_1y_2)S(y_1y_2) &= \alpha_3, \end{aligned}\right\} \quad \ldots \;\ldots (7),$$

where the mean values of the α's for the population are 0.

Let us denote the product moment corresponding to $y_1^m y_2^n y_3^p y_4^q$ by β_{mnpq} if all factors belong to the same class and in the opposite case let us insert 'd' or 's' as denoting different or same class.

* In the sums S all factors of a product are supposed to belong to different classes except those which are denoted by an 's' inserted between them, as belonging to the same class.

We find then

$$\left.\begin{aligned}
\Sigma(y_1^4) &= nq\beta_4\\
\Sigma(y_1^3 y_2) &= nq(q-1)\beta_{31}\\
\Sigma(y_1^2 y_2^2) &= \tfrac{1}{2}nq(q-1)\beta_{22}\\
\Sigma(y_1^2 y_2 y_3) &= \tfrac{1}{2}nq(q-1)(q-2)\beta_{211}\\
\Sigma(y_1 y_2 y_3 y_4) &= \tfrac{1}{24}nq(q-1)(q-2)(q-3)\beta_{1111}\\
S(y_1^2 y_2^2) &= \tfrac{1}{2}n(n-1)q^2\beta_{\underset{d}{2\,2}}\\
S(y_1^2 \underset{s}{y_2} y_3) &= \tfrac{1}{2}n(n-1)q^2(q-1)\beta_{\underset{d\ s}{2\,1\,1}}\\
S(y_1 \underset{s}{y_2} y_3 \underset{s}{y_4}) &= \tfrac{1}{4}n(n-1)q^2(q-1)^2\beta_{\underset{s\ d\ s}{1\,1\,1\,1}}
\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (8)$$

Till now no suppositions have been made as to the law of distribution of the y's, but *in the following calculation we shall suppose that the distribution is normal and the correlation between individuals of the same class normal.*

For the general case of normal correlation between n variables the product moments have been determined by Sverker Bergström *. Taking the standard deviations as units of the variable and denoting the correlation coefficients by r_{12}, r_{23} where for instance r_{23} means the correlation coefficient between the 2nd and 3rd variable of a product moment β'_{mnpq} he finds the following formulae for the product moments of the 4th order:

$$\left.\begin{aligned}
\beta'_4 &= 3\\
\beta'_{31} &= \beta'_{13} = 3r_{12}\\
\beta'_{22} &= 2r_{12}^2 + 1\\
\beta'_{211} &= 2r_{12}r_{13} + r_{23}\\
\beta'_{1111} &= r_{12}r_{34} + r_{13}r_{24} + r_{14}r_{23}
\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (9)$$

Substituting our special values for the correlation coefficient we find

$$\left.\begin{aligned}
\beta_4 &= 3s^4\\
\beta_{31} &= 3rs^4\\
\beta_{22} &= (2r^2+1)s^4\\
\beta_{211} &= r(1+2r)s^4\\
\beta_{1111} &= 3r^2s^4
\end{aligned}\right. \qquad \ldots\ldots\ldots\ldots (10).$$

and further

$$\left.\begin{aligned}
\beta_{\underset{d}{2\,2}} &= s^4\\
\beta_{\underset{d\ s}{2\,1\,1}} &= rs^4\\
\beta_{\underset{s\ d\ s}{1\,1\,1\,1}} &= r^2s^4
\end{aligned}\right\}$$

We are now by means of (8) and (10) in a position to evaluate the mean values of the products put down under (6) and (7).

* *Vide* S. Bergström: *Biometrika*, Vol. XII 1918, p. 177.

We find

$$\begin{aligned}
&\overline{(\Sigma(y_1^2))^2} &&= nq\,\{nq + 2 + 2(q-1)\,r^2\}\,s^4 \\
&\overline{(\Sigma(y_1 y_2))^2} &&= \tfrac{1}{2}nq\,(q-1)\,\{1 + 2(q-2)\,r + [\tfrac{1}{2}nq\,(q-1) + q^2 - 3q + 3]\,r^2\}\,s^4 \\
&\overline{\Sigma(y_1^2)\,\Sigma(y_1 y_2)} &&= \tfrac{1}{2}\,nq\,(q-1)\,\{nq + 4 + 2\,(q-2)\,r\}\,rs^4 \qquad\qquad (11). \\
&\overline{(S(y_1 y_2))^2} &&= \tfrac{1}{2}n\,(n-1)\,q^2\,\{1 + (q-1)\,r\}^2\,s^4 \\
&\overline{\Sigma(y_1^2)\,S(y_1 y_2)} &&= 0 \quad\text{and}\quad \overline{\Sigma(y_1 y_2)\,S(y_1 y_2)} = 0
\end{aligned}$$

The calculation of $\overline{\sigma^4}$ may now be continued. We find, by substituting the above mean values in (5),

$$n^2q^2\overline{\sigma^4} = s^4\,\{n^2q^2 - 1 - 2\,(nq+1)\,(q-1)\,r + (q-1)\,[2nq - (q-1)]\,r^2\}.$$

From (3) is found

$$n^2q^2\,(\overline{\sigma^2})^2 = s^4\,\{n^2q^2 - 2nq + 1 - 2\,(nq-1)\,(q-1)\,r + (q-1)^2\,r^2\},$$

and accordingly

$$\sigma^2_{\sigma^2} = \overline{\sigma^4} - (\overline{\sigma^2})^2 = \frac{2s^4}{n^2q^2}\,\{nq - 1 - 2\,(q-1)\,r + (q-1)\,(nq - q + 1)\,r^2\},$$

or arranged according to powers of nq

$$\sigma^2_{\sigma^2} = \frac{2s^4}{nq}\left\{1 + (q-1)\,r^2 - \frac{1}{nq}\,[1 + r\,(q-1)]^2\right\} \quad \ldots\ldots\ldots\ldots (12)$$

This formula for the S.D. of the squared standard deviations is thus exact, supposing that the correlation be normal.

For great values of n or rather of $\dfrac{nq}{1 + (q-1)\,r^2}$ we may consider the S.D. of σ^2 a differential, so that

$$\sigma'^2 = \sigma^2 + \delta\sigma^2 = \sigma^2 + 2\sigma\,\delta\sigma.$$

From $\delta\sigma^2 = 2\sigma\,\delta\sigma$ we find by squaring and taking mean value for a great number of samples,

$$\sigma^2_{\sigma^2} = 4\sigma^2\sigma_\sigma^2,$$

and by substituting the value of $\sigma^2_{\sigma^2}$, omitting the last term,

$$\sigma_\sigma^2 = \frac{s^2}{2nq}\,\{1 + (q-1)\,r^2\},$$

or, as with the accuracy obtainable we have

$$s^2 = \sigma^2,$$

it follows that:
$$\sigma_\sigma^2 = \frac{\sigma^2}{2nq}\,\{1 + (q-1)\,r^2\}.$$

We notice when comparing this formula with (1) that only for $r = 1$ and $r = 0$ does the rule

$$\sigma_\sigma^2 = \tfrac{1}{2}\sigma_v^2$$

hold good.

The fraternal correlation coefficient ρ for the present sample is, when all the $\frac{1}{2}q(q-1)$ pairs of siblings are used for the calculation, defined by

$$\rho = \frac{\Pi}{\sigma^2},$$

where

$$\Pi = \frac{2}{nq(q-1)} \Sigma(y_1 y_2) - \bar{y}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

To determine the S.D. of ρ one requires in addition to $\sigma^2_{\sigma^2}$, the S.D. of Π and the product moment for Π and σ^2.

(d) *Mean Value and Standard Deviation of the Product Moment* Π.

Taking mean value of (13) for a great number of samples we find as

$$\overline{\Sigma(y_1 y_2)} = \tfrac{1}{2} nq(q-1) r s^2 \text{ and } (\bar{y}^2) = \sigma_{\bar{y}}^2,$$

$$\bar{\Pi} = s^2 \left\{ r - \frac{1}{nq}[1 + (q-1) r] \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

For calculating the mean value of Π^2 (13) may be written

$$n^2q^2(q-1)\,\Pi = -(q-1)\Sigma(y_1^2) + 2(nq-q+1)\Sigma(y_1 y_2) - 2(q-1) S(y_1 y_2) \ldots(15),$$

from which follows

$$n^4q^4(q-1)^2\,\Pi^2 = (q-1)^2(\Sigma(y_1^2))^2 + 4(nq-q+1)^2(\Sigma(y_1 y_2))^2 +$$
$$- 4(q-1)(nq-q+1)\overline{\Sigma(y_1^2)\Sigma(y_1 y_2)} + 4(q-1)^2(S(y_1 y_2))^2,$$

the mean values of the two products being 0 according to (11). Substituting the rest of the values from (11) we find

$$(q-1) n^2q^2\overline{\Pi^2} = s^4 \{2nq - (q-1) + 2r[nq(q-3) - (q-1)^2]$$
$$+ r^2[n^2q^2(q-1) - 2nq(q-2) - (q-1)^2]\} \ldots\ldots (16),$$

and by squaring (14) is found

$$(q-1) n^2q^2(\bar{\Pi})^2 = s^4\{q - 1 - 2r[nq(q-1) - (q-1)^2]$$
$$+ r^2[n^2q^2(q-1) - 2nq(q-1)^2 + (q-1)^3]\}.$$

By subtraction of this equation from (16) we arrive at

$$\sigma^2_\Pi = \overline{\Pi^2} - (\bar{\Pi})^2 = \frac{2s^4}{nq(q-1)} \left\{ 1 - \frac{q-1}{nq} + 2r\left[q - 2 - \frac{(q-1)^2}{nq}\right] \right.$$
$$\left. + r^2\left[q^2 - 3q + 3 - \frac{(q-1)^3}{nq}\right]\right\},$$

or arranged according to nq

$$\sigma^2_\Pi = \frac{2s^4}{nq(q-1)} \left\{ 1 + 2r(q-2) + r^2(q^2 - 3q + 3) - \frac{q-1}{nq}[1 + r(q-1)]^2 \right\},$$

which may also be written

$$\sigma^2_\Pi = \frac{2s^4}{nq(q-1)} \left\{ [1 + r(q-2)]^2 + r^2(q-1) - \frac{q-1}{nq}[1 + r(q-1)]^2 \right\} \ldots(17)$$

(e) *The Product Moment*, $\Pi_{\Pi\sigma^2}$, *of* Π *and* σ^2.

By multiplication of (4) and (15) and taking mean value for a great number of samples we find for the mean value of the product $\Pi\sigma^2$

$$n^4q^4(q-1)\,\overline{\Pi\sigma^2} = -(nq-1)(q-1)(\overline{\Sigma(y_i^2)})^2 - 4(nq-q+1)(\overline{\Sigma(y_1y_2)})^2$$
$$+ 2\{n^2q^2 - nq^2 + 2(q-1)\}\,\overline{\Sigma(y_i^2)\Sigma(y_1y_2)} + 4(q-1)(\overline{S(y_1y_2)})^2,$$

the mean values of the two products being zero according to (11). Introducing the rest of the mean values from (11), we have

$$n^2q^2\,\overline{\Pi\sigma^2} = s^4\{-nq-1+r[n^2q-nq(q-4)-2(q-1)]+r^2[nq(q-3)-(q-1)^2]\}.$$

From (3) and (14) is found

$$n^2q^2\,\overline{\Pi}\,.\,\overline{\sigma^2} = s^4\{-nq+1+r[n^2q^2-nq^2+2(q-1)]+r^2[-nq(q-1)+(q-1)^2]\}.$$

As
$$\Pi_{\Pi\sigma^2} = \overline{\Pi\sigma^2} - \overline{\Pi}\,.\,\overline{\sigma^2},$$

it follows from the two foregoing equations that

$$\Pi_{\Pi\sigma^2} = \frac{2s^4}{n^2q^2}\{-1+2r[nq-(q-1)]+r^2[nq(q-2)-(q-1)^2]\},$$

or
$$\Pi_{\Pi\sigma^2} = \frac{2s^4}{nq}\left\{r[2+(q-2)r] - \frac{1}{nq}[1+(q-1)r]^2\right\} \quad \ldots\ldots\ldots\ldots \quad (18)$$

(f) *The Standard Deviation of the Fraternal Correlation Coefficient*

If the sample is great in proportion to $(q-1)r$ the errors of Π and σ^2 can be treated as differentials and we have for the correlation coefficient calculated from a sample

$$\rho = \frac{\Pi+\delta\Pi}{\sigma^2+\delta\sigma^2} = \frac{\Pi}{\bar{\sigma}^2} + \frac{1}{\sigma^2}\delta\Pi - \frac{\Pi}{(\sigma^2)^2}\delta\sigma^2$$

and
$$\bar{\rho} = \frac{\overline{\Pi}}{\overline{\sigma^2}} = \frac{r - \dfrac{1}{nq}\{1+(q-1)r\}}{1-\dfrac{1}{nq}\{1+(q-1)r\}}.$$

and therefore neglecting the term containing $\frac{1}{nq}$ which according to these suppositions cannot be evaluated

$$\bar{\rho} = r.$$

From $\delta\rho = \frac{1}{\bar{\sigma}^2}\left\{\delta\Pi - \frac{\Pi}{\bar{\sigma}^2}\delta\sigma^2\right\}$ we find by squaring and forming mean value

$$\sigma^2_\rho = \frac{1}{(\sigma^2)^2}\left\{\sigma^2_\Pi + \left(\frac{\Pi}{\sigma^2}\right)^2\sigma^2_{\sigma^2} - 2\frac{\Pi}{\sigma^2}\Pi_{\Pi\sigma^2}\right\}.$$

When the values from (3), (12), (14), (17) and (18) are introduced in this formula and the terms containing the higher power of $\frac{1}{nq}$ are neglected, we get

$$\sigma_\rho^2 = \frac{2}{nq(q-1)}\{[1+r(q-2)]^2+r^2(q-1)\} + \frac{2r^2}{nq}\{1+(q-1)r\}^2 - \frac{4r^2}{nq}\{2+(q-2)r\},$$

from which is found

$$\sigma_\rho^2 = \frac{2}{nq(q-1)}\{1 + r(q-2) - r^2(q-1)\}^2$$

and

$$\sigma_\rho = \sqrt{\frac{2}{nq(q-1)}}(1-r)\{1+(q-1)r\} \quad \text{.................(19).}$$

For $q = 2$ this formula coincides with the usual formula for the standard deviation of a correlation coefficient calculated from two series of values of two variables corresponding in pairs, the values of each series being mutually uncorrelated.

(g) *Numerical Evaluation of the Formula for the S.D. of a Fraternal Correlation Coefficient.*

The number, N, of observed pairs of observations being equal to $\frac{1}{2}nq(q-1)$ the formula (19) may also be written

$$\sigma_\rho = \frac{1}{\sqrt{N}}(1-r)\{1+(q-1)r\}.$$

Comparing materials of observations with different number of siblings q, we see that for the calculation of fraternal correlation information of each available pair of siblings has a value inversely proportional to $\{1+(q-1)r\}^2$. The ratio $v_q = \left(\frac{1+r}{1+(q-1)r}\right)^2$ serves as a measure for the value which must be attributed to information of an observed pair among q siblings, supposed that all of the $\frac{1}{2}nq(q-1)$ pair of siblings are used for the calculation, and supposed that the value of information of a pair of siblings for $q = 2$ is put equal to 1. On the other hand $\frac{1}{v_q}$ indicates the ratio between the numbers of pairs of siblings which are required for obtaining the same accuracy in the correlation coefficient in the case of q and in the case of two siblings from each family. Table I gives the numerical values of v for different values of r and q.

TABLE I.

$$v_q = \left(\frac{1+r}{1+(q-1)r}\right)^2.$$

	$r = 0{\cdot}1$	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
2	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
3	·840	·735	·660	·605	·563	·524	·502	·179	·460
4	·716	·563	·468	·405	·360	·327	·301	·280	·264
5	·617	·444	·349	·290	·250	·221	·200	·184	·171
6	·538	·300	·270	·218	·184	·160	·143	·130	·119
7	·473	·298	·216	·170	·141	·121	·107	·096	·088
8	·419	·260	·176	·136	·111	·095	·083	·074	·068
9	·373	·213	·146	·111	·090	·076	·066	·059	·054
10	·335	·184	·123	·093	·074	·063	·054	·048	·044

We notice that for values of r somewhat greater than 0·5, such as are usually found for mammals, v_3 has already decreased to about $\frac{1}{2}$ and v_4 to about $\frac{1}{3}$. By giving the same weight to each pair of siblings when forming fraternal correlation tables from a material consisting of fraternities of different size, we therefore fail very largely to pay due regard to the observations With material under consideration, as for example anthropometric data, which according to its nature consists of small groups of siblings of varying number, and which is not so numerous that we can afford to omit observations from the calculation to make q constant for all fraternities, the rational proceeding must be to sort the material according to the number of siblings and calculate the correlation coefficient of each group separately.

It is then possible to effect considerable saving of time and labour in the investigation of correlation by avoiding the forming of fraternal correlation tables and using instead the formula

$$r = \frac{1}{q-1}\left(q\frac{\sigma_q^2}{\sigma^2} - 1\right)^*,$$

where σ_q is the directly calculated S.D. for mean values of fraternities. The results found by the formula are identical with those of the defining formula, so that the only objection to this method of calculation is the lack of opportunity to examine the shape of the regression curve.

From the correlation coefficients found for different values of q†, it is finally possible with knowledge of their S.D.'s to calculate a mean value of the fraternal correlation coefficient and its S.D.

In investigations of inheritance with animals with numerous offspring where a great number of siblings are available, we have to face the problem of deciding what number of siblings it is profitable to employ for the investigation.

We shall state provisionally the problem as follows: with which value of q do we, provided the number of examined offspring individuals (nq) be fixed, obtain the most accurately determined fraternal correlation coefficient? Or in other words for which value of q is

$$\frac{1}{q-1}\{1 + r(q-1)\}^2 \text{ a minimum?}$$

* *Vide* K. Smith, *Comptes-Rendus des Trav. du Lab. Carlsb.* Vol. XIV. No. 11, 1921, p. 8, where the formula is deduced for the special case $q = 10$.

† In the memoir quoted it is shewn (p. 29) that the above formula may also be written

$$r = 1 - \frac{q}{q-1}\frac{\sigma^2_{f,q}}{\sigma^2},$$

$\sigma^2_{f,q}$ being the squared S.D. inside fraternities of q siblings and being calculated as a mean of such values obtained from each of the n fraternities. We may here instead of $\sigma_{f,q}$ introduce the presumptive S.D. inside a fraternity ${}_p\sigma_f$ that is the S.D. we expect to find in fraternities consisting of a great number of siblings. The relation is

$${}_p\sigma_f^2 = \frac{q}{q-1}\sigma^2_{f,q}.$$

so that we find

$$r = 1 - \frac{{}_p\sigma_f^2}{\sigma^2},$$

which shews that the value of r arrived at must be expected to be independent of q.

The condition of minimum is

$$q = 1 + \frac{1}{r}.$$

Corresponding to the values $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{2}{3}$ for r the values of q are 5, 4, 3 and $\frac{5}{2}$.

In examining the question of the most profitable number of siblings, attention must also be paid to the determination of the parental correlation and the question will therefore be further discussed in the following section. Besides it cannot be left out of consideration that, as a rule, it will be easier to examine the same number of individuals distributed among a smaller than among a greater number of fraternities. When regard only is had to fraternal correlation, the values of q obtained above must therefore be considered the minimum values.

For a more detailed illustration of the variation of the S.D. of the fraternal correlation coefficient with the number of siblings Table II has been calculated. The table gives the values of the S.D. for 1000 observations distributed among from 500 to 100 fraternities, the sizes of which therefore vary from 2 to 10.

TABLE II.

The Standard Deviation of a Fraternal Correlation Coefficient calculated from 1000 observed Individuals.

q	$r=\frac{1}{4}$	$r=\frac{1}{3}$	$r=\frac{1}{2}$	$r=\frac{2}{3}$
2	·0419	·0398	·0335	·0286
3	·0356	·0351	·0316	·0278
4	·0339	·0344	·0323	·0289
5	·0335	·0348	·0335	·0304
6	·0337	·0356	·0350	·0320
7	·0342	·0365	·0365	·0336
8	·0349	·0376	·0380	·0352
9	·0356	·0387	·0395	·0367
10	·0363	·0398	·0410	·0382

The table does not show a rapid increase of the S.D. when the number of siblings increases beyond the most profitable number found above. But a comparison of the values for $q=5$ and for $q=10$ still shows that the latter are respectively 8 %, 14 %, 22 % and 25 % greater than the former, so that when there are 10 siblings in each fraternity respectively 18 %, 31 %, 50 % and 58 % more individuals are required to obtain the same accuracy than when there are only 5 siblings from each family.

(h) *Application of the Formula to previous Calculations of Correlation.*

In an investigation* concerning the characters, *number of vertebrae* ('Vert.'), *number of rays in the pectoral fins* ('Pd.' and 'Ps') and *number of pigment spots* ('Pigm.') in *Zoarces viviparus* from the station Nakkehage in Isefjord, Denmark,

* K. Smith, *Comptes-Rendus des Trav. du Lab. Carlsberg*, Vol. XIV No. 11, 1921.

the fraternal correlation coefficient was calculated for 6 (for pigment spot only 5) samples from different years consisting of fraternities of 10 siblings. In this case the probable error of the fraternal correlation coefficient is according to (19)

$$\text{P.E.}\,(r) = \frac{0{\cdot}67449}{\sqrt{45n}}\,(1-r)(1+9r).$$

Table III gives for each sample the values of n, r and P.E. (r), as well as r for all the samples each weighted according to the S.D.

TABLE III.

Fraternal Correlation.

Year when sample taken	Vert.		Pd.		Pigm.	
	n	$r \pm$ P.E.	n	$r \pm$ P.E.	n	$r \pm$ P.E.
1914	138	0·1590 ± ·0238	132	0·3169 ± ·0231		
1915	168	0·4693 ± ·0215	174	0·4196 ± ·0211	75	0·3175 ± ·0306
1916	123	0·5108 ± ·0248	122	0·3985 ± ·0251	87	0·3118 ± ·0289
1917	177	0·4715 ± ·0209	176	0·3634 ± ·0206	127	0·4112 ± ·0247
1918	153	0·4801 ± ·0225	156	0·3329 ± ·0215	113	0·3071 ± ·0247
1919	98	0·4066 ± ·0281	98	0·2893 ± ·0260	86	0·3722 ± ·0296
From total samples		0·4689 ± ·0095	—	0·3561 ± ·0092		0·3517 ± ·0122

For the mean values of r probable errors have previously been calculated based on the 6 or 5 values found. These probable errors had for

Vert. Pd. and Pigm. respectively

the values 0·0094 0·0137 and 0·0128,

which for Vert. and Pigm. agree extremely well with the theoretical values now found, while for Pd. the error had been estimated somewhat too great.

II. Parental Correlation.

For investigation of parental correlation we have a sample consisting as above of nq offspring values $y_1, y_2, y_3 \ldots\ldots y_{nq}$ distributed in n classes with q in each, and in addition, containing for each class an observed parental value x. We aim at finding the correlation between x and y's of the same class.

Let the parental correlation be r_p and the S.D. for x's s' in the population which we may imagine that the sample represents, and let us choose the mean value of the population as zero point for x.

The parental correlation coefficient is from the sample determined by

$$\rho_p = \frac{\Pi_{xy}}{\sigma\sigma'},$$

where σ' is the S.D. of x calculated from the sample, and Π_{xy} is the product moment for x and y determined by

$$\Pi_{xy} = \frac{1}{nq}\Sigma(x_1y_1) - \bar{x}\,\bar{y} \quad\text{.............................(20)}.$$

As in the previous section Σ denotes a sum of products each of which consists of factors from the same class. In the sums S each product contains factors from at least two classes, and when two factors belong to the same class it is indicated by an 's' inserted between them.

For evaluation of the standard deviation of ρ_p the S.D. of Π_{xy}, σ and σ' are required, as well as the product moments for each pair of these three functions.

(*a*) *Mean Value and Standard Deviation of the Product Moment* Π_{xy}.

The equation (20) may also be written

$$\Pi_{xy} = \frac{n-1}{n^2q}\Sigma(x_1y_1) - \frac{1}{n^2q}S(x_1y_1) \quad\text{.......... ....(21)}.$$

By taking the mean value for a great number of samples we therefore find

$$\overline{\Pi}_{xy} = \frac{n-1}{n}r_p ss' \quad\text{........... .. ..(22)}.$$

From (21) we find by squaring and taking mean value

$$n^4q^2\,\overline{\Pi^2_{xy}} = (n-1)^2\,\overline{(\Sigma(x_1y_1))^2} + \overline{(S(x_1y_1))^2} - 2(n-1)\,\overline{\Sigma(x_1y_1)S(x_1y_1)} \quad\text{...(23)}$$

Together with the determination of the mean values occurring here, we shall determine the other mean values of products required for the evaluation of σ_{ρ_p}. They are such as arise from multiplication of $\Sigma(x_1y_1)$ and $S(x_1y_1)$ with each of the two groups $\Sigma(y_1^2)$, $\Sigma(y_1y_2)$, $S(y_1y_2)$ and $\Sigma(x_1^2)$, $S(x_1x_2)$ and also those which contain a factor of each of the two latter groups. As in the foregoing section, we need, however, not consider products of a Σ and an S, because such products may be developed into sums of products all containing a factor uncorrelated with all the other factors of the product, from which it follows that the mean value for a great number of samples is zero for each of these sums of products. It remains to determine the following products:

$$\left.\begin{aligned}
(\Sigma(x_1y_1))^2 &= \Sigma(x_1^2y_1^2) + 2\Sigma(x_1^2y_1y_2) + 2S(x_1\underset{s}{\,}y_1x_2\underset{s}{\,}y_2)\\
(S(x_1y_1))^2 &= S(x_1^2y_1^2) + 2S(x_1^2y_1\underset{s}{\,}y_2) + 2S(x_1\underset{s}{\,}y_1x_2\underset{s}{\,}y_2) + \epsilon_1\\
\Sigma(x_1y_1)\,\Sigma(y_1^2) &= \Sigma(x_1y_1^3) + \Sigma(x_1y_1^2y_2) + S(x_1\underset{s}{\,}y_1y_2^2)\\
\Sigma(x_1y_1)\,\Sigma(y_1y_2) &= \Sigma(x_1y_1^2y_2) + 3\Sigma(x_1y_1y_2y_3) + S(x_1\underset{s}{\,}y_1y_2\underset{s}{\,}y_3)\\
S(x_1y_1)\,S(y_1y_2) &= S(x_1\underset{s}{\,}y_1y_2^2) + 2S(x_1\underset{s}{\,}y_1y_2\underset{s}{\,}y_3) + \epsilon_2\\
\Sigma(x_1y_1)\,\Sigma(x_1^2) &= \Sigma(x_1^3y_1) + S(x_1^2x_2\underset{s}{\,}y_1)\\
S(x_1y_1)\,S(x_1x_2) &= S(x_1^2x_2\underset{s}{\,}y_1) + \epsilon_3\\
\Sigma(x_1^2)\,\Sigma(y_1^2) &= \Sigma(x_1^2y_1^2) + S(x_1^2y_1^2)\\
\Sigma(x_1^2)\,\Sigma(y_1y_2) &= \Sigma(x_1^2y_1y_2) + S(x_1^2y_1\underset{s}{\,}y_2)\\
S(x_1x_2)\,S(y_1y_2) &= S(x_1y_1x_2y_2) + \epsilon_4
\end{aligned}\right\}\quad\text{........ (24)}$$

ϵ_1, ϵ_2, ϵ_3 and ϵ_4 are sums, the means of which are 0. The product moments are, as in the previous section, denoted by β and the indices concerning "x's" are placed in front of β, for instance $\frac{1}{nq}\Sigma(x^2y^2)$ is denoted by ${}_2\beta_2$. We thus find for the mean values of the sums occurring in (24):

$$\left.\begin{aligned}
&\Sigma(x_1^3 y_1) && = nq\,{}_3\beta_1\\
&\Sigma(x_1^2 y_1^2) && = nq\,{}_2\beta_2\\
&\Sigma(\overline{x_1^2 y_1 y_2}) && = \tfrac{1}{2}nq(q-1)\,{}_2\beta_{11}\\
&\Sigma(x_1 y_1^3) && = nq\,{}_1\beta_3\\
&\Sigma(\overline{x_1 y_1^2 y_2}) && = nq(q-1)\,{}_1\beta_{21}\\
&\Sigma(x_1 y_1 y_2 y_3) && = \tfrac{1}{6}nq(q-1)(q-2)\,{}_1\beta_{111}\\
&S(x_1^2 \underset{s}{x_2} y_1) && = n(n-1)q\,{}_{\underset{d}{2}\,\underset{s}{1}}\beta_1\\
&S(x_1^2 y_1^2) && = n(n-1)q\,{}_{\underset{d}{2}}\beta_2\\
&S(\overline{x_1^2 \underset{s}{y_1} y_2}) && = \tfrac{1}{2}n(n-1)q(q-1)\,{}_2\beta_{\underset{l}{1}\,\underset{s}{1}}\\
&S(\overline{\underset{s}{x_1} y_1 \underset{s}{x_2} y_2}) && = \tfrac{1}{2}n(n-1)q^2\,{}_{\underset{d}{1}\,\underset{s}{1}}\beta_{\underset{d}{1}}{}^{*}\\
&S(\overline{\underset{s}{x_1} y_1 y_2^2}) && = n(n-1)q^2\,{}_1\beta_{\underset{s}{1}\,\underset{d}{2}}\\
&S(\underset{s}{x_1} y_1 \underset{s}{y_2} y_3) && = \tfrac{1}{2}n(n-1)q^2(q-1)\,{}_1\beta_{\underset{s}{1}\,\underset{d}{1}\,\underset{s}{1}}
\end{aligned}\right\} \quad \text{(25)}.$$

From Bergström's formulae (9) we find, when introducing r_p, r and 0 for the correlation coefficients and remembering that in his formulae s and s' are taken as units for y and x:

$$\left.\begin{aligned}
&{}_3\beta_1 && = 3r_p s'^3 s\\
&{}_2\beta_2 && = (2r_p^2+1)s'^2s^2\\
&{}_2\beta_{11} && = (2r_p^2+r)s'^2s^2\\
&{}_1\beta_3 && = 3r_p s's^3\\
&{}_1\beta_{21} && = r_p(1+2r)s's^3\\
&{}_1\beta_{111} && = 3rr_p s's^3\\
&{}_{\underset{d}{2}\,\underset{s}{1}}\beta_1 && = r_p s'^3 s\\
&{}_{\underset{d}{2}}\beta_2 && = s'^2s^2\\
&{}_2\beta_{\underset{d}{1}\,\underset{s}{1}} && = rs'^2s^2\\
&{}_{\underset{d}{1}\,\underset{s}{1}}\beta_{\underset{d}{1}}{}^{*} && = r_p^2 s'^2 s^2\\
&{}_1\beta_{\underset{s}{1}\,\underset{d}{2}} && = r_p s's^3\\
&{}_1\beta_{\underset{s}{1}\,\underset{d}{1}\,\underset{s}{1}} && = rr_p s's^3
\end{aligned}\right\} \quad \text{(26)}.$$

* In this single case the notation fails, as it ought to be indicated that the first x and the last y belong to the same class.

Applying (25) and (26) we find for the mean values of the products under (24) the following values

$$
\left.
\begin{aligned}
(\Sigma(x_1 y_1))^2 &= nq\{q(n+1)r_1 + 1 + (q-1)r\}\, s'^2 s^2 \\
(S(x_1 y_1))^2 &= nq(n-1)\{qr_p^2 + 1 + (q-1)r_1\}\, s'^2 s^2 \\
\Sigma(x_1 y_1)\Sigma(y_1^2) &= nq\{nq + 2 + 2(q-1)r\}\, r_p s'^2 s^2 \\
\Sigma(x_1 y_1)\Sigma(y_1 y) &= \tfrac{1}{2}nq(q-1)\{2 + (nq+2q-2)r\}\, r_p s'^2 s^2 \\
S(x_1 y_1)S(y_1 y) &= n(n-1)q^2\{1 + (q-1)r\}\, r_p s'^2 s^2 \\
\Sigma(x_1 y_1)\Sigma(x^2) &= nq(n+2)r_1 s'^2 s^2 \\
S(x_1 y_1)S(x_1 x) &= n(n-1)qr_1 s'^2 s^2 \\
\Sigma(x_1^2)\Sigma(y^2) &= nq(n+2r_1^2)s'^2 s^2 \\
\Sigma(x_1^2)\Sigma(y_1 y) &= \tfrac{1}{2}nq(q-1)(2r_1^2 + nr)s'^2 s^2 \\
S(x_1^2)S(y_1 y) &= \tfrac{1}{2}n(n-1)q\, r_p\, s'^2 s^2
\end{aligned}
\right\} \qquad (27)
$$

We may now continue the calculation of Π_{xy}. Introducing the mean values in (23) we get

$$n q \Pi_{xy} = (n-1)\{nqr_1 + 1 + (q-1)r\}\, s'^2 s^2.$$

From (22) we find

$$n q (\Pi_{x})^2 = (n-1)\, qr_1^2\, s'^2 s^2,$$

and when this equation is subtracted from the foregoing

$$\sigma^2_{\Pi_{xy}} = \Pi_{xy} - (\Pi_{x})^2 = \frac{n-1}{nq}\{qr_1 + 1 + r(q-1)\}\, s'^2 s^2. \qquad (28)$$

(b) The Product Moment $\Pi_{\Pi_{xy}\sigma}$ of Π_{xy} and σ.

Multiplication of (1) and (21) gives

$$n^2 q\, \Pi_{xy}\, \sigma = (nq-1)(n-1)\Sigma(y^2)\Sigma(x_1 y) - 2(n-1)\Sigma(x_1 y)\Sigma'(y\,y) + 2S(x_1 y_1)S(y\,y) + \gamma_1$$

where γ_1 consists of terms $S \cdot \Sigma$, the mean values of which are zero.

Taking the mean value and applying (27) we therefore find

$$n q\, \Pi_{xy}\, \sigma = (n-1)r_1\{nq(nq+1) + rnq(q-1)\}\, s^2 s.$$

For $\Pi_{x}\, \sigma$ we get from (3) and (22)

$$n q\, \Pi_{xy}\, \sigma = (n-1)r_1\{nq(nq-1) - rnq(q-1)\}\, s'^2 s,$$

and accordingly from the two latter equations

$$\Pi_{\Pi_{xy}\sigma^2} = \Pi_{xy}\,\sigma - \Pi_{x}\,\sigma = \frac{2(n-1)}{nq}\, r_1\{1 + r(q-1)\}\, s'^2 s. \qquad (29)$$

(g) *Numerical Evaluation of the Formula for the S.D. of a Parental Correlation Coefficient.*

We shall first examine how valuable a material consisting of n groups of q siblings with corresponding parental values is compared with nq pairs of values from different families. Denoting the S.D.'s of ρ_p calculated from the two materials by $\sigma_{q\rho_p}$ and $\sigma_{1\rho_p}$ we find by applying (36)

$$v_q' = \frac{\sigma_1^2\rho_p}{\sigma_q^2\rho_p} = \frac{(1-r_p^2)^2}{q(1-r_p^2)^2-(q-1)(1-r)\left\{1-r_p^2\,\frac{3-r}{2}\right\}} \quad \text{.........(37).}$$

This ratio indicates the value of an observed pair, when the parental value also occurs combined with $(q-1)$ other offspring values, in proportion to the value of an observed pair when the parental value only occurs once in the calculation.

The numerical values of (37) are, for values of r_p and r, fairly well representative of the values met with in investigations of inheritance given in Table IV.

TABLE IV.

$$v_q' = \sigma^2{}_1\rho_p : \sigma^2{}_q\rho_p.$$

	$r_p = \cdot 3$ $r = \cdot 4$	$r_p = \cdot 4$ $r = \cdot 5$	
1	1·000	1·000	1·000
2	·735	·698	·666
3	·581	·536	·499
4	·481	·435	·399
5	·410	·366	·332
6	·357	·316	·285
7	·316	·278	·249
8	·284	·248	·221
9	·258	·224	·199
10	·236	·204	·181

It appears that entering into the same parental correlation table families with numbers of offspring varying from, for example, 1 to 5 the same weight is given to pairs of observations which according to Table IV ought to vary in weight from 1 to $\frac{1}{3}$.

It is therefore a more rational proceeding to sort the families according to the number of offspring and deal with each group separately. The work may then be shortened by calculating the correlation coefficient between the parental value and the mean for the offspring from which the parental correlation for individuals is obtained by multiplying with $\frac{\sigma_q}{\sigma}$, σ_q being as above (see I (g)) the S.D. for means of fraternities of q individuals. It is then possible to calculate the correlation coefficient with S.D. for each group of families and finally calculate a mean value for the correlation coefficient.

In investigations of inheritance with animals with numerous offspring it is as a rule easier to provide information of a given number of individuals among a small number of families than to examine the same number of individuals if they belong to a larger number of families. The labour required is therefore not proportional to the number of individuals and it must be estimated for the individual materials whether the encumbrance of dealing with a relatively large number of families is duly compensated for by the reduction of the number of individuals hereby permissible.

It does not seem at the outset probable, but it may be possible, that, even in cases in which parent and offspring are equally easily available for investigation, a shortening of labour, that is, a diminution of the total number of observed individuals, may be obtainable by examining several offspring individuals of each family. We will therefore examine for which value of q, $\sigma^2_{\rho_p}$ is a minimum when $n(q+1)$ is put equal to a constant k. We find the condition

$$(1-r_p^2)^2 - \left(1+\frac{1}{q}\right)(1-r)\left\{1-r_p^2\,\frac{3-r}{2}\right\} = 0,$$

from which follows

$$(1-r)\left\{1-r_p^2\,\frac{3-r}{2}\right.$$

$$(1-r_p^2)^2-(1-r)\left\{1 \qquad 2\right\}$$

To obtain a survey we introduce a few sets of values for r_p and r for which we give the result in Table V.

TABLE V.

0·20	0·25	1·8
0·30	0·40	1·3
0·50	0·60	1·0

It will be seen, that for sufficiently small values of r and r_p it is profitable to examine several siblings of each family in those cases where the examination of an offspring individual requires the same labour as that of a parent.

As a guide for the choice of the number of offspring in the more frequently occurring case when it is easier to provide data of offspring than of parent, we give in Table VI for some values of r_p and r the number of observations which, for varying values of q, yield the same accuracy in the parental correlation coefficient as 1000 parents with 1000 offspring.

It appears from the table that while the number of offspring increases evenly with increasing q the number of parents decreases more and more slowly, so that the compensation obtained in this way for the increased total number of offspring

tends to be very small for increasing q. Already by increasing q from 5 to 6 we find, for $r_p = \cdot 3$ and $r = \cdot 4$, that to outweigh the augmentation of 360 in the number of offspring, we only get a diminution of 21 in the number of parents.

TABLE VI.

Number of Parental and Offspring Individuals which for varying q yield the same Accuracy to ρ_p.

	$r_p = \cdot 3$, $r = \cdot 4$, $\cdot 0288$		$r_p =$ [illegible], $r = \cdot 5$, σ_ρ $\cdot 0266$		$r_p = \cdot 5$, $r = \cdot 6$, σ_{ρ_p} $\cdot 0237$	
	Number of Parents	Number of Offspring	Number of Parents	Number of Offspring	Number of Parents	Number of Offspring
	1000	1000	1000	1000	1000	1000
	680	1360	717	1133	751	1502
	573	1720	622	1866	668	2004
	520	2081	575	2299	627	2507
	488	2411	546	2732	602	3009
	467	2801	528	3166	585	3511
7	452	3161	511	3599	573	4013
8	440	3522	504	4032	565	4516
9	431	3882	496	4465	558	5018
10	424	4242	490	4898	552	5520

For fraternal correlation we have found (see Table II) that the most profitable number of offspring was 3–4 for the values of r now considered, and that a somewhat greater number was not substantially opposed to economy of work. Whether the number ought to be increased beyond 3–4 or confined to even fewer offspring individuals from each family depends in each investigation upon the relative difficulty of observing parents and offspring.

(*h*) *Application of the Formula to previous Calculations of Correlation.*

For the investigation of *Zoarces viviparus* mentioned in the previous section, in which 10 offspring individuals were examined for each mother, we have according to (36) the following formula for the probable error of the maternal correlation coefficient:

$$\text{P.E.}\ (r_p) = \frac{0{\cdot}67449}{\sqrt{n}} \left\{ (1 - r_p^2)^2 - \frac{9}{10}(1 - r)\left[1 - r_p^2 \frac{3 - r}{2}\right] \right\}^{\frac{1}{2}}.$$

In Table VII are found the values of r_p for the three characters examined as well as their probable errors calculated from this formula. Giving each of these values of r_p its due weight we have calculated a mean value and its probable error.

TABLE VII.

Maternal Correlation.

Year when sample taken	Vert. $r_p \pm$ P.E.	Pd. $r_p \pm$ P.E.	Pigm. $r_p \pm$ P.E.
1914	0·3513 ± ·0343	0·2409 ± ·0332	
1915	0·4375 ± ·0281	0·3215 ± ·0303	0·3762 ± ·0381
1916	0·4139 ± ·0355	0·2116 ± ·0387	0·3622 ± ·0373
1917	0·3775 ± ·0298	0·2824 ± ·0293	0·3722 ± ·0332
1918	0·4382 ± ·0298	0·2928 ± ·0298	0·3710 ± ·0308
1919	0·3674 ± ·0378	0·1851 ± ·0387	0·3380 ± ·0398
From total samples	0·4021 ± ·0131	0·2654 ± ·0133	0·3654 ± ·0158

It appears that these probable errors agree extremely well with those originally calculated* on the basis of the 5 or 6 values of the correlation coefficient obtained from 5 or 6 samples.

Summary.

In the first section we dealt with fraternal correlation and a formula was deduced for the standard deviation of the fraternal correlation coefficient for the case when the material of observation consists of equal numbers of offspring from each family and when each available pair of siblings is introduced into the calculation. The formula is calculated on the supposition of normal distribution and normal fraternal correlation.

It is shewn by means of the formula that forming fraternal correlation tables for fraternities of different numbers and giving each pair of observations the same weight we disturb very highly the distribution of weight which the observations must claim according to their nature. We find further from the formula that when the number of observed offspring from each family may be freely chosen the best determination of fraternal correlation from a given number of observations is obtained by taking $\left(1 + \frac{1}{r}\right)$ offspring individuals from each family (r = frater. corr. coeff.)

In the second section we deduce, also supposing normal distribution and normal correlation, the S.D. of the parental correlation coefficient calculated from a material comprising equal numbers of offspring from each family. The formula shews that forming parental correlation tables of a material consisting of families of different sizes we also in an unfortunate manner disturb the due distribution of weight among the pairs of observation. It is shewn that if observations of

* *Vide l.c.*, p. 24, Table 6.

parents are as easily produced as those of offspring it is, for determination of parental correlation, only for small values of the corr. coeffs., for instance $r_p < \frac{1}{4}$ and $r < \frac{1}{3}$, profitable to include more than one offspring individual from each family in the calculation. For the case more frequently occurring, when the observation of parents represents more labour or greater cost than that of offspring, we have for certain values of r_p and r and varying sizes of fraternities calculated such numbers of parents and of offspring which yield the same accuracy to the parental correlation as 1000 parents with corresponding 1000 offspring. Table VI shews that when the number of siblings exceeds 4–5, there is not much gained by increasing it.

Considering both fraternal and parental correlation we may therefore generally conclude that an essential increase in the number of offspring beyond $1 + \frac{1}{r}$, i.e. in practice 3–4, is only then to be recommended, when it causes a relatively insignificant increase in labour.

This research has been occasioned by the investigations of inheritance carried out by the Carlsberg Laboratorium København and I am much indebted to Dr. Johs. Schmidt for the interest he has taken in my work.

ON THE VARIATIONS IN PERSONAL EQUATION AND THE CORRELATION OF SUCCESSIVE JUDGMENTS.

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CONTENTS.

1. Introduction.

Starting from Bessel's discovery, in the early part of the last century, of the existence of a definite relative personal equation for two observers recording transits by the eye and ear method, there has been a continuous discussion among astronomers on the errors which such personal equations may introduce, and on the methods of eliminating them or correcting for them*. In such discussions it has been the usual practice to take the yearly mean personal equation, whether relative or absolute, of different observers and to use this mean personal equation as the basis of any correction to be applied to observations made in that year. From a comparison of the yearly means it is admitted that there may be gradual secular changes in personal equation, but it is found that for experienced observers there is usually very little variation. In text-books on Practical Astronomy brief mention of the subject is usually made, and the conclusion drawn is that for an observer in normal health, the personal equation in any one type of observation will remain sensibly constant for "short periods" of time; an exact definition of the words "short period" is not and clearly cannot be attempted†. It is further assumed that variations from the personal equation are due to accidental errors and may be taken as randomly distributed in accordance with the Gaussian Law. With the recent introduction of photography and mechanical methods of record, the interest of the astronomer in the subject has to some extent diminished, but there are many fields of scientific observation where the human element cannot be eliminated, and in the modern researches of the psychologist we find a study is made of problems of this type for their own interest and for the light which they may throw on the working of the human machine.

One very important aspect of the problem of personal equation, and of particular import to the astronomer, was discussed in detail in a paper entitled "On the Mathematical Theory of Errors of Judgment, with Special Reference to the Personal Equation," published in the *Phil. Trans.* (Vol. 198 A, p. 235). In this case various series of experiments were carried out simultaneously by three observers under identical conditions and it was found that there was a marked correlation between the variations in absolute personal equation of the different observers. This in itself was sufficient to show that the judgments of any one observer were not randomly distributed about his mean personal equation. The purpose of the present paper is to discuss the variations in judgment of *one* observer, and to inquire how far the evidence of four or five experiments suggests that the theory of personal equation and of errors of judgment, as usually accepted, requires modification.

The subject is a large one, and much beyond the scope of a single paper; but by making careful inquiries of this type with the help of statistical methods, it

* For example, *Monthly Notices*, Vol. XL. 1880, pp. 75, 165, 302 (Discussion of Greenwich Observations of the Moon); *Monthly Notices*, Vol. XLIV. 1884, pp. 1 and 39 (Greenwich Observations of the Sun); *Monthly Notices*, Vol. LVII. 1897, p. 504 (General Discussion of relative personal Equations).

† For example, in Campbell's *Elements of Practical Astronomy*, 1899, p. 157; Young's *General Astronomy*, Revised Edn. § 114, and Chauvenet's *Spherical and Practical Astronomy*, 4th Edn. II. p. 189.

may be possible to construct a more generalised theory of errors of judgment than that which has hitherto been adopted, and although the practical corrections which such a theory will impose may not be large, yet a more detailed knowledge of the nature of the variations and perhaps some insight into the psychological and physiological factors which underlie them, will give the observer a clearer idea of the precautions to be taken to avoid error and a greater justification for confidence in his results.

II. GENERALISED THEORY OF PERSONAL EQUATION.

Before proceeding to the reduction of the Experiments which have been carried out, I will consider whether it is not possible to make a very general, and yet simple, analysis of personal equation. Let us suppose that we have a large number, N, of observations, which have been made in separate groups, or at what may be termed separate *sessions*. For the astronomer, a session will be a night's work; for the physicist or psychologist, one continuous set of readings or observations. Any particular observation y may be designated (1) by τ, a function of the time when it was recorded, measured from some fixed epoch, or (2) by the number of the session in which it was made, and t, the time of record measured from the commencement of that session. E.g. an observation made in the pth session may be written either as y_τ or ${}_py_t$. We will suppose that the secular change can be represented by the function $\phi(\tau)$, but in addition to this change there may be another of a different type which may be termed the *sessional change*, and will be represented by the function $f_p(t)$. The fundamental difference between a secular and sessional change is this: if there is a break of some hours or perhaps days between two series of observations, the sessional change of the first series will have no influence on the judgments of the second series, while the secular change will continue from series to series. The sessional change is thus peculiar to its own session or series of observations, although it is very possible that the same type of change may be repeated in session after session: it may be a change resulting simply from fatigue or perhaps from more complex causes. Figure 3 (p. 46) provides a good illustration of secular and sessional changes; the centres of the small circles represent the mean values of twenty different series of observations, and it will be seen that the general tendency is for a drop in mean judgment from left to right of the diagram; this is the secular change. The sessional changes are represented by the continuous lines drawn through the centres of the circles and the slope of these lines is on the whole seen to be very constant throughout the twenty series. In this case the secular and sessional changes are acting in the same direction, but they may well act in opposite directions.

We have thus seen that an observation y may be expressed in the form

$$y = \phi(\tau) + f_p(t) + Y_t \quad \text{..........................(i)},$$

where Y_t is the residual after the removal of secular and sessional changes. The duration of the session is likely to be so short compared with the period over which the secular change is measured, that τ may be taken as practically constant

for any one session, and $\phi(\tau_p)$ may be described as the secular term in the observations of the pth session. It remains therefore to consider the function $f_p(t)$. Supposing that there were n observations made in a session, it would of course be possible to fit an $(n-1)$th order parabola on which all the observations would lie, so that the values of Y_t would all be zero, but such a curve would be entirely useless. If the observations are made at finite intervals so that we can imagine that one may be interpolated between two others, owing to the mass of random errors to which each judgment is subject, we should not for a moment expect that the interpolated error would lie on, or even close to the $(n-1)$th order parabola. A curve of far lower order would probably give a much better fit. If the sessional change is a sign of some physiological change of state which is affecting the observer's judgment, it is natural to suppose that it can be represented fairly closely by some simple curve—a low order parabola if not a straight line, or perhaps, if periodic, a sine curve. Suppose that in a practical case, a first or second order parabola has been fitted to the observations of a session; then it will be easy to test whether the residuals Y_t follow a Gaussian distribution; a simple practically sufficient, if not theoretically sufficient test would be to find whether

$$\left.\begin{aligned} &\sum_{t=1}^{n}(Y_t)=0, \quad \sum_{t=1}^{n}(Y_t^3)=0 \;\ldots\ldots\ldots\text{(ii)} \\ &\beta_2=\frac{\mu_4}{(\mu_2)^2}=\frac{\Sigma(Y_t^4)}{\{\Sigma(Y_t^2)\}^2}=3 \;\ldots\ldots\;\text{(iii)} \end{aligned}\right\}\text{ approximately.}$$

But there is a further possibility; it may be found that although the relations (ii) and (iii) hold approximately, the Y_t's are not randomly distributed in time, and that there is in fact a correlation between the successive values of Y_t, so that

$$r_{Y_t Y_{t+k}}=\frac{\sum_{t=1}^{n}(Y_t Y_{t+k})}{\sqrt{\sum_{t=1}^{n}(Y_t^2)\sum_{t=1}^{n}(Y_{t+k}^2)}}\neq 0$$

for perhaps several positive integral values of k from 1 upwards.

To emphasise the importance of the different terms in the relation

$$_py_t=\phi(\tau_p)+f_p(t)+Y_t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i) } bis,$$

let us take the case of an astronomer who makes a number of observations, often at many days' interval. He will take a mean

$$\bar{y}=\text{mean }\phi(\tau_p)+\text{mean }f_p(t),$$

but he must not suppose that the quantities

$$_py_t-\bar{y}=\phi(\tau_p)-\text{mean }\phi(\tau_p)+f_p(t)-\text{mean }f_p(t)+Y_t$$

follow a Gaussian distribution. It will be only a part of the expression that does so, the Y_t's, and it is possible that even these may not be true.

Further it is clear that successive values of $_py_t-\bar{y}$ will not be independent; correlation will arise from the inclusion of both the secular and sessional terms,

and perhaps too from a relationship between the successive $\bar{Y}_t$'s. There may be no large scale sessional change, and it may be possible to correct for a secular change in personal equation, but even then the mean of a small number "m" of successive observations, subject to its probable error $\cdot 6745\sqrt{\frac{1}{m}}\,\sigma_m$, will not be a satisfactory approximation to the true value of the quantity observed, if these "m" observations are correlated. Suppose for example that the points in Figure 14 (p. 76) represent a series of successive observations which have been corrected for any secular change in personal equation; the linear sessional change is small and has been represented by the continuous straight line, while the dotted straight line represents the mean value of the 63 observations. Yet many sets of 10 consecutive observations could be taken, the difference between the mean of which and that of the whole 63 would be far greater than would be anticipated from the value of the probable error calculated from the expression above. This is because the observations are not randomly distributed in time.

In addition to secular and sessional changes in the value of an estimation, there may be similar changes in the standard deviation; the judgments may become more erratic or less so A sessional change giving an increase in standard deviation would suggest the effect of fatigue, and secular change decreasing the standard deviation might be the indication of increased accuracy with experience. An example of secular change in personal equation and standard deviation is illustrated in the diagram on p 84, the details of this will be discussed more fully in the reduction of Experiment D, but it is here sufficient to say that the central curve represents the smoothed personal equation, while the distance between any point on this curve and either of the outer curves gives the smoothed standard deviation at that point or period in the series of observations. It will be seen that the standard deviation increases in the later observations.

It would be out of place at this point to enter further into the details of variation in personal equation and correlation of judgments, but I think that enough has been said to indicate the general lines of enquiry. In choosing the experiments which will be described in the following sections, the aim has been to select those in which there was likely to be considerable variation in judgment, and where consequently the secular and sessional changes, if present, would be clearly recognizable and the correlation of successive judgments easy to measure. It was also important that the errors in measurement should be small compared with the variations in judgment.

It may of course be urged that the experiments should have been carried out by an observer who was unaware of the lines of enquiry and therefore not liable to bias of any form, but this was not practicable, and in fact none of the reductions had been completed nor the general theory developed before all the experiments had been carried out, and I do not think that the observations could have been affected by any conscious or unconscious prejudice.

III. The Experiments.

The present paper is based on the reduction of the following Experiments:

A. Estimation of the value of a Third, or Trisection Experiment.

B. Estimation of the value of a Half, or Bisection Experiment.

C. Estimation of Time, by counting of Ten Seconds.

D. Estimation of Ten Seconds without intermediate counting.

E. Some repeated measurements of fine structure in a Stellar Spectrum, with a Zeiss Comparator.

The first four Experiments were carried out by the writer in accordance with a uniform scheme; each Experiment was divided into 20 series of 63 observations, making 1260 observations in all. Only one series (or 63 observations) was done at a sitting to avoid as far as possible the effect of fatigue; in the case of Experiments *A* and *B* the sequence of the series was much broken, spreading over some weeks, but *C* and *D* were carried out within four consecutive days. The dates of the series are given with the detailed discussion of the observations below.

(*a*) *Experiments A and B.*

Figure 1 is a copy of one of the printed forms used for these experiments; the longer line was used for *A*; distance between inner edges of bounding marks 7·53 inches; the shorter line was used for *B*; distance between inner edges of bounding marks 5·94 inches.

The lines were on the same form simply for convenience in printing, etc. and that not used was concealed while the observation on the other was being made; a fresh line was used for each of the 1260 observations. In carrying out a series a pile of 63 forms was placed on a table slightly tilted up towards the observer, and straight in front of him, with a good light coming from the left-hand side, the pencil being in his right hand. He then made a short pencil stroke across the line at the point which he estimated was one-third way along the line from the left-hand end (Experiment *A*), or at the point which he considered to bisect the line (Experiment *B*). He then turned the form over, face downwards at his side, and proceeded to deal with the next form in the same manner, continuing until the 63 were finished *. The pencil stroke was made after a rapid eye estimate, the aim being to record the first impression of third or half formed upon seeing the fresh line, and to avoid hesitation; the average time taken in going through a series of 63 observations was 5 minutes 40 seconds for Trisection, 5 minutes 22 seconds for Bisection, or 5·4 seconds and 5·1 seconds respectively between judgments.

To avoid bias, it would have been desirable to complete all the observations of an experiment before commencing the measurement of any of the series, but

* Actually in Experiments *A* and *B* 70 forms were marked in each series; the first 7 were to enable the observer to "get his eye in," and the measures of them were not used at all in the reduction.

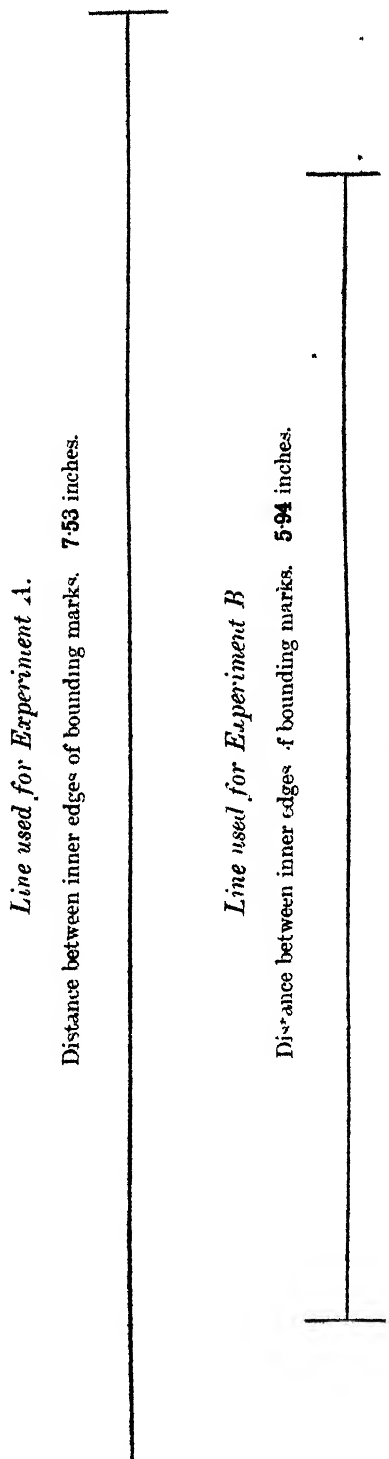


Fig. 1.

from considerations of time and as all the forms were not printed at the commencement this was not done. In some cases therefore a series was measured directly after it had been marked, and if the observer happened to remember that its estimates were considerably too large or too small, his judgment would almost certainly be influenced when marking the next succeeding series; the correlation of judgments within this second series would hardly be altered, but any natural secular change which had been occurring from series to series might be broken*.

The measures of the observations were made with a ruler divided to fiftieth's of an inch, so that readings could be taken to one hundredth of an inch with fair accuracy.

(b) *Experiments C and D.*

These two experiments were carried out with the help of a chronograph. The instrument was run by clockwork, and had a paper tape on which records could be made independently by two pens worked by small electromagnets. One pen was put in circuit with a second's pendulum, a platinum pointer at the end of which made contact at each swing through the vertical position by cutting through a bead of mercury, the other pen was connected with a tapping key. The rate of the driving clock was not quite uniform, and the pendulum second-marks on the tape were therefore necessary in reckoning the intervals between the marks made by the other pen, corresponding to taps of the key. As the estimate in both experiments was one of 10 seconds, it was found that except for a few cases in Experiment D†, the true value of the time interval between the taps could be represented with sufficient accuracy by the factor e/p, where,

* See p. 49, remark in Table I, regarding Series IX and X.

† In Experiment D, some of the estimates had values nearer 20 seconds than 10 seconds, and here half the distance on the tape between the nearest corresponding 20 seconds was taken for p.

e was the distance measured on the tape between consecutive marks of the key.

p the length on the tape of the nearest corresponding 10 seconds recorded by the pendulum pen.

Had the pendulum been beating exactly one second, $10 \times \frac{e}{p}$ seconds would have been the true length of the estimate; actually the period as found by comparison for a long run with a watch was,

$$\left.\begin{array}{lll} \text{before Experiments } C \text{ and } D & (\ 6\text{th December}) & 1{\cdot}020 \text{ seconds} \\ \text{after} \quad ,, \quad ,, & (16\text{th} \quad ,, \quad) & 1{\cdot}019 \quad ,, \end{array}\right\},$$

so that the length of estimate with sufficient accuracy is $10{\cdot}2 \times \frac{e}{p}$ seconds. It is the factor $\frac{e}{p}$ that will be used throughout the reductions.

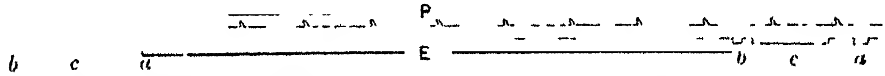


Fig. 2. Shows a small piece of tape, and the points from which the measurements were made.

If the amplitude of the pendulum was rather small, it was sometimes noticeable that the intervals between the second marks were alternately longer and shorter; this was due either to slight deformation in the shape of the mercury bead or (what is really the same thing) from the centre of the bead not having been placed exactly under the equilibrium position of the platinum pointer. But in taking for measurement the even number of 10 seconds, such errors would be inappreciable

In both experiments the beginning and end of the estimate were recorded by sharp taps on the key (at a and b respectively in Figure 2); a long drawn tap (c in figure) then followed to make a break before the next estimate was recorded. The interval between the b tap of one observation and the a tap of the following varied from $1\frac{1}{2}$ to $2\frac{1}{2}$ seconds. This method of record soon became quite automatic, and very few mistaps occurred.

The measurements on the tape were made from the sharp beginnings of the marks, which correspond to the making of the electric contact at the beginning of the tap on the key.

In Experiment C the counting was "sotto voce," the first tap being made on the count "nought," the last on "ten"; in order that the counts might be quite uniform the word "sen" was used instead of the two-syllabled "seven." The counting was usually done in step to a slight beat of the thumb on the key (not hard enough, of course, to make contact), and it was fairly easy to keep the attention concentrated during the counts. In Experiment D there was no counting and it was far harder to keep one's mind fixed; in fact the mental effort required was quite noticeable, and I found that a greater interval of rest was required

between each series than for C. It is mainly by reference to the passing of external events, to changes the duration of which we can infer from previous experience, that we estimate any but the shortest intervals of time. In the counting experiment, the second-intervals between each of the 10 counts which made up the observation were comparatively short, and the beating of the thumb or fingers became almost mechanical; the interval of course varied but was not subject to violent fluctuations. But while most people are able to estimate a second interval with fair accuracy, it would need very much practice to estimate a 10 second interval, and in my case I found it quite impossible to concentrate attention for 10 seconds, solely on the passing of time. I soon found myself imagining that I saw the seconds' hand of a watch, passing usually from the position where 60 is marked on the dial to the 10; but it was not another case of counting, for I did not note the passing of each individual second mark, only having a vague idea of the position of the 5 second division line. If I tried to think of nothing, my thoughts probably wandered on to other subjects, until I came up with a start, and realising that I had very little idea of how long before I had pressed the key to start the observation, pressed it to finish, with the greatest uncertainty. To keep attention fixed, it appeared that I must try to record the stages of the passage of 10 seconds, and this I was doing vaguely on the imaginary clock face, but I must say that the seconds' hand was very refractory, at times appearing to stop or even move backwards, and was often so slow that I had to close the observation before it reached the 10 second mark.

I have given the above description at some length in order to shew that there was an essential difference between Experiments C and D, which is borne out by the figures of the reduction given later in this paper. The observer with the key sat in a separate room where the beats of the chronograph could not be heard. Experiment D was actually carried out in the week previous to C, before starting, a few trials at estimating 10 seconds had been made with a watch, but these were not repeated after the commencement. Again, some 10 second counts were made with a watch before starting on C, but no comparison with a watch or clock was made during the course of the experiment. The measuring up of C and D was left until both experiments were completed, so that the chance of some bias to the judgment, which occurred in the case of A and B was avoided

(c) *Experiment E.*

This consists of nine series of readings made with a Zeiss Comparator at the Solar Physics Observatory, Cambridge, on photographic plates of the spectrum of Nova Aquilae III. The readings were taken in the first place in order to calculate the Probable Errors of the measurements of certain types of structure featuring in the broad emission bands, and each series consists of readings taken from 51 consecutive settings on a particular marking, either a maximum or the edge of a maximum. Although the number of readings is not sufficient for any great weight to be attached to the results, they are, I think, of sufficient interest to be included. In the instrument used, the plate to be measured is fixed to a slide,

which is moved horizontally in a greased slot by pressure with the hand; the measurer looks through one eyepiece and pushes the slide until the feature on the plate of which he is wishing to measure the position, comes under a cross wire in the focus of the eyepiece; then looking through a second eyepiece at the scale attached to the slide, he takes the reading, the last two figures of which are read from a graduated wheel attached to a micrometer screw-head. In making a measurement there are therefore two adjustments:

(1) The setting of the marking in the plate under the cross wire in the first eyepiece.

(2) The shifting of two very close parallel wires by a micrometer screw in the second eyepiece, until a line of division on the scale appears to lie exactly in the centre between them.

Far the greater source of error arises from the first setting, particularly if the marking on the plate is not clear cut. In taking a series of measurements, the observer should always move the slide from the same direction—that is he should always push it or always pull it, until he thinks that the marking is bisected or "edged" by the cross wire, and then he should stop; if he obviously overshoots the mark he should start again, and not hesitatingly move the slide backwards and forwards in search of what he thinks may be the best setting. By shifting the slide into position from the same direction, the measures may be all subject to a fairly constant personal equation due to "over push" or "under push," "over pull" or "under pull" of the slide, but this effect may be eliminated by reversing the plate in the instrument, making a fresh series of measures, and taking the mean of the two. In this particular set of readings the slide was always "pulled" into its final position.

(*d*) It is hoped that the results of some further experiments of a different type in estimating length which were kindly undertaken for me by Mr E. A. Milne of Trinity College, and Mr L. J. Comrie of St John's College, Cambridge, will be included in a future paper.

IV. Terminology.

Experiments A, B, C and D were arranged in accordance with a uniform scheme, each Experiment being divided into 20 "series" consisting of 63 observations. The series will be designated by the Roman numerals I, II...XX in the order in which they were carried out, and the 63 observations* in a series by the letters

$$y_1, y_2, \ldots y_t \ldots y_{63}.$$

In dealing with each Experiment one of the first objects will be to ascertain whether there is any correlation between successive judgments, and the manner in which this correlation, if existent, falls off as the interval between the judgments correlated is increased. To obtain these coefficients of correlation it is necessary

* The first 7 observations, see footnote, p. 28, being always disregarded.

to divide the observations of each series into "groups," and thus we have the 50 observations

$y_1, y_2, \dots y_{50}$ form Group 1 with mean d_1 and standard deviation σ_1,

$y_2, y_3, \dots y_{51}$ " " 2 " " d_2 " " " σ_2,

....

$y_k, y_{k+1}, \dots y_{50+k-1}$ " " k " " d_k " " " σ_k,

....

$y_{14}, y_{15}, \dots y_{63}$ " " 14 " " d_{14} " " " σ_{14}.

By "the correlation of successive judgments at intervals of one," I shall understand the correlation of the 50 observations of Group 1 of a series with the 50 corresponding observations of Group 2 of that series; this will be expressed as ρ_1. Similarly "the correlation of successive judgments at intervals of k," or ρ_k, is the correlation of the corresponding observations in Groups 1 and $k+1$. In fact ρ_k is given by

$$\rho_k = \frac{\frac{1}{50}\sum_{t=1}^{50} y_t y_{t+k} - d_1 d_{k+1}}{\sigma_1 \cdot \sigma_{k+1}} \quad \text{.........................(iv).}$$

When these constants are to be referred to some particular series, say the pth, the prefix p will be placed before them, e.g. ${}_p\sigma_1$, ${}_p\sigma_k$, ${}_p\rho_k$, etc.

A comparison of the d's, σ's and ρ's of the different series will be instructive, but as each of these constants has been calculated from 50 observations only, to obtain quantities with smaller probable errors we must combine the observations of the 20 series. Thus we shall obtain

$$D_k = \frac{1}{mn}\sum_m \sum_{t=1}^{n} y_{t+k-1} = \frac{1}{m}\sum_m (d_k) \quad \text{......................(v),}$$

where $n = 50$, the number in a group,

$m = 20$, the number of series,

and $\sum_m$ indicates summation for all 20 series

$$P_k = \frac{1}{mn}\sum_m \sum_{t=1}^{n} y_t y_{t+k} - D_1 D_{k+1}$$

$$= \frac{1}{m}\sum_m (\rho_k \sigma_1 \sigma_{k+1} + d_1 d_{k+1}) - D_1 D_{k+1}.$$

$$P_k = \frac{1}{m}\sum_m (\rho_k \sigma_1 \sigma_{k+1}) + \frac{1}{m}\sum_m (D_1 - d_1)(D_{k+1} - d_{k+1}) \text{ in view of (v) ...(vi).}$$

Putting $k=0$, in (vi) we have as the square of the standard deviation

$$S_1^2 = \frac{1}{m}\sum_m (\sigma_1^2) + \frac{1}{m}\sum_m (D_1 - d_1)^2$$

and similarly .(vii),

$$S_{k+1}^2 = \frac{1}{m}\sum_m (\sigma_{k+1}^2) + \frac{1}{m}\sum_m (D_{k+1} - d_{k+1})^2$$

and finally the coefficient of correlation $\mathbf{R}_k$ is given by

$$\mathbf{R}_k = \frac{P_k}{S_1 S_{k+1}}, \quad \text{......................(viii).}$$

D_k and S_k are the mean and standard deviation of the combined observations—1000 in all—of the 20 Groups k, while $\mathbf{R}_k$ is the correlation between the 1000 observations in the 20 Groups 1 and the corresponding 1000 observations in the 20 Groups $k+1$, where it must be remembered that owing to the break between each series the 50th observation in Series I is correlated with the $(50+k)$th observation in that series, and not with the kth in Series II, etc.

It will be seen from the equations (vi) and (viii) that it is possible for $\mathbf{R}_k$ to have a large value even though the coefficients of correlation of successive judgments for the separate series are negligible. For though $\sum_m (\rho_k \sigma_1 \sigma_{k+1})$ may be zero for $k \geqslant p$, let us say, where p may perhaps be 3 or 4, it is clear that the coefficients for the combined series, $\mathbf{R}_k$, will not vanish as k increases unless

$$L_k = \frac{\sum_m (D_1 - d_1)(D_{k+1} - d_{k+1})}{S_1 S_{k+1}} \rightarrow 0.$$

In fact if L_k (and therefore $\mathbf{R}_k$) does not vanish for values of k for which the ρ_k's of the individual series vanish, this is a sign of the existence of a secular change running through the series; the means of the separate series differ significantly from the mean of the combined 1000 observations, that is to say they differ significantly from each other. Now it is important to obtain a measure of the correlation of successive judgments, when freed from this secular term. First I define S_k' by the relation

$$S_k' = \sqrt{\frac{1}{m} \sum_m (\sigma_k^2)} \quad \text{......................(ix),}$$

($m = 20$, $\sum_m$ indicating summation for the 20 series); it is the standard deviation of the 1000 observations in the combined Groups k after the secular change has been removed. Then $\mathbf{R}_k'$ is given by

$$\mathbf{R}_k' = \frac{\frac{1}{m} \sum_m (\rho_k \sigma_1 \sigma_{k+1})}{S_1' S_{k+1}'} \quad \text{......................(x),}$$

this is the correlation of successive judgments freed from secular change; before correlating the observations we are in fact fitting the series means together, by subtracting ${}_1d_1 - D_1$ from the observations of the 1st Group of Series I, ${}_1d_2 - D_2$ from the 2nd Group and so on, and again subtracting ${}_1d_{k+1} - D_{k+1}$ from the observations of the $(k+1)$th Group of Series I, etc.

Again it may be desirable to examine the residuals after a sessional change has been removed from the observations of each series, in addition to the general secular term. Suppose that an observation in the pth Series can be expressed in the form introduced on page 25

$${}_py_t = \phi(\tau_p) + f_p(t) + {}_pY_t \quad \text{......................(i) bis,}$$

where $\phi(\tau_p)$ represents the secular term which we take as constant for all the observations of the pth Series, and $f_p(t)$ gives the sessional change, then S_1'' will be the standard deviation of the 1000 residuals in the twenty 1st Groups, S_k'' of the 1000 residuals in the twenty kth Groups, etc., so that

$$S_k'' = \sqrt{\frac{1}{mn} \sum_m \sum_{t=1}^{n} (Y^2_{t+k-1})} \quad \text{...........................(xi)},$$

the mean of the residuals being zero, and $m = 20$, $n = 50$ again; while the correlation of the successive residuals at intervals of k, after the removal of secular and sessional terms, or $\mathbf{R}_k''$ will be given by

$$\mathbf{R}_k'' = \frac{\frac{1}{mn} \sum_m \sum_{t=1}^{n} (Y_t Y_{t+k})}{S_1'' . S''_{k+1}} \quad \text{..............................(xii)}.$$

TABLE OF CONSTANTS.

In the following table definitions are given of the most important of the constants referred to in the preceding section and of others to be introduced in the sequel.

1. The kth Group of the pth Series consists of the 50 observations

$${}_py_k,\ {}_py_{k+1},\ \dots\dots\ {}_py_{k+50-1}.$$

As each Series consists of 63 observations, there are 14 Groups in each of the 20 Series,

n will often be used for 50, the number of observations in a Group,

m " " 20 " " Series.

2. *The crude Observations.*

(*a*) For the pth Series.

$\bar{d}$ = mean of the whole 63 observations.

${}_pd_k$ = mean of observations in kth Group.

${}_p\sigma_k$ = standard deviation of observations in kth Group.

${}_p\rho_k$ = coefficient of correlation between corresponding observations of Groups 1 and $k+1$, i.e. between ${}_py_1$ and ${}_py_{k+1}$, ${}_py_2$ and ${}_py_{k+2}$, etc.

${}_p\sigma_\delta$ = standard deviation of the first forward differences of the observations in Group 1, i.e. of ${}_py_2 - {}_py_1$, ${}_py - {}_py_2 \dots {}_py_{51} - {}_py_{50}$.

${}_pb$ = slope of the straight line $y - {}_pd_1 = {}_pb\left(t - \frac{n+1}{2}\right)$ which fits "best" the 50 observations ${}_py_1, {}_py_2, \dots {}_py_t, \dots {}_py_n$ of Group 1.

${}_p\sigma_k'$ = standard deviation of residuals left after the ordinates of this "best" fitting straight line have been subtracted from the observations of Group k.

${}_p\rho_k'$ = coefficient of correlation between these residuals of Group 1 and Group $k+1$.

In the reduction of the results of the experiments, unless it is necessary to specify a particular series, the prefix p before these constants will usually be omitted for brevity.

(*b*) For the combined 20 series.

$\bar{D}$ = mean of the whole 1260 (= 20 × 63) observations of an experiment.

D_k = mean of the 1000 observations in the combined kth Groups of the 20 series.

S_k = standard deviation of the 1000 observations in the combined kth Group of the 20 series.

$\mathbf{R}_k$ = coefficient of correlation between the 1000 observations in the 1st Groups and the 1000 corresponding observations in the $k+1$th Groups.

${}_sR_k$ = coefficient of correlation between the 1000 sth forward differences of the observations in the 1st Groups and the corresponding differences of the observations in the $k+1$th Groups.

S_δ = standard deviation of the 1000 first forward differences of the observations in the 1st Groups.

3. *The Observations freed from the Secular Change.*

The "secular term" in the observation ${}_py_t$ considered as a member of the kth Group is ${}_pd_k$. Thus the mean of the 1000 observations in the kth Groups each freed from its secular term will be zero.

S_k' = standard deviation of the 1000 observations (freed from secular term) in the kth Groups.

$\mathbf{R}_k'$ = coefficient of correlation between the 1000 observations in the 1st Groups and the 1000 corresponding observations in the $k+1$th Groups (all freed from secular term).

4. *The Observations freed from both Secular and Sessional Change.*

$y = f_p(t)$ is the curve representing the sessional change in the pth Series, so that $f_p(t)$ is the "sessional term" in ${}_py_t$, the tth observation in the pth Series.

${}_pY_t$ = the residual left after removing the secular and sessional terms from ${}_py_t$.

S_k'' = standard deviation of the 1000 Y's in the kth Groups.

$\mathbf{R}_k''$ = coefficient of correlation between the 1000 Y's in the 1st Groups and the corresponding 1000 Y's in the $k+1$th Groups.

${}_p\alpha_t$ = the part of ${}_pY_t$ representing the actual estimate which the observer wishes to record.

${}_p\beta_t$ = the part of ${}_pY_t$ representing a complex of accidental errors superimposed on ${}_p\alpha_t$ in the process of record.

G_k = standard deviation of the sessional terms in the 1000 observations of the kth Groups.

F_k = 1st order product moment coefficient about the mean of these sessional terms in the 1st Groups and the corresponding terms in the $k+1$th Groups.

V. ON METHODS OF REDUCTION.

(a) *Variate Difference Correlation.*

It will become evident in the detailed discussion of the results of the experiments, that a considerable part of the correlation of the successive judgments is due to a secular change with time, occurring from series to series, and in the case of the Trisections, to a sessional change as well occurring within the series; I therefore propose to consider at this point how far the Variate Difference Correlation Method is applicable in this type of problem, and to do this will approach the matter from a slightly more general point of view than that of "Student" in *Biometrika*, Vol. x. p. 179.

Suppose, that x and y are the two variables to be correlated, with corresponding values

$$x_1, x_2, \ldots x_t, \ldots x_v \ldots,$$
$$y_1, y_2, \ldots y_t, \ldots y_v \ldots,$$

and that we may express x_t and y_t in the form

$$x_t = F_1(t) + X_t,$$
$$y_t = F_2(t) + Y_t,$$

where $F_1(t)$ and $F_2(t)$ are polynomials of degree n in t, the unit of t being the interval of time or space between the successive values of the variates, which is supposed equal and constant; X_t and Y_t are independent of the secular or sessional change represented by F_1 and F_2.

Let us now obtain a general expression for

(1) $r_{\Delta_n x_t;\, \Delta_n y_t}$ or ${}_nR$, the correlation of the nth forward differences of x_t and y_t.

(2) $r_{\Delta_n X_t;\, \Delta_n Y_t}$ or ${}_nR'$ " " " X_t " Y_t.

Now

$$\Delta_n x_t = (1-\epsilon)^n x_{n+t} = x_{n+t} - n x_{n+t-1} \ldots (-1)^s \frac{n!}{s!\,(n-s)!} x_{n+t-s} \ldots (-1)^n x_t \ldots \text{(xiii)},$$

where the operator ϵ is defined by $\epsilon^s x_t = x_{t-s}$, etc.

Further we must assume that

$$(a) \quad \sum_{t=1}^{v} x_{t+h} = \text{constant for all values of } h \text{ small compared with } v,$$
$$= 0, \text{ by suitable choice of origin,}$$
$$\sum_{t=1}^{v} y_{t+h} = 0,$$

from which it follows that $\sum_{t=1}^{v} \Delta_n x_{t+h} = 0 = \sum_{t=1}^{v} \Delta_n y_{t+h}$,

$$(b) \quad \sum_{t=1}^{v} (x^2_{t+h}) = \text{constant} = v\sigma_x^2 \text{ for all values of } h \text{ small compared with } v,$$

$$\sum_{t=1}^{v} (y^2_{t+h}) = v\sigma_y^2 \quad \text{" " " " "}$$

$$(c)\quad \sum_{t=1}^{v} (x_{t+h}\, x_{t+h+k}) = v \times {}_x\rho_k \sigma_x^2 \text{ for all values of } h \text{ small compared with } v,$$

$$\sum_{t=1}^{v} (y_{t+h}\, y_{t+h+k}) = v \times {}_y\rho_k \sigma_y^2 \quad ,, \quad ,, \quad ,, \quad ,,$$

$$\sum_{t=1}^{v} (x_{t+h+k}\, y_{t+h}) = v \times {}_{xy}\rho_k \sigma_x \sigma_y \quad ,, \quad ,, \quad ,, \quad ,,$$

Similar relations will hold for the residuals X and Y.

Then a little consideration shews that the sum of the coefficients of the products of the x's and y's whose indices differ by p in the expression

$$\Delta_n x_t\, \Delta_n y_t \;\text{ or }\; (1-\epsilon)^n x_{n+t} (1-\epsilon)^n y_{n+t}$$

is the coefficient of $v \times {}_{xy}\rho_p \sigma_x \sigma_y$ in the product moment

$$\sum_{t=1}^{v} \Delta_n x_t \,.\, \Delta_n y_t\,;\ \text{call this coefficient } a_p.$$

Now

$$\left.\begin{array}{llll} \epsilon^r & \text{operating on } x_{n+t} & \text{gives} & x_{n+t-r} \\ \epsilon^{r'} & ,, \quad ,, \quad y_{n+t} & ,, & y_{n+t-r'} \end{array}\right\}$$

and if $(n+t-r)-(n+t-r')=p$, then $r'-r=p$; hence a_p is the sum of the coefficients of the products $\epsilon_1^r\,\epsilon_2^{r'}$ in the expansion of $(1-\epsilon_1)^n(1-\epsilon_2)^n$ for which $r-r'=p$, or the coefficient of ϵ^p in

$$\left(1-\frac{1}{\epsilon}\right)^n (1-\epsilon)^n,$$

or of ϵ^{n+p} in

$$(-1)^n (1-\epsilon)^{2n},$$

so that

$$a_p = (-1)^p \frac{2n\,!}{(n+p)\,!\,(n-p)\,!}.$$

Hence finally writing $j=n+p$ we have

$$\frac{1}{v}\sum_{t=1}^{v} \Delta_n x_t\, \Delta_n y_t = \sigma_x \sigma_y \sum_{j=0}^{2n} (-1)^{n+j} \frac{2n\,!}{(2n-j)\,!\,j\,!}\, {}_{xy}\rho_{j-n} \quad \text{.........(xiv)},$$

where negative values of the subscript of ρ imply that the subscript of x is less than that of y; e.g. ${}_{xy}\rho_{-p}$ is the correlation between x_t and y_{t+p}.

Similarly for the standard deviations of the nth differences

$$\frac{1}{v}\sum_{t=1}^{v} (\Delta_n x_t)^2 = \sigma_x^2 \sum_{j=0}^{2n} (-1)^{n+j} \frac{2n\,!}{(2n-j)\,!\,j\,!}\, {}_x\rho_{j-n} \quad \text{...........(xv)},$$

$$\frac{1}{v}\sum_{t=1}^{v} (\Delta_n y_t)^2 = \sigma_y^2 \sum_{j=0}^{2n} (-1)^{n+j} \frac{2n\,!}{(2n-j)\,!\,j\,!}\, {}_y\rho_{j-n} \quad \text{...........(xvi)},$$

and for the correlation between the differences

$${}_nR = \frac{\displaystyle\sum_{j=0}^{2n} (-1)^{n+j} \frac{2n\,!}{(2n-j)\,!\,j\,!}\, {}_{xy}\rho_{j-n}}{\sqrt{\left\{\displaystyle\sum_{j=0}^{2n} (-1)^{n+j} \frac{2n\,!}{(2n-j)\,!\,j\,!}\, {}_x\rho_{j-n}\right\}\left\{\displaystyle\sum_{j=0}^{2n} (-1)^{n+j} \frac{2n\,!}{(2n-j)\,!\,j\,!}\, {}_y\rho_{j-n}\right\}}} \quad \text{...(xvii)}.$$

The correlation of the nth forward differences of the residuals X_t and Y_t or ${}_nR'$ will equal an exactly similar expression to the last, in which ${}_{XY}\rho$, ${}_X\rho$ and ${}_Y\rho$ are

substituted for ${}_{xy}\rho$ ${}_{x}\rho$ and ${}_{y}\rho$. But as $F_1(t)$ and $F_2(t)$ are polynomials of degree n in t, we know that

$$\left.\begin{array}{l}\Delta_n x_t = \Delta_n X_t + \text{constant}\\ \Delta_n y_t = \Delta_n Y_t + \text{constant}\end{array}\right\},$$

and therefore

$$_nR = r_{\Delta_n x_t;\,\Delta_n y_t} = r_{\Delta_n X_t;\,\Delta_n Y_t} = {}_nR',$$

that is to say we may equate $_nR$ to an expression similar to that on the right hand side of (xvii) above, except that the correlation coefficients of the residuals, namely: ${}_{XY}\rho$, ${}_{X}\rho$ and ${}_{Y}\rho$ are to be substituted for ${}_{xy}\rho$, ${}_{x}\rho$ and ${}_{y}\rho$.

Now in the usual problem to which the Variate Difference Method is applied it is assumed that after taking a sufficient number of differences we shall approach a state in which the corresponding values of X_t and Y_t, the residuals left after the ordinates of an nth order parabola have been subtracted from x_t and y_t, are mutually at random in time or space; or that

$${}_{XY}\rho_p = 0,\quad {}_{X}\rho_p = 0,\quad {}_{Y}\rho_p = 0,$$

for all values of p other than zero, and that

$${}_{X}\rho_0 = 1 = {}_{Y}\rho_0,\qquad {}_{XY}\rho_0 = r_{XY},$$

i.e. the correlation between X_t and Y_t. Upon this assumption it follows at once from the modified form of (xvii) that

$$_nR = {}_{XY}\rho_0 \quad \text{or} \quad r_{\Delta_n x_t\,\Delta_n y_t} = r_{XY},$$

the fundamental relation of the original Variate Difference Correlation Method.

Let us now turn to the particular type of problem in which we wish to correlate *the successive values of the same variate.* If we are correlating the values at intervals of k, we shall have as corresponding variables, not x_t and y_t but y_t and y_{t+k} so that

$$\left.\begin{array}{llllll}{}_{xy}\rho_{j-n} & \text{becomes} & \rho_{j+k-n} & \text{and} & {}_{XY}\rho_{j-n} & \text{may be written } \rho_{j+k-n}{}^{(n)}\\ {}_{x}\rho_{j-n} & \text{,,} & \rho_{j-n} & \text{,,} & {}_{X}\rho_{j-n} & \text{,,}\quad\text{,,}\quad \rho_{j-n}{}^{(n)}\\ {}_{y}\rho_{j-n} & \text{,,} & \rho_{j-n} & \text{,,} & {}_{Y}\rho_{j-n} & \text{,,}\quad\text{,,}\quad \rho_{j-n}{}^{(n)}\end{array}\right\}$$

where as in the notation of page 35 ρ_p is the correlation of successive values of the variate at intervals of p, and $\rho_p{}^{(n)}$ the correlation of successive residuals (at intervals of p) which are left after the subtraction of the ordinates of an nth order parabola representing the secular change. Hence we have from equation (xvii) that $_nR_k$, or the correlation between the nth forward differences of y_t and y_{t+k} is given by

$$_nR_k = \frac{\sum_{j=0}^{2n} (-1)^{n+j} \frac{2n!}{(2n-j)!\,j!}\,\rho_{k+j-n}}{\sum_{j=0}^{2n} (-1)^{n+j} \frac{2n!}{(2n-j)!\,j!}\,\rho_{j-n}} \quad \text{...... ....(xviii)},$$

$$= \frac{\sum_{j=0}^{2n} (-1)^{n+j} \frac{2n!}{(2n-j)!\,j!}\,\rho_{k+j-n}{}^{(n)}}{\sum_{j=0}^{2n} (-1)^{n+j} \frac{2n!}{(2n-j)!\,j!}\,\rho_{j-n}{}^{(n)}} \quad \text{................(xix)},$$

where negative values of the subscript of ρ and $\rho^{(n)}$ are to be treated as positive: e.g. if $k=1$, $n=5$, $j=1$, then $\rho_{k+j-n}=\rho_{-3}=\rho_3$.

We are again supposing that this secular change can be represented by $y=f(t)$, a polynomial of degree n in t, but we cannot expect that after removing a parabola of even 5th or 6th order*, the residuals $Y_1, Y_2, \ldots Y_t, \ldots Y_v$ will be mutually at random in time or space; if we anticipate correlation between Y_t and Y_{t+k}, we must also be prepared for correlation between Y_t and Y_{t+k-1}, and in any case the correlation between Y_t and Y_t or ρ'_{k+j-n} where $j=n-k$, will be unity. Hence we cannot make the assumptions of the first problem (that ${}_{XY}\rho_p=0$, etc.), in fact

$$r_{\Delta_n Y_t \Delta_n Y_{t+k}} \text{ is not equal to } r_{Y_t Y_{t+k}}.$$

Now consider the use which may be made of equations (xviii) and (xix). If the values of the ρ_p's have been calculated from the crude values of the variate, the quickest method of finding the correlations of differences ${}_nR_k$ is not by direct calculation but by putting these known values of the ρ_p's into the right hand side of (xviii). Then using (xix) we have a number of equations connecting the $\rho_p{}^{(n)}$'s, and the question that at once arises is whether there are sufficient equations to determine these coefficients? It will be seen at once that there cannot be; if we are proceeding to nth differences, we can obtain q equations by putting $k=1, 2, \ldots q$, but these will contain coefficients $\rho_1{}^{(n)}$, to $\rho_{n+q}{}^{(n)}$; in fact n more equations are required. By using the appropriate equations for the Product Moments and for the Standard Deviation of nth differences corresponding to (xiv), (xv) and (xvi) we could obtain one further equation, but at the same time we introduce one further unknown, the standard deviation of the residuals.

That these equations will be indeterminate, can be seen from another standpoint; the nth difference correlation equations (xviii) and (xix) will be satisfied not only by the ρ_p's and $\rho_p{}^{(n)}$'s as defined above, but by the correlation of the residuals left after the ordinates of a parabola of any order less than n, have been subtracted from the crude observations. Nor can further equations for the $\rho_p{}^{(n)}$'s, be obtained by proceeding to $n+1$, or higher differences; the further relations obtained will not be independent, for example

$${}_{n+1}R_1 = \frac{-1+2\,{}_nR_1 - {}_nR_2}{2\,(1-{}_nR_1)} \text{ etc.}$$

The possible application of these difference correlation equations is considered in the next section.

(*b*) *The Application of the Results of the preceding Section.*

Although the correlation of differences does not appear to provide a general method for obtaining the correlation of successive values of a variate after secular changes have been removed, the equations (xviii) and (xix) will be found of considerable assistance in certain cases.

* The figures will probably not warrant the taking of differences of much higher orders than 5th or 6th.

The results of the analysis given in the three illustrative problems below will be used in obtaining the values of various constants in the reduction of the experiments in the later sections. It seemed desirable to collect the algebra together in this way, but in reading this paper the reader may find it more convenient to pass on and refer back to the theory when occasion arises for the numerical application of the results.

Problem 1. In this and the following illustrations of the methods of the preceding section, the notation of Section IV for the correlation of judgment will be used.

I shall suppose that we have m series of observations through the course of which there is some form of secular change; the means of the different series, or the values of ${}_1d$, varying considerably. The coefficients of correlation for the combined series, $\mathbf{R}_1, \mathbf{R}_2, \ldots \mathbf{R}_k \ldots \mathbf{R}_s$ have been calculated, and also the single coefficient $\mathbf{R}_1'$, the correlation of the successive values of the observations (at intervals of 1) after the series means have been fitted together—i.e. after removal of secular change.

It is clear that $\Delta_1 y_t = \Delta_1 Y_t'$, where $y_t = d_1 + Y_t'$, within any one series, and

$$\sum_m \sum_{t=1}^{n} (\Delta_1 y_t \,.\, \Delta_1 y_{t+k}) = \sum_m \sum_{t=1}^{n} (\Delta_1 Y'_t \,.\, \Delta_1 Y'_{t+k}) \text{ etc.},$$

where Σ again stands for summation for the m series, so that the 1st difference correlation equations (xviii) and (xix) are applicable, and become

$${}_1R_1 = \frac{-1 + 2\mathbf{R}_1 - \mathbf{R}_2}{2(1-\mathbf{R}_1)}, \quad {}_1R_k = \frac{-\mathbf{R}_{k-1} + 2\mathbf{R}_k - \mathbf{R}_{k+1}}{2(1-\mathbf{R}_1)} \qquad k = 2, \text{ to } s-1 \quad \ldots\text{(xx)},$$

$$= \frac{-1 + 2\mathbf{R}_1' - \mathbf{R}_2'}{2(1-\mathbf{R}_1')}, \quad = \frac{-\mathbf{R}'_{k-1} + 2\mathbf{R}_k' - \mathbf{R}'_{k+1}}{2(1-\mathbf{R}_1')} \qquad k = 2, \text{ to } s-1 \quad \ldots\text{(xxi)}.$$

From (xx) we get the values of ${}_1R_k, k = 1, 2 \ldots s-1$, and using these and value of $\mathbf{R}_1'$ already supposed to be known, the $s-1$ equations (xxi) will give the $s-1$ unknowns $\mathbf{R}_2', \ldots \mathbf{R}_s'$.

The accuracy of this method will of course depend on the errors involved in the assumptions (a), (b), and (c) of page 37 above.

Problem 2. To obtain the coefficients of correlation of the successive residuals left after the ordinates of the "best" fitting straight lines have been subtracted from each of m series of observations, that is, after the removal of a linear sessional change as well as a secular change. In the notation of p. 35 these coefficients may therefore be written

$$\mathbf{R}_1'', \ \mathbf{R}_2'' \ldots \mathbf{R}_k'' \ldots.$$

In the first place let us obtain the constants of the straight line "best" fitting the 50 observations of Group 1 of a series; this can be done by the method of Least Squares.

If for any series the equation to the line is

$$y = d + b\left(t - \frac{n+1}{2}\right) \qquad (n = 50 \text{ as before}) \quad \ldots\ldots\ldots\ldots\text{(xxii)},$$

where the tth observation is

$$y_t = d + b\left(t - \frac{n+1}{2}\right) + Y_t,$$

we have that

$$K = \sum_{t=1}^{n} Y_t^2$$
$$= \sum_{t=1}^{n} \left\{y_t - d - b\left(t - \frac{n+1}{2}\right)\right\}^2, \text{ is to be a minimum,}$$

therefore

$$\frac{\partial K}{\partial d} = 0 \text{ and } \frac{\partial K}{\partial b} = 0,$$

or

$$\sum_{t=1}^{n} Y_t = 0 \text{ whence } \sum_{t=1}^{n} y_t = nd,$$

and

$$\sum_{t=1}^{n} \left\{y_t - d - b\left(t - \frac{n+1}{2}\right)\right\}\left(t - \frac{n+1}{2}\right) = 0$$

giving

$$\sum_{t=1}^{n} \left\{y_t\left(t - \frac{n+1}{2}\right)\right\} = b \sum_{t=1}^{n} \{t^2 - (n+1)\,t + \tfrac{1}{4}(n+1)^2\}.$$

Or, the first order product moment coefficient about the mean of y_t and t

$$\bar{p}_{11} = b\,\frac{(n^2-1)}{12},$$

giving for the constants of the best fitting line

$$d = d_1 = \frac{1}{n}\sum_{t=1}^{n} y_t$$

$$b = \frac{12}{(n^2-1)}\bar{p}_{11}.$$

The next step is to obtain the correlation of the successive residuals left after the ordinates of this line have been subtracted from the observations

We shall have that

$$n\sigma_1\sigma_2\rho_1 = \sum_{t=1}^{n}\left\{d + b\left(t - \frac{n+1}{2}\right) + Y_t\right\}\left\{d + b\left(t + 1 - \frac{n+1}{2}\right) + Y_{t+1}\right\}$$
$$- nd\left(d + \frac{y_{n+1} - y_1}{n}\right)$$

$$= d\sum_{t=1}^{n}(y_t + y_{t+1}) - nd^2 + b\sum_{t=1}^{n}\left\{\left(t + 1 - \frac{n+1}{2} - 1\right)(y_{t+1} - d)\right.$$
$$\left. + \left(t - \frac{n+1}{2} + 1\right)(y_t - d)\right\} - b^2\sum_{t=1}^{n}\left(t - \frac{n+1}{2}\right)\left(t - \frac{n-1}{2}\right)$$
$$+ \sum_{t=1}^{n} Y_t Y_{t+1} - nd^2 - d\,(y_{n+1} - y_1)$$

$$= \sum_{t=1}^{n} Y_t Y_{t+1} + b\left\{2n\bar{p}_{11} + \frac{n+1}{2}(y_{n+1} - d) + \frac{n-1}{2}(y_1 - d) - y_{n+1} + y_1\right\}$$
$$- b^2\sum_{t=1}^{n}\left\{t^2 - nt + \frac{n^2-1}{4}\right\}$$

$$= \sum_{t=1}^{n} Y_t Y_{t+1} + b\left\{2n\bar{p}_{11} - nd + \frac{n+1}{2}y_1 + \frac{n-1}{2}y_{n+1}\right\} - b^2\frac{n(n^2-1)}{12}$$

$$= \sum_{t=1}^{n} Y_t Y_{t+1} + \frac{b^2}{12}n(n^2-1) + b\left\{\frac{n+1}{2}y_1 + \frac{n-1}{2}y_{n+1} - nd\right\},$$

and if ρ_1' be the correlation of the successive residuals and σ_1' and σ_2' the corresponding standard deviations in Groups 1 and 2, we have finally

$$\rho_1'\sigma_1'\sigma_2' = \rho_1\sigma_1\sigma_2 - \frac{b^2}{12}(n^2-1) - \frac{b}{2n}\{(n+1)y_1 + (n-1)y_{n+1} - 2nd\} \quad \ldots\text{(xxiii)}.$$

Similarly we have

$$n\sigma_1^2 = \sum_{t=1}^{n}\left\{d + b\left(t - \frac{n+1}{2}\right) + Y_t\right\}^2 - nd^2$$

$$= 2d\sum_1^n (y_t) - 2nd^2 + 2b\sum_{t=1}^{n}\left\{\left(t-\frac{n+1}{2}\right)(y_t - d)\right\} - b^2\sum_{t=1}^{n}\left(t - \frac{n+1}{2}\right)^2 + \sum_{t=1}^{n} Y_t^2$$

$$= \sum_{t=1}^{n} Y_t^2 + 2bn\bar{p}_{11} - \frac{b^2}{12}n(n^2-1)$$

$$= \sum_{t=1}^{n} Y_t^2 + \frac{b^2}{12}n(n^2-1),$$

whence it follows that

$$\sigma_1'^2 = \sigma_1^2 - \frac{b^2}{12}(n^2-1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxiv)}.$$

And again,

$$n\sigma_2^2 = \sum_{t=1}^{n}\left\{d + b\left(t + 1 - \frac{n+1}{2}\right) + Y_{t+1}\right\}^2 - n\left(d + b + \frac{Y_{n+1} - Y_1}{n}\right)^2$$

$$= 2d\sum_{t=1}^{n} y_{t+1} - 2nd^2 + 2b\sum_{t=1}^{n}\left\{\left(t + 1 - \frac{n+1}{2}\right)(y_{t+1} - d)\right\} - b^2\sum_{t=1}^{n}\left(t - \frac{n-1}{2}\right)^2$$

$$+ \sum_{t=1}^{n} Y^2_{t+1} - n\left(\frac{Y_{n+1} - Y_1}{n}\right)^2 - nb^2 - 2nbd - 2(b+d)(y_{n+1} - y_1 - nb)$$

$$= n\sigma_2'^2 + 2bn\bar{p}_{11} + nb^2 - b^2\sum_{t=1}^{n}\left(t^2 - (n-1)t + \frac{(n-1)^2}{4}\right)$$

$$+ 2b\left(\frac{n+1}{2}y_1 + \frac{n-1}{2}y_{n+1} - nd\right)$$

$$= n\sigma_2'^2 + \frac{b^2}{12}n(n^2-1) + b\{(n+1)y_1 + (n-1)y_{n+1} - 2nd\}$$

$$\sigma_2'^2 = \sigma_2^2 - \frac{b^2}{12}(n^2-1) - \frac{b}{n}\{(n+1)y_1 + (n-1)y_{n+1} - 2nd\} \ldots\ldots\ldots\ldots\text{(xxv)}$$

If the values of ρ_1' have been calculated by this means for each of the m series, we shall have for the combined series,

$$R_1'' = \frac{\sum_m(\rho_1'\sigma_1'\sigma_2')}{\sqrt{\sum_m(\sigma_1'^2)\sum_m(\sigma_2'^2)}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(xxvi)},$$

a modified form of equation (xii).

As we are subtracting the ordinates of a *different* straight line from each series, a modification of the first-difference equations may be necessary. The

1st order product moment coefficient, for the m combined series*, of successive first differences at intervals of k is given by

$$
\begin{aligned}
{}_1P_k &= \frac{1}{mn}\sum_m\sum_1^n (y_t - y_{t+1})(y_{t+k} - y_{t+k+1}) - \frac{1}{m^2}\left\{\sum_m \frac{y_1 - y_{n+1}}{n}\right\}\left\{\sum_m \frac{y_{k+1} - y_{k+n+1}}{n}\right\} \\
&= \frac{1}{mn}\sum_m\sum_1^n (Y_t - Y_{t+1} - b)(Y_{t+k} - Y_{t+k+1} - b) \\
&\qquad - \frac{1}{m^2}\left\{\sum_m \left(-b + \frac{Y_1 - Y_{n+1}}{n}\right)\right\}\left\{\sum_m \left(-b + \frac{Y_{k+1} - Y_{k+n+1}}{n}\right)\right\} \\
&= \frac{1}{mn}\sum_m\sum_{t=1}^n (Y_t - Y_{t+1})(Y_{t+k} - Y_{t+k+1}) - \left\{\sum_m \frac{Y_1 - Y_{n+1}}{nm}\right\}\left\{\sum_m \frac{Y_{k+1} - Y_{k+n+1}}{nm}\right\} \\
&\qquad - \frac{1}{m}\sum_m \left(b\,\frac{y_1 - y_{n+1} + y_{k+1} - y_{n+k+1}}{n}\right) + \left\{\sum_m \frac{b}{m}\right\}\left\{\sum_m \frac{y_1 - y_{n+1} + y_{k+1} - y_{n+k+1}}{mn}\right\} \\
&\qquad - \sum_m \frac{b^2}{m} + \left\{\sum_m \frac{b}{m}\right\}^2 .
\end{aligned}
$$

Or finally,

$$ {}_1P_k = (-\mathbf{R}_{k-1}'' + 2\mathbf{R}_k'' - \mathbf{R}_{k+1}'')S''^2 - Q_k - \overline{b^2} \quad \ldots\ldots (\text{xxvii}), $$

making the assumptions (a), (b), and (c) of p. 37 and where

1. $Q_k = \frac{1}{m}\sum_m \left(b\,\frac{y_1 - y_{n+1} + y_{k+1} - y_{n+k+1}}{n}\right) - \left\{\sum_m \frac{b}{m}\right\}\left\{\sum_m \frac{y_1 - y_{n+1} + y_{k+1} - y_{n+k+1}}{mn}\right\}$.

2. $\sqrt{\overline{b^2}}$ is the standard deviation of the b's.

There will be similar corrected expressions for the *standard deviations* of the combined first differences.

If we are justified in neglecting terms of the order of $Q_k + \overline{b^2}$, we may use the first difference equations,

$$
\left.
\begin{aligned}
{}_1R_k &= \frac{-\mathbf{R}_{k-1} + 2\mathbf{R}_k - \mathbf{R}_{k+1}}{2(1 - \mathbf{R}_1)} \\
&= \frac{-\mathbf{R}_{k-1}'' + 2\mathbf{R}_k'' - \mathbf{R}_{k+1}''}{2(1 - \mathbf{R}_1'')}, \quad k = 1, 2 \ldots s-1
\end{aligned}
\right\} \ldots\ldots\ldots (\text{xxviii}),
$$

where, as in Problem 1, the known $\mathbf{R}_k$'s will give the ${}_1R_k$'s, and it will only be necessary to calculate directly the one quantity $\mathbf{R}_1''$, in order to obtain

$$ \mathbf{R}_2'', \mathbf{R}_3'' \ldots \mathbf{R}_s''. $$

Problem 3. In the last illustration it may happen that while $Q_k + \overline{b^2}$ is so small as to cause only a negligible error in the value of $\mathbf{R}_2''$ found from

$$ {}_1R_1 = \frac{-1 + 2\mathbf{R}_1'' - \mathbf{R}_2''}{2(1 - \mathbf{R}_1'')}, $$

* ${}_pb$ is the slope of best fitting line in the pth Series.

the cumulative effect of this error may be considerable in the value found for $\mathbf{R}_s''$ ($s = 12$, say). If then we take second differences

$$
\begin{aligned}
{}_2P_k &= \frac{1}{mn}\sum_m \sum_{t=1}^{n} (y_t - 2y_{t+1} + y_{t+2})(y_{t+k} - 2y_{t+k+1} + y_{t+k+2}) \\
&\qquad - \frac{1}{m^2}\left\{\sum_m \frac{y_1 - y_2 - y_{n+1} + y_{n+2}}{n}\right\}\left\{\sum_m \frac{y_{k+1} - y_{k+2} - y_{k+n+1} + y_{k+n+2}}{n}\right\} \\
&= \frac{1}{mn}\sum_m \sum_{t=1}^{n} (Y_t - 2Y_{t+1} + Y_{t+2})(Y_{t+k} - 2Y_{t+k+1} + Y_{t+k+2}) \\
&\qquad - \frac{1}{m^2}\left\{\sum_m \frac{Y_1 - Y_2 - Y_{n+1} + Y_{n+2}}{n}\right\}\left\{\sum_m \frac{Y_{k+1} - Y_{k+2} - Y_{k+n+1} + Y_{k+n+2}}{n}\right\} \\
&= (\mathbf{R}_{k-2}'' - 4\mathbf{R}_{k-1}'' + 6\mathbf{R}_k'' - 4\mathbf{R}_{k+1}'' + \mathbf{R}_{k+2}'')\,S'^2,
\end{aligned}
$$

and is independent of the differing values of the b's.

The appropriate equations are in fact of type

$$
{}_2R_k = \frac{\mathbf{R}_{k-2}'' - 4\mathbf{R}_{k-1}'' + 6\mathbf{R}_k'' - 4\mathbf{R}_{k+1}'' + \mathbf{R}_{k+2}''}{2(3 - 4\mathbf{R}_1'' + \mathbf{R}_2'')} \qquad \text{...........(xxix)},
$$

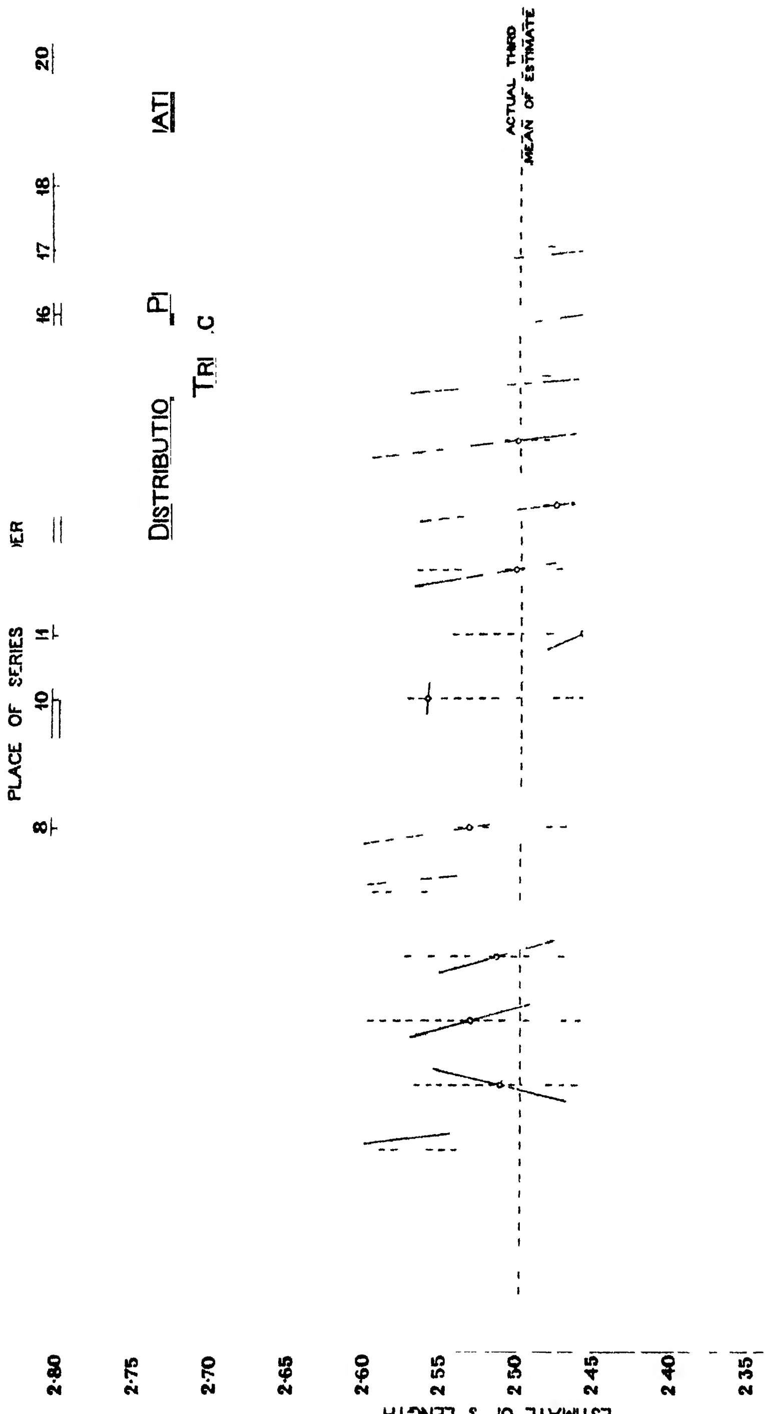
for $k = 1, 2, 3 \ldots s-2$, where $\mathbf{R}_{-1}'' = \mathbf{R}_1''$ etc. and $\mathbf{R}_0'' = 1$. Then using the known value of $\mathbf{R}_1''$, and that of $\mathbf{R}_2''$, found as in Problem 2 from the first difference equation, these $s-2$ equations will give the $s-2$ unknowns $\mathbf{R}_3'' \ldots \mathbf{R}_s''$.

It is clear that similar methods could be applied in the case of sessional changes of higher order, but I have taken the algebra in these three Problems, as the results will be used in the reduction of the experiments later on. The general explanation and equations may have appeared long, but the actual calculation in any particular case of such quantities as ${}_1R_1, {}_1R_2, \ldots {}_1R_k$, or ${}_2R_1, \ldots {}_2R_k$, and then of $\mathbf{R}_2', \ldots \mathbf{R}_k'$, and $\mathbf{R}_2'', \ldots \mathbf{R}_k''$, is exceedingly simple, and far shorter than a direct calculation from the crude figures would be. In two cases the correlations were calculated both by the difference correlation method and directly without approximation, and the agreement of the former results with the latter established confidence in this method of approximation.

VI. EXPERIMENT A (TRISECTION). REDUCTION OF OBSERVATIONS.

(a) *The individual Series.*

The observations of this Experiment have been reduced in more detail than in the other cases; the values of ρ_k, $k = 1, 2, \ldots 13$, were found separately for each series, and these and the values of d and σ—the means and standard deviations of the Groups—are given in Tables I, II and III. Several points of interest will be noted; in the first place the observations have a marked tendency to decrease (i.e. for the estimate of a third to become smaller) both in the course of a series (as is seen by the general decrease of d_k as k increases) and also in passing from the earlier to the later series. These are examples of what have been termed Sessional and Secular Changes. These changes are illustrated in Figure 3 where the centres of the circles give the values of d_1 for each Series, the length of the dotted lines from either side of these points representing the standard deviations σ_1, and the


PLACE OF SERIES
ACTUAL THIRD
MEAN OF ESTIMATE
2·80
2·75
2·70
2·65
2·60
2·55
2·50
2·45
2·40
2·35
ESTIMATE OF ⅓ LENGTH

Fig

continuous lines through the points representing the "best" fitting straight lines for the 50 observations of Group 1, the slopes of these last lines, or constants $_pb$, have been calculated by the Least Square method as in Problem 2, p. 41, and their values are given in the 3rd column of Table IV.

Another way of examining the sessional change, and of obtaining a typical representation of it, is to calculate the average values for the 20 series of y_t the tth observation in a series, thus

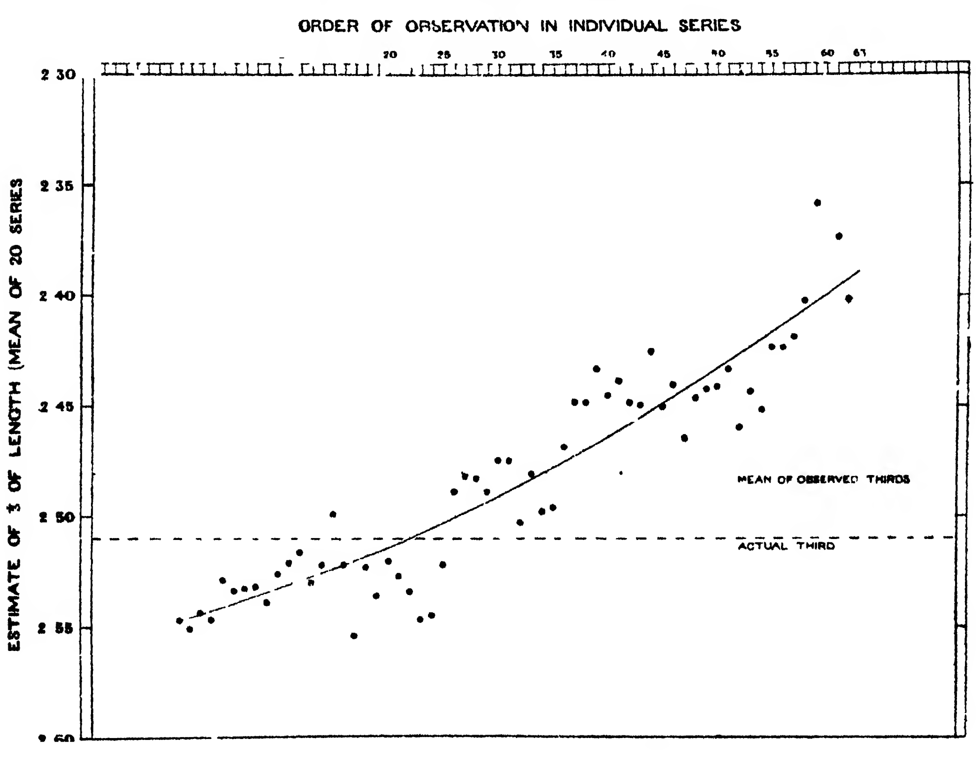
$$\bar{y}_t = \frac{1}{m}\sum_m y_t = \frac{1}{m}\sum_m (d + Y_t)$$

$$= \bar{D} + \frac{1}{m}\sum_m Y_t,$$

where $_p\bar{d}$ stands for the mean of the pth series (63 observations) as opposed to $_pd_k$, the mean of a particular Group k of that series.

The values of Y_t represent the sessional variation in any series about the mean of that series or session of observations, and the sequence $\bar{y}_t - \bar{D}$, $t = 1, 2, 3, \ldots 63$, will clearly represent the mean sessional change. The values of $\bar{y}_t$ are given at the end of Table II and have been plotted in Figure 4, where they have been fitted with the second order parabola (calculated by least squares)

$$y = \cdot 486 + \cdot 00255t - \cdot 0000189t^2 \quad \ldots\ldots\ldots (xxx).$$



TRISECTION EXPERIMENT MEAN SESSIONAL CHANGE

Fig. 4.

Figures 3 and 4 together show very clearly the marked sessional change; while the former shows that except in a few series, notably Series I, IV and X, the regression is remarkably constant in its value, the latter indicates that the sessional change is better represented by a parabola than by a straight line.

The sessional change can also be represented numerically with the help of the correlation ratio of y_t upon t. If we are dealing with the observations freed from the secular change, that is after the removal of the means $_p\bar{d}$ from the 63 observations of the pth series we have η_{y_t} given by

$$\eta_{y_t}^2 = \frac{\sum\limits_{t=1}^{63} (y_t - \bar{D})^2}{63 S'^2}, \text{ where } S'^2 = \frac{1}{1260} \sum_m \sum_{t=1}^{63} ({}_py_t - {}_p\bar{d})^2,$$

or S' is the standard deviation of the whole 1260 observations after the removal of the secular term*. Then the ratio of the mean square distance of every observation from the regression line or line of means y_t, to the standard deviation of the observations is

$$\frac{\sqrt{\frac{1}{1260} \sum\limits_{t=1}^{63} \sum\limits_m (y_t - y_t)^2}}{S'} = \sqrt{1 - \eta_{y_t}^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxxi)},$$

where $\sum\limits_m$ indicates summation for the 20 series.

This is a measure of the closeness of fit of the observations in a series to the mean sessional change as represented by the values y_t; the larger η_{y_t} and therefore the smaller $\sqrt{1 - \eta_{y_t}^2}$ is, the more nearly does a sessional change of the same form recur in series after series. A comparison of the values of $\sqrt{1 - \eta_{y_t}^2}$ for the different experiments will show the relative significance of their mean sessional changes.

In the present case the value of η_{y_t} is found to be $\cdot579 \pm \cdot013$, while

$$\sqrt{1 - \eta_{y_t}^2} = \cdot815.$$

It would be an interesting problem to obtain the correlation of the successive residuals left after the ordinates of the "best" fitting parabola for each series had been subtracted from the observations of that series; but although this has not been done, a fair idea of the degree to which the correlation of the successive judgments in the individual series is due to the sessional change can be obtained by removing the "best" fitting straight lines from each series. The values calculated for the $_pb$'s have been referred to above, and using these and the equations (xxii)–(xxiv) of pp. 41—43, the values of σ_1' and ρ_1', or the standard deviations and correlations of successive observations freed from the linear sessional changes, have been calculated and are given in the 4th and 6th columns of Table IV. The ρ_1''s are all less than the corresponding ρ_1's, except in Series X where they are

* **Actually it is only the values of the Group Standard Deviations S_1', S_2' . . S_{14}' which have been calculated; they are not all equal (as shown in Table V) owing to the sessional change in standard deviation, but an approximation to S' sufficiently accurate for the purpose will be given by taking**

$$S'^2 = \tfrac{1}{14}\{S_1'^2 + S_2'^2 + \ldots + S_{14}'^2\}.$$

TABLE I. TRISECTION.

Coefficients of Correlation for Separate Series, Dates, Times and Remarks.

													D[illegible] 1920	Rem[illegible]	Time taken
+·3008	+·2435	−·0416	+·1813	+·0189	+·0021	+·030	+	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	a.m. [illegible] July	—	—
+·5485	+·5134	+·4577	+·4283	+·4391	+·3557	+·36	+ 73	+·1842	+·1716	+·2197	+·2756	+·2265	a.m. 9 ,,	—	—
+·5560	+·5065	+·3653	+·3981	+·3778	+·2284	+·26	+ 44	+·4221	+·3317	+·3953	−·4397	+·5145	a.m. 16 ,,	—	—
−·0460	+·2168	+·0399	−·0942	+·0948	−·0273	+·29	+ 83	−·0510	+·0082	+·0505	−·1315	+·0307	a.m. 17 ,,	After measuring plates for 1½—2 hrs at Observatory	—
+·3234	+·2309	+·2022	+·1691	+·0485	+·0963	+·157	−·0 47	−·0776	−·0442	−·1614	−·0936	·1219	a.m. 22 ,,	—	—
+·3390	+·4403	+·1749	+·1472	+·2966	−·0580	+·148	+·0 33	+·0663	+·1874	+·1848	−·4314	+·3999	a.m. 23 ,,	—	—
+·6457	+·4532	+·3681	+·2193	+·2849	+·2142	+·107	+·0 19	+·0186	+·0746	−·1200	+·1348	+·2051	a.m. 11 July	Tired and not very fit	6m 45s
+·6089	+·5938	+·3917	+·3340	+·3702	+·3829	+·266	+·4 45	+·3246	+·3862	+·4091	+·3602	+·3013	p.m.		7 0
+·7075	+·4058	+·3596	+·3458	+·3359	+·4011	+·37	+·3 06	+·3637	+·3376	+·2206	+·1252	+·1420	a.m. 12 ,,		5 45
+·7151	+·6392	+·3890	+·1505	+·0283	·0735	+·646	+·0 15	+·1381	+·2571	+·2252	+·2707	+·0433	p.m.	Was aware of under-estimation in IX	6 15
															5 30
+·7381	+·4814	+·3095	+·1658	+·0993	+·0295	−·108	−·2 38	−·3844	−·3775	−·4303	−·3428	−·4122	a.m. 13 ,,	—	6 15
+·6360	+·5137	+·4434	+·4119	+·2917	+·2638	+·229	+·0 11	+·0482	+·1065	+·0569	−·1401	−·0515	p.m.	—	6 0
+·6897	+·5563	+·3618	+·3391	+·3109	+·2061	+·324	+·3 10	+·3204	+·2917	+·2201	−·1660	+·0862	a.m. 14 ,,	—	6 15
+·7965	+·6875	+·6018	+·4370	+·2968	+·2458	−·218	+·1 41	+·1273	+·1751	+·2697	+·3373	+·1034	p.m.	Eyes tired from much reading	5 45
+·8568	+·8256	+·7196	+·6029	+·5277	+·3665	+·292	+·1 50	+·1249	+·1016	+·1219	+·1821	+·2741	p.m. 19 ,,	Plate measuring in morning at	
+·7412	+·5754	+·4567	+·3267	+·2352	+·2133	+·081	+·0 44	+·1275	+·1142	−·0066	−·0560	−·0397	a.m. 20 ,,	— [Observatory	6 0
+·6556	+·6168	+·4732	+·5055	+·4488	+·5434	+·538	+·5 20	+·3595	+·4370	+·2971	+·4283	+·2851	a.m. 21 ,,	—	6 15
+·3144	+·0829	−·0376	−·1662	−·2014	−·1357	+·040	+·0 07	+·0665	−·0834	−·0554	+·0658	+·1440	a.m. 22 ,,	—	6 30
+·7219	+·5919	+·6033	+·5623	+·5148	+·4560	+·114	+·4 80	+·3376	+·1793	+·1527	+·1000	−·0227	a.m. 24 ,,	—	6 15
+·5072	+·2875	+·2672	+·2536	+·1140	+·1257	+·258	+·2 81	+·2465	+·3125	+·3233	+·2468	+·2321	a.m. 27 ,,	—	6 45

Probable Errors of Coefficient of Correlation calculated from 50 Pairs of the Variates.

Value of ρ ...	·80	·70	·60	·50	·40	·30	·20	·10
Value of P.E.	±·0343	±·0486	±·0610	±·0715	±·0801	±·0868	±·0916	±·0944

Mean time taken for a series of 70 observations including the 7 preliminary trials*) 6m 14s

Mean interval between records of judgment ... ... 5s·4

* See p. 28 footnote.

TABLE TR ON

Se. ies Gro fi ·000

	G up 1	Gro 2	Group 3	Group 4	Group 5	Group 6	Group	Group 8	Group 9	G p 10	Group 11			
I	+·1238	+·1192	+·1208	+·1176	+·1184	+·1188	+·1222	+·1234	+·1226	+·1216	+·1206	+·1202	+·1164	+·1122
II	+·2036	+·2032	+·1996	+·1930	+·1882	+·1826	+·1754	+·1718	+·1678	+·1648	+·1598	+·1542	+·1516	+·1484
III	+·1350	+·1350	+·1312	+·1280	+·1268	+·1230	+·1210	+·1212	+·1168	+·1106	+·1022	+·0962	+·0936	+·0924
IV	+·0114	+·0124	+·0128	+·0112	+·0126	+·0170	+·0206	+·0210	+·0206	+·0178	+·0178	+·0156	+·0130	+·0140
V	+·0308	+·0298	+·0292	+·0284	+·0262	+·0256	+·0232	+·0184	−·0138	+·0078	+·0036	−·0042	−·0100	−·0152
VI	+·0132	+·0110	+·0078	+·0074	+·0040	+·0012	·0000	−·0030	−·0062	−·0122	−·0168	−·0206	−·0254	−·0298
VII	+·1448	+·1390	+·1374	+·1378	+·1372	+·1384	+·1392	+·1362	+·1346	+·1292	+·1274	+·1230	+·1220	+·1214
VIII	+·0314	+·0270	+·0234	+·0210	−·0184	+·0120	+·0086	+·0040	+·0028	−·0038	−·0054	−·0078	−·0106	−·0132
IX	−·1596	−·1630	−·1662	−·1708	−·1730	−·1744	−·1760	−·1760	−·1780	−·1786	−·1804	−·1806	−·1816	−·1811
X	+·0590	+·0582	+·0580	+·0624	+·0680	+·0710	+·0716	+·0736	+·0732	+·0722	+·0652	+·0610	+·0534	+·0488
XI	−·0418	−·0414	−·0384	−·0368	−·0364	−·0386	−·0424	−·0470	−·0464	−·0470	−·0456	−·0128	−·0408	−·0404
XII	+·0014	+·0004	+·0004	−·0004	−·0036	−·0040	−·0070	·0114	−·0130	−·0140	−·0146	−·0182	−·0186	−·0192
XIII	−·0248	−·0264	−·0266	−·0258	−·0264	−·0281	−·0322	·0356	−·0384	−·0412	−·0452	−·0490	−·0510	−·0520
XIV	·0000	−·0022	−·0052	−·0078	−·0126	−·0178	−·0226	·0274	−·0334	−·0398	−·0462	−·0486	−·0522	−·0568
XV	−·0710	−·0760	−·0814	−·0858	−·0920	−·0968	−·1032	·1080	−·1126	−·1184	−·1216	−·1224	−·1244	−·1254
XVI	−·0610	−·0632	−·0648	−·0690	−·0732	−·0770	−·0824	·0870	−·0910	−·0946	−·0982	−·1016	−·1022	−·1026
XVII	−·0746	−·0808	−·0856	−·0892	−·0936	−·0976	−·1004	·1010	−·1044	−·1106	−·1142	−·1170	−·1184	−·1218
XVIII	−·1056	−·1086	−·1122	−·1142	−·1132	−·1144	−·1144	·1116	−·1176	−·1194	−·1206	−·1222	−·1246	−·1272
XIX	−·1300	−·1332	−·1362	−·1412	−·1460	−·1510	−·1538	·1548	−·1564	−·1608	−·1638	−·1684	−·1696	−·1726
XX	−·1334	−·1330	−·1330	−·1348	−·1370	−·1386	−·1404	·24	·1450	−·1456	−·1476	−·1496	−·1496	−·1512
ean D	·023'	−·0046:		·008			−·014	·33		·02310	−·02618	−·02914	−·03145	−·03358

V of the "ith" Obser at each Series.

				$\bar{y}$
·547	22	·534	43	2·450
·551	23	·547	44	·426
·544	24	·545	45	·451
·547	25	·522	46	·441
·529	26	·489	47	·465
·534	27	·482	48	·447
·533	28	·483	49	·443
·532	29	·489	50	·442
·539	30	·475	51	·434
·526	31	·475	52	·460
·521	32	·503	53	·444
·516	33	·481	54	·452
·530	34	·498	55	·424
·522	35	·496	56	·424
·499	36	·469	57	·419
·522	37	·449	58	·403
·554	38	·449	59	·359
·523	39	·434	60	·371
·536	40	·446	61	·374
·520	41	·439	62	·402
·527	42	·449	63	2·423

Mean o. ob 803

ABLE TRISECTION

Standard Deviations of Series Groups (in inches).

	Group 1	Group 2	Group 3	Group 4	5	Group 6	Group 7		Group 9	Group 10	Group 1	Group		
I	·06093	·06575	·06591	·06739	·06682	·06672	·0638[illegible]	·06263	·06343	·06195	·06565	·06635	·06926	·07217
II	·09184	·09197	·0951[illegible]	·09992	·10373	·11079	·11767	·11960	·12158	·12161	·12354	·12654	·12813	·12962
III	·09561	·09561	·09474	·09506	·09427	·09586	·09548	·09526	·09840	·10851	·11789	·12381	·12526	·12385
IV	·05902	·05857	·05828	·06024	·05935	·05900	·05908	·05941	·05948	·06084	·06084	·06509	·06856	·06726
V	·07210	·07204	·07191	·07167	·07113	·07156	·07530	·07661	·07962	·08117	·08376	·08602	·08920	·08963
VI	·05921	·05984	·06087	·06104	·06274	·06590	·06699	·06935	·07230	·07718	·08184	·08260	·08358	·08629
VII	·11154	·11190	·11043	·11095	·11036	·11104	·11198	·10962	·10823	·10695	·10457	·10181	·10178	·10182
VIII	·12060	·12592	·13248	·13538	·13719	·13882	·13871	·14170	·14216	·14421	·14526	·14515	·14478	·14430
IX	·10233	·10443	·10490	·10777	·10804	·10793	·10711	·10711	·10528	·10486	·10479	·10474	·10372	·10402
X	·11141	·11226	·11229	·10950	·10768	·10445	·10360	·10258	·10326	·10400	·11265	·11697	·11872	·11543
XI	·09435	·08450	·08814	·09026	·09068	·09086	·09229	·09181	·09176	·09194	·09227	·09239	·09112	·09086
XII	·07105	·07071	·07071	·07082	·07028	·06977	·06978	·06920	·06914	·06888	·06795	·06755	·06732	·06693
XIII	·08214	·08116	·08098	·08154	·08153	·08257	·08278	·08446	·08422	·08489	·08521	·08240	·08269	·08219
XIV	·08141	·07961	·07697	·07658	·07781	·07747	·07807	·08197	·08267	·08744	·09428	·09510	·09659	·09756
XV	·10711	·10387	·09996	·09621	·09334	·08997	·08332	·08005	·07560	·07401	·07226	·07212	·07341	·07395
XVI	·07819	·07511	·07395	·06914	·06565	·06084	·05767	·05182	·05341	·05587	·05799	·06168	·06162	·06154
XVII	·08594	·08501	·07836	·07756	·07576	·07651	·07507	·07511	·07292	·07601	·07351	·07097	·07012	·07036
XVIII	·04644	·04821	·05247	·05273	·05365	·05379	·05377	·05390	·05655	·05863	·06065	·06217	·06087	·06040
XIX	·06920	·06866	·06997	·06678	·06579	·06552	·0632[illegible]	·06198	·06157	·05949	·05706	·05696	·05550	·05351
XX	·04443	·04446	·04446	·04531	·04277	·04285	·04350	·04343	·04291	·04319	·04379	·04477	·04477	·04403
	·08450	·08478	·08499	·08524	·08502	·08533	·08535	·08545	·08576	·08722	·08908	·090[illegible]0		·09065

The values of S_t at the bottom of the Table have been taken from Table VI below.

equal, but though there is in general a considerable reduction, it is clear that neither a linear sessional change nor a parabolic one of the form represented by equation (xxx) account for the greater part of the correlation of successive judgments.

The coefficients ρ_1 and also ρ_1' vary considerably from series to series, but there is no very marked progressive secular change. On the whole both ρ_1 and ρ_1' are large when the standard deviation is large, and a measure of this correspondence will be given by the correlation of ρ and σ. This can be obtained most readily, and with sufficient accuracy for the purpose, from the correlation of the ranks of these variates, by the method referred to in *Biometrika*, Vol. X. p. 416*.

The results are

$$\text{correlation between } \rho_1 \text{ and } \sigma_1 + \cdot52 \pm \cdot11 = r_{\rho\sigma},$$

$$\text{,,} \qquad \text{,,} \qquad \rho_1' \text{ and } \sigma_1' + \cdot60 \pm \cdot10 = r'_{\rho\sigma}.$$

The difference is not significant, and we may draw the conclusion which could not have been assumed *a priori*, that the correlation of successive judgments is larger when the variations in judgment are larger, and that this relationship does not appear to be reduced when the large linear sessional change has been removed Large values of σ might have implied erratic observation and small relation between

TABLE IV. *Constants of Individual Series (Trisection).*

(The definition of these constants is given on p. 35.)

Series	d_1						
I	2·6238	+ ·000673	+ ·2925	+ ·3008	·06015	·06093	·0721
II	·7036	·002964	+ ·4149	+ ·5485	·08125	·09182	·0873
III	·6350	·003626	+ ·3643	+ ·5560	·08001	·09561	·0901
IV	·5114	+ ·001718	·2521	− ·0160	·05356	·05902	·0854
V	·5309	− ·001555	+ ·2520	+ ·3234	·06853	·07210	·0839
VI	·5132	·001529	+ ·2270	+ ·3390	·05495	·05921	·0681
VII	·6148	·004241	+ ·4918	+ ·6457	·09322	·11154	·0939
VIII	·5314	− ·002788	+ ·5478	+ ·6089	·11369	·12060	·1067
IX	·3404	− ·004477	+ ·4979	+ ·7075	·07935	·10233	·0783
X	·5590	·000036	+ ·7151	+ ·7154	·11141	·11141	·0841
XI	·4582	·000972	+ ·7320	+ ·7381	·08317	·08435	·0610
XII	·5014	·002720	+ ·1851	+ ·6360	·05923	·07105	·0606
XIII	·4752	− ·003594	+ ·5161	+ ·6897	·06409	·08244	·0649
XIV	·5000	·003818	+ ·6433	+ ·7965	·05993	·08141	·0519
XV	·4290	− ·005588	+ ·6810	+ ·8568	·07051	·10711	·0573
XVI	·4390	− ·003071	+ ·6408	+ ·7112	·06441	·07819	·0562
XVII	·4251	·004369	+ ·2569	+ ·6556	·05840	·08594	·0713
XVIII	·3944	·000580	+ ·2870	+ ·3144	·04568	·04641	·0544
XIX	·3700	− ·003236	+ ·4935	+ ·7219	·05107	·06920	·0516
XX	2·3666	− ·001725	+ ·2850	+ ·5072	·03680	·04443	·0441

Mean value of b - ·002425.

* The theory is based on the hypothesis that the variates follow a normal distribution, and though this may not be strictly true for the ρ_1's and σ_1's the method probably gives a sufficiently accurate approximation to the value of their correlation.

successive judgments, and at the same time high correlation might have been expected to result in small variation. The significance of this will be discussed in the concluding sections of this paper.

In Table III giving the σ's, it will be seen that in general the standard deviations increase in the later groups; though this may be due in part to the parabolic form of the sessional change, with its tendency to an increasing drop towards the end of the series, it is possible that it also indicates a fatigue effect setting in, and causing the later observations in a series to be more erratic; the same phenomenon appears in the Bisection Experiment where there is no appreciable sessional change within the series. It may in fact be looked on as a sessional change in the standard deviation.

At the end of Table I are given the dates on which the different series were carried out, remarks noted at the time as to the condition of the observer, and, for the last 14 series, the time taken to mark off the 70 forms*. It will be seen that there was a large gap between the times of carrying out the first six and the last fourteen series, and this interval of nearly two months broke the continuity of the secular change in the means of the series. In Figure 5 the means of Group 1 (or

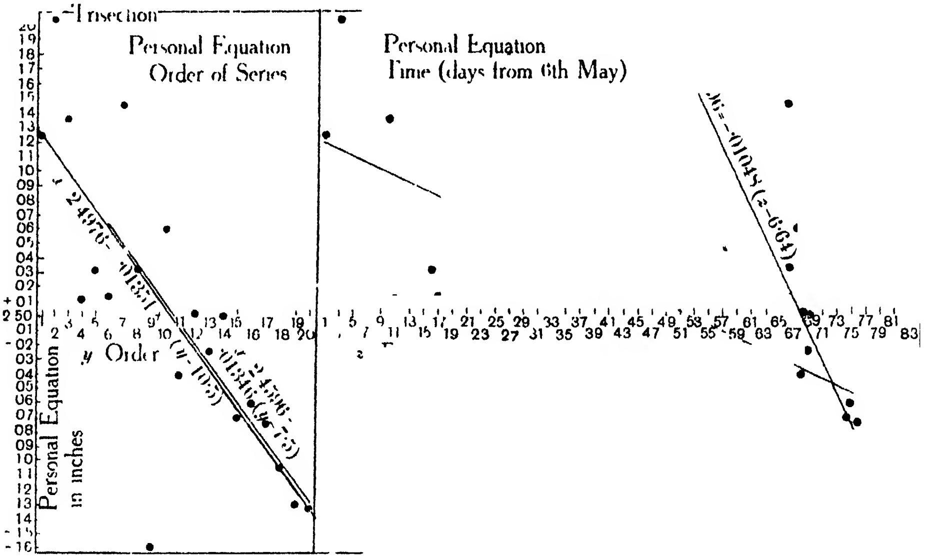


Fig. 5.

the d_1's of each series) have been plotted firstly with the order of series and secondly with the date of series.

If x is the personal equation, or mean value of the observations of Group 1 of a series measured in inches,

y the number or order of a Series,

z the number of days between the 6th May and the date on which the series was carried out,

* Reference to the 7 trial forms first marked, in addition to the 63 of the Series proper, is made on p. 28 in footnote.

we have for the regression straight line of x on y

$$x - 2{\cdot}4976 = -{\cdot}01351\,(y - 10{\cdot}5) \quad \text{(xxxii)},$$

and for the regression of x on z

$$x - 2{\cdot}4976 = -{\cdot}00233\,(z - 53{\cdot}05) \quad \text{(xxxiii)},$$

and these lines have been drawn in the diagram.

The corresponding coefficients of correlation between (1) personal equation and order, (2) personal equation and time, and (3) time and order, a meaningless coefficient but required to find the partial correlations, are

$$(1)\quad r_{xy} = -{\cdot}800 \pm {\cdot}054,$$

$$(2)\quad r_{xz} = -{\cdot}692 \pm {\cdot}079,$$

$$(3)\quad r_{yz} = +{\cdot}882 \pm {\cdot}033,$$

and the partial correlations are

$$r_{xy.z} = -{\cdot}559 \pm {\cdot}104,$$

$$r_{xz.y} = +{\cdot}049 \pm {\cdot}150.$$

But the interval between the May and July series was so large, that the series should perhaps be considered as forming two groups, one of six and the other of fourteen. Taking the last fourteen series, we have the regression lines

$$x - 2{\cdot}4596 = -{\cdot}01346\,(y - 7{\cdot}5),$$

the Series VII being given the order 1, VIII, 2 etc., and

$$x - 2{\cdot}4596 = -{\cdot}01048\,(z - 6{\cdot}64)$$

z being the days between 10th July and date of Series. These lines have also been drawn on the diagrams.

The correlations are

$$(1)\quad r_{xy} = -{\cdot}674 \pm {\cdot}098,$$

$$(2)\quad r_{xz} = -{\cdot}673 \pm {\cdot}099,$$

$$(3)\quad r_{yz} = +{\cdot}956 \pm {\cdot}016,$$

giving partial correlations

$$r_{xy.z} = -{\cdot}143 \pm {\cdot}177,$$

$$r_{xz.y} = -{\cdot}138 \pm {\cdot}177.$$

The point of interest is this: there is a secular change in personal equation from series to series; is this change more closely related to the number of series or sessions that have gone before (that is, almost, to the experience gained), or is it due to some change with time in the observer's outlook? Suppose that it was arranged to carry out observations on a number of different days with varying intervals of time between them, and that on each day a certain number of different series of observations or sessions were undertaken at regular intervals of perhaps an hour or less; any series could then be classified as the pth series of the qth day. Then $r_{xy.z}$ (the partial correlation of personal equation and order, time being kept constant) would give a measure of the relationship between change in personal equation and order of series in any one day. This will not necessarily be the same

as the sessional change, for it has been supposed that this latter occurs only during the course of a sitting, and is broken by the interval of rest in between. On the other hand if we take all the pth series of the various days, then $r_{xz.y}$ (the partial correlation of personal equation and time, order being constant) gives the relation between change in personal equation and time, taken over a long period.

The long break in the middle of the Trisection Experiment takes away any real significance from the difference between $r_{xy.z}$ ($-\cdot 559$) and $r_{xz.y}$ ($+\cdot 049$) for the twenty series, and in the case of the last fourteen series these coefficients are equal ($-\cdot 143$ and $-\cdot 138$), because the intervals between the series were nearly uniform. In the Timing Experiments, C and D, the arrangement of the series in groups on consecutive days leads to considerably more interesting results*.

A comparative measure of the consistency of the consecutive judgments in the different series, is the standard deviation of first differences, or

$$\sigma_\delta = \sqrt{\frac{\sum\limits_{t=1}^{n}(y_{t+1}-y_t)^2}{n}} = \sigma_1\sqrt{2(1-\rho_1)}$$

approximately. The values of this expression are given in the 8th column of Table IV.

Now suppose we compare the constants in Table IV, the dates and remarks at the end of Table I and the diagrammatic representation of seven of the series given in Figure 6. The first series to be remarked on is IV; most of the series were carried out at the beginning of the morning before any other work, and it is possibly the fact that IV was done soon after a spell of measuring spectrograms with a Zeiss comparator that explains the exceptional values of ρ_1 and ρ_1', namely $\rho_1 = -\cdot 0460$, $\rho_1' = -\cdot 2521$. The σ_δ, or standard deviation of 1st differences, is no higher than for the other series done at about the same time (in May), and the σ_1 is lower. The first graph in Figure 6 gives the diagram of this series; the rapid fluctuations in judgment about a very steady, if slowly changing, personal equation may perhaps have some physiological significance. In the second and third graphs of Figure 6 are represented two of the four Series VII–X which were done when the observer was not very fit; they have large values for σ_1, and the σ_δ's are large compared with those of the ten series which follow, showing that the judgments were rather erratic; the correlation is however high. In VIII there is a great jump between the 44th and 45th judgments, from 2·22 to 2·66, and the gradual drop down, which follows, to 2·20 (for the 52nd judgment) is a good example of a way in which successive judgments are correlated. In Series XI (not represented among the graphs) there appears to be a periodic variation, for the correlation falls steadily from $\rho_1 = +\cdot 7381$ to $\rho_{12} = -\cdot 4428$.

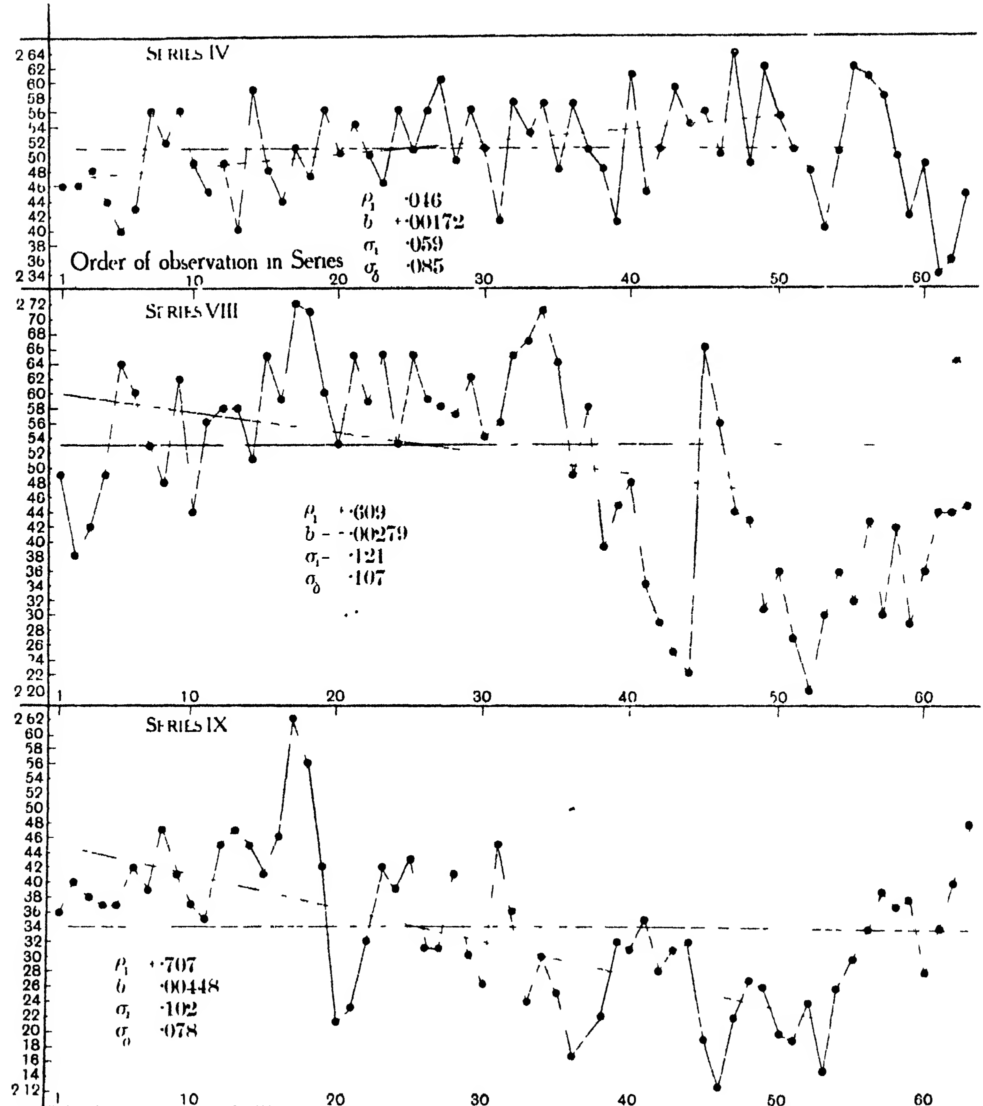
XIII, XV and XVI are typical highly correlated series with large sessional changes; the σ_δ's as well as the σ_1's are considerably smaller than in the series VII–X. In examining the fourth to sixth graphs we notice what may be called the large scale correlated variations superimposed upon the linear sessional change;

* See pp. 70, 75 and 83, and below.

it is due to these variations that the values of ρ_1' remain so large, and it is their absence that makes the correlation in IV so low. The last graph (Series XX) is that for which the σ_1 and σ_δ are least, and yet ρ_1 is quite high (+ ·507).

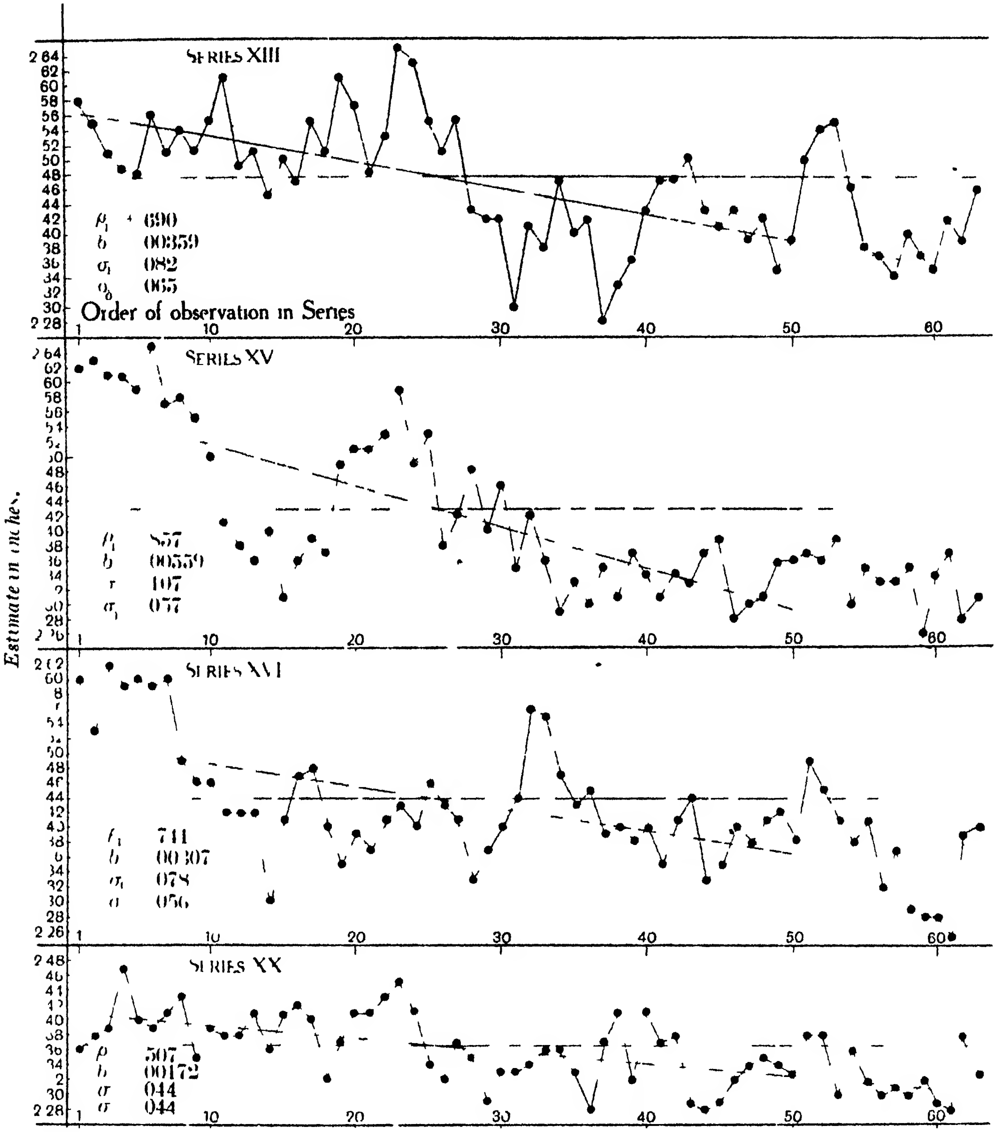
As a last instance of these points, we may compare the constants* for XV and XVIII:

	b	ρ_1	ρ_1	σ_1	σ_δ
XV	− ·005588	+ ·6810	+ ·8568	·10711	·0573
XVIII	− ·000580	+ ·2870	+ ·3144	·04644	·0544



The horizontal line intersecting each graph gives the mean of the first 50 observations in that series.
Fig. 6. Trisection. Diagrams representing variations in judgment.

* For definitions of these constants the table on page 35 may be referred to.



The horizontal line intersecting each graph gives the mean of the first 50 observations in that series

Fig. 6. Trisection Diagrams representing variations in judgment (continued)

XV has a large linear sessional change, but superimposed on this there must be considerable correlated variations, for the removal of the best fitting straight line only reduces ρ_1 (·8768) to ρ_1' (·6810). XVIII has variations altogether on a smaller scale; the correlation of successive judgments is low and it is barely affected by the removal of the linear sessional change. And yet though the σ_1 for XV is more than twice as great as for XVIII, the σ_δ's, or measures of the average jump in estimation from judgment to judgment, are practically identical; the importance of this constant σ_δ as a measure of variability of judgment is discussed on p. 69 below.

TABLE

Constants of Combined Series (Trisection).

	·50													2·50
	·00237	−·00463	−	·00845		5	−·01	−·01693	−·019	·02310	0261	−·02914	−·03 5	−·03358
	·12892 ±·00191	·12926 ±·00195	·12961 ±·00195	·13003 ±·00196	13042 ±00197	3 3 ±00 8	·131 ±·001	·13165 ±·00199	·13195 ±·00199	·13270 ±·00200	1334 0020	·13354 ±·00201	·13338 ±·00201	·13310 ±·00201
$\mathbf{R}_k$	+·8385 ±·0063	·7946 ±·0079	7361 ±·0098	·7059 ±·0107	·6904 ±0112	669 ±01 8	·66 2 ±·0119	·6498 ±·0123	·6333 ±·0128	·6346 ±·0127	2 0	6215 ± 31	·6151 ±·0133	
S_k'	·08450 ±·00127	·08478 ±·00128	·08499 ±·00128	·08524 ±·00129	·08502 ± 128	·08 33 ±·00 29	·0853 ±·0012	·08545 ±·00129	·08576 ±·00129	·08722 ±·00132	·08908 ±·00134	·09010 ±·00136	·09072 ±·00137	·09065 ±·00137
$\mathbf{R}_k'$ (1)	+·6246 ±·0130	·5231 ±·0155	·3878 ±·0181	·3155 ±·0192	·2 12 ±·0 96	·2324 ±·0202	·221 ±·02	·1910 ±·0206	·1647 ±·0208	·1826 ±·0206	·1681 ±·0207	·1719 ±·0207	·1603 ±·0208	
$\mathbf{R}_k'$:	+·6246	·5226	3867	·3166			·22	1866	·1483	·1514	·1250	·1211	·1062	
${}_1R_k$	−·3641	+·0452	−·0876	−·0455	+		+·02	+·0065	−·0551	+·0393	·0300	+·0146		
${}_2R_k$	−·6500	+·1987	−·0641	−·0075			+·036	+·0145	−·05 2	+	8			

TAB E

efficients c at Succ e Observ ... nea Sessi ange.

	=1	2		4										
$\mathbf{R}_k'$ from 1st Differ ence Equations	+·892 ±·162	+·3 ±·0		+·1654 ±·0207	+·0699 ±·0212	+·0209 ±·0213	−·0 5 ±·02 3	−·06 ±·02	−·107 ±·021	−·1595 ±·0208	−·15 ±·02	−·1912 ±·0206	−·1905 ±·0205	−·2167 ±·02
$\mathbf{R}_k'$ from 2nd Diffe ence Equations				·1655	+·0703	+·0218			6	−·522	−	·1765	−·17 0	−·19

(*b*) *The Combination of Series.*

Having discussed the reduction of the individual series, I will proceed to consider the results of combining the 20 series. The formulae (v) to (viii) on pp. 33 and 34 give the values of D_k, S_k, and $\mathbf{R}_k$ which are tabled below. Remembering that D_1 and S_1 are the mean and standard deviation of the combined observations $y_1, y_2 \dots y_{50}$ from each of the 20 series, D_2 and S_2 the mean and standard deviation of the combined observations $y_2, y_3 \dots y_{51}$, and finally D_{14} and S_{14} of $y_{14}, y_{15} \dots y_{63}$, we see that the progressive decrease in D_k as k increases indicates the shortening in the estimate of a third during the course of a sitting, while the increase in S_k may perhaps be partly due to increasing variability of judgment due to fatigue. The values of $\mathbf{R}_k$ are large, but this is to be expected owing to the large changes in personal equation from series to series; in fact for $k = 13$ it will be found that the limiting expression L_k of page 34 gives

$$L_{13} = +\cdot 5435, \text{ while } \mathbf{R}_{13} = \cdot 6151.$$

The reason for this difference between L_{13} and $\mathbf{R}_{13}$ is that $\sum\limits_m (\rho_{1,}\sigma_1\sigma_{14})$, and therefore $\mathbf{R}_{13}'$, does not vanish. The next step is to obtain the values of S_k' and $\mathbf{R}_k'$, or the standard deviations and correlations of successive judgments after the secular change has been removed. They are found from Equations (ix) and (x) of p. 34 and are given in Table V (5th and 6th rows).

There is here an opportunity of testing the accuracy of the Difference Correlation method discussed in Section V (*b*); the case is that of Problem 1, page 41, the values of $\mathbf{R}_1, \mathbf{R}_2 \dots \mathbf{R}_{13}$ are known and give the correlation of 1st differences, ${}_1R_1, {}_1R_2 \dots {}_1R_{13}$, these together with the coefficients of correlation of 2nd differences to be used later, are given at the bottom of Table V. Then using the value $+\cdot 6246$ for $\mathbf{R}_1'$, we get the values of the 12 quantities $\mathbf{R}_2' \dots \mathbf{R}_{13}'$, which have been inserted in the 7th row of Table V. It will be seen that the values obtained by this approximate method agree well with the others, the differences being within the probable error of the $\mathbf{R}_k'$'s up to and including $\mathbf{R}_9'$; beyond this the approximate values become rapidly too small, the error, from the form of Equations (xx) and (xxi), being clearly cumulative. This failure is certainly largely due to the fact that the errors involved in the assumptions (*a*), (*b*) and (*c*) of p. 37 are not negligible when the later groups enter into the correlation, for we have already seen that both D_k and S_k change steadily with k.

The values of S_k' and $\mathbf{R}_k'$ in Table V correspond to the average values of the standard deviations and correlations of successive judgments in the individual series, i.e. of the σ's and ρ's given in Tables III and I. Owing to the sessional change which occurs during the course of nearly all the series, $\mathbf{R}_k'$ does not vanish as k increases, but appears to approach a limiting value in the neighbourhood of $+\cdot 16$. By obtaining for the separate series, the coefficient of correlation, ρ_1', of the successive observations at intervals of one, freed from the *linear* sessional change, a step has been made towards the further reduction of the problem. $\mathbf{R}_1''$, the

coefficient for the combined series corresponding to ρ_1' of the individual series, is given by

$$R_1'' = \frac{\underset{m}{\Sigma}(\rho_1'\sigma_1'\sigma_2')}{\sqrt{\left\{\underset{m}{\Sigma}(\sigma_1'^2)\right\}\left\{\underset{m}{\Sigma}(\sigma_2'^2)\right\}}} \quad \text{.......................... (xxvi) bis,}$$

and taking ρ_1', σ_1' and σ_2' calculated from Equations (xxiii), (xxiv) and (xxv) we find that

$$R_1'' = +\cdot 48922 \pm \cdot 01623.$$

Then R_2'', $R_3''\ldots R_{12}''$ can be found by the method of Problem 2, p. 41; or again using the value of R_2'' found from the first difference equations, we may proceed to second differences as in Problem 3, and so obtain $R_3''\ldots R_{12}''$. In this particular case there is no need to use the second difference equations*, but the values of the R_k'''s have been worked out by both methods, as numerical examples of the theory of Sections V (*a*) and (*b*). A comparison of the values given in Table VI shows that there is no significant difference between the results of the two methods†, and the agreement found earlier in this section between the values of R_k' calculated directly and those found from the difference equations, warrants confidence in the results for R_k''. Although the negative values of R_k'' are probably too large for the higher values of k (just as the later positive values of R_k' in Table V, row 7, were too small), there is no doubt I think, that the correlation of the successive observations freed from the linear sessional change, does become negative at $k=5$, 6 or 7 and remain negative for the higher values of k. A word of qualification is necessary; the linear sessional change to be removed has been represented by the line "best" fitting the *first 50 observations* of each series, and a glance at Figure 4 shows that the mean values of the later observations in the series of 63 would lie well off this line because of the parabolic form of the sessional change; the negative values found for R_8'', R_9'', etc. may probably be largely accounted for by this fact. A more satisfactory approximation to the correlation of successive judgments freed from sessional change will be obtained in Section XI below.

As $\sigma_\delta = \sigma_1\sqrt{2(1-\rho_1)}$, referred to on p. 55 above, gives the standard deviation of the first differences of consecutive judgments in a single series, we shall have as a corresponding measure for the combined twenty series

$$S_\delta = S_1'\sqrt{2(1-R_1')}.$$

For the Trisection Experiment

$$S_\delta = \cdot 0732.$$

* To get an idea of the order of the terms Q_k and b^2 which are being neglected, the values were calculated for two values of k, with the result

$$\left.\begin{aligned} k&=1,\ Q_k+b^2=\cdot 000001064\\ k&=9,\ Q_k+b^2=+\cdot 000000192\end{aligned}\right\}.$$

† The probable errors in the Table have been calculated from the usual formula $\epsilon = \pm\cdot 6745\dfrac{1-R^2}{\sqrt{N}}$, and do not cover the errors arising from the method of approximation.

(c) *On the possible Effect of shifting the Head during the course of a Series.*

It was suggested to me that the correlation of successive judgments in this and in the Bisection Experiment might be due to periodic shifting of the head from side to side during the course of a series, some parallax effect of the two eyes making corresponding variations in the estimation of a third (or a half) of the line on the form. Now such an explanation might account for part of the correlation in these two experiments, but it could not explain the regular secular and sessional changes in the Trisection, except by the highly improbable hypothesis that the observer's head leant over increasingly to one side during the course of a sitting, and that he started with it more on one side in the later series than in the earlier ones. But beyond this, the fact that correlation is found also in the timing experiments suggests that it is of deeper and more complex origin. It is likely to arise from many unknown causes affecting the environment and condition of the observer, and if one of them is a relative shifting of the eyes, it is of interest, for it will enter into many kinds of observations, where the observer who takes the readings is not looking through a fixed eyepiece.

To test the effect of a relative shift between head and paper, 42 of the forms were taken, and trisected in the usual way, but for alternate groups of seven the paper was shifted 4 inches relatively from side to side. The measures of the estimates and their means are in Table VII. The three sets of seven under the heading I, were made with the forms in one position, the three sets under II with the forms shifted 4 inches to the right. The difference is noticeable at once; readings I are smaller than II, and at the same time the curious effect of sessional change is appearing the later readings of I and again of II, being on the whole smaller than the earlier ones. Now in carrying out the observations of the Trisection and Bisection Series, the body and head were kept as steady as possible, and it is unlikely that frequent shifts as large as 4 inches could have occurred; further the differences between the means of readings I and II are much smaller than the actual variations in judgment shown in the diagrams of Figure 6.

But as a further test, a series of 63 forms were marked off, with the head fixed mechanically, the results are given in Table VIII with the usual notation. The correlations are not as high as many of those in Table I, but they are comparable with those of Series I, V, VI, XVIII. The sessional change is also indicated by the decrease in d_k as k increases*. Without carrying through a good number of series with fixed head, no useful comparison can be made, but I think that the evidence of this one series is sufficient to justify the assertion that a shifting of the head from side to side cannot account for the greater part of the correlation of successive judgments.

(d) *Summary.*

First considering the individual series, it was noticed that there was a secular change in Personal Equation with time—i.e. the means decreased in passing from

* The value of σ_δ or ·074 may be compared with that of S_δ for the ordinary series of the Experiment *A*, which was ·0732.

TABLE VII.

Experiments on Shift of Head.

Order of Obser-vations	I		Order of Obser-vations	II		Order of Obser-vations	I		Order of Obser-vations	II		Order of Obser-vations	I		Order of Obser-vations	II	
1	2·39		8	2·48		15	2·45		22	2·49		29	2·36		36	2·45	
2	2·39		9	2·50		16	2·43		23	2·49		30	2·41		37	2·38	
3	2·39		10	2·57		17	2·39		24	2·55		31	2·38		38	2·40	
4	2·44	Mean 2·416	11	2·47	Mean 2·500	18	2·46	Mean 2·414	25	2·48	Mean 2·483	32	2·42	Mean 2·377	39	2·39	Mean 2·410
5	2·45		12	2·44		19	2·40		26	2·47		33	2·37		40	2·46	
6	2·44		13	2·53		20	2·39		27	2·44		34	2·35		41	2·37	
7	2·41		14	2·51		21	2·38		28	2·46		35	2·35		42	2·42	

TABLE VIII.

	$k=1$	2	3	4	5	6	7	8	9	10	11	12	13	14
d_k	2·5194	2·5188	2·5160	2·5144	2·5136	2·5096	2·5050	2·5048	2·5040	2·5052	2·5064	2·5036	2·5022	2·5008
σ_k	·07004	·06965	·07034	·06955	·06948	·07071	·06546	·06558	·06588	·06598	·06678	·06904	·06652	·06602
ρ_k	+·4438	+·4084	+·2856	+·2121	+·0719	+·0107	+·0298	−·0017	+·1521	+·1287	+·0430	+·1097	+·0619	—

$$\sigma_\delta = \sigma_1 \sqrt{2(1-\rho_1)} = \cdot 074$$

the earlier to the later series; in addition there was a remarkably constant sessional change within each series, this change being again a decrease from the earlier to the later observations. There was something in these changes almost analogous to an elastic strain, during the course of a series the estimation of a third drops, in the interval between the series there is a recovery, but not a complete recovery, for the first judgments in the succeeding series start at a little lower level than the first, but well above the last judgments in the series before; this slight "permanent deformation" caused by the "strain" represented in the sessional change, results in the secular fall. The figure below gives an ideal representation of this.

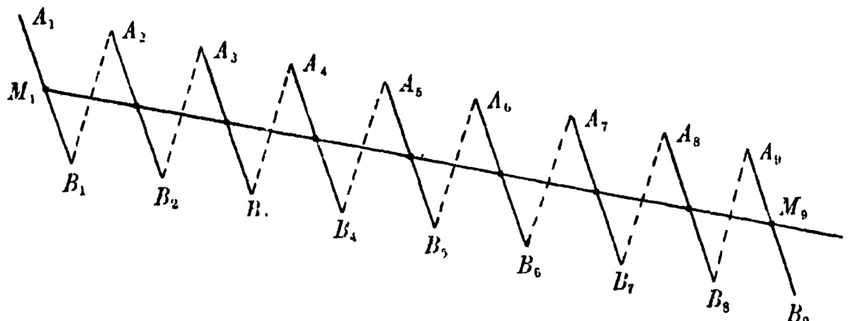


Fig. 7.

A_1B_1, A_2B_2, sessional change in Series 1, 2 etc.
B_1A_2, B_2A_3, "recovery" during interval between Series 1 and 2, 2 and 3 etc.
M_1M_9 the resulting secular change.

Then combining the twenty series, in order to get more reliable results, the coefficients of correlation of successive judgments, $\mathbf{R}_k$, were obtained; owing to the secular and sessional changes these coefficients had very high values and as k increased, apparently tended to a limit at about $+\cdot 60$. By fitting the means of the series together, the secular change was eliminated, and a series of coefficients $\mathbf{R}_k'$ obtained, which represented the average value of the correlation in a series: owing to the sessional change the $\mathbf{R}_k'$'s did not appear to tend to zero as k increased but to a limit at $+\cdot 16$ or $+\cdot 15$. The correlation of successive values of the residuals, left after subtracting the ordinates of the straight line "best" fitting the first 50 observations of each series from the observations of that series, gave a set of coefficients $\mathbf{R}_k''$, which fell off very rapidly and became negative when k equalled 6 or 7; the large negative values of the coefficients for the high values of k were probably due in part to the method of approximation used, and also to the fact that the straight line fitting the first 50 observations in a series did not represent satisfactorily the sessional change.

The values of $\mathbf{R}_k'$ calculated (up to $k=13$), gave no evidence of any tendency to periodicity in this coefficient, although there was evidence of this occurring in some of the individual series; periodicity in $\mathbf{R}_k'$ would indicate marked variations of roughly the same period occurring at any rate in a large number of the series.

It will be shown in a later section that the values of $\mathbf{R}_k'$, $k = 1 \ldots 13$, can be fitted very closely by a curve of the type $y = p + qr^k$, where p, q, and r are constants.

Finally it was shown that the correlation of successive judgments could not be due to a shifting of the head during the course of observation, although this might perhaps be one of many contributory causes.

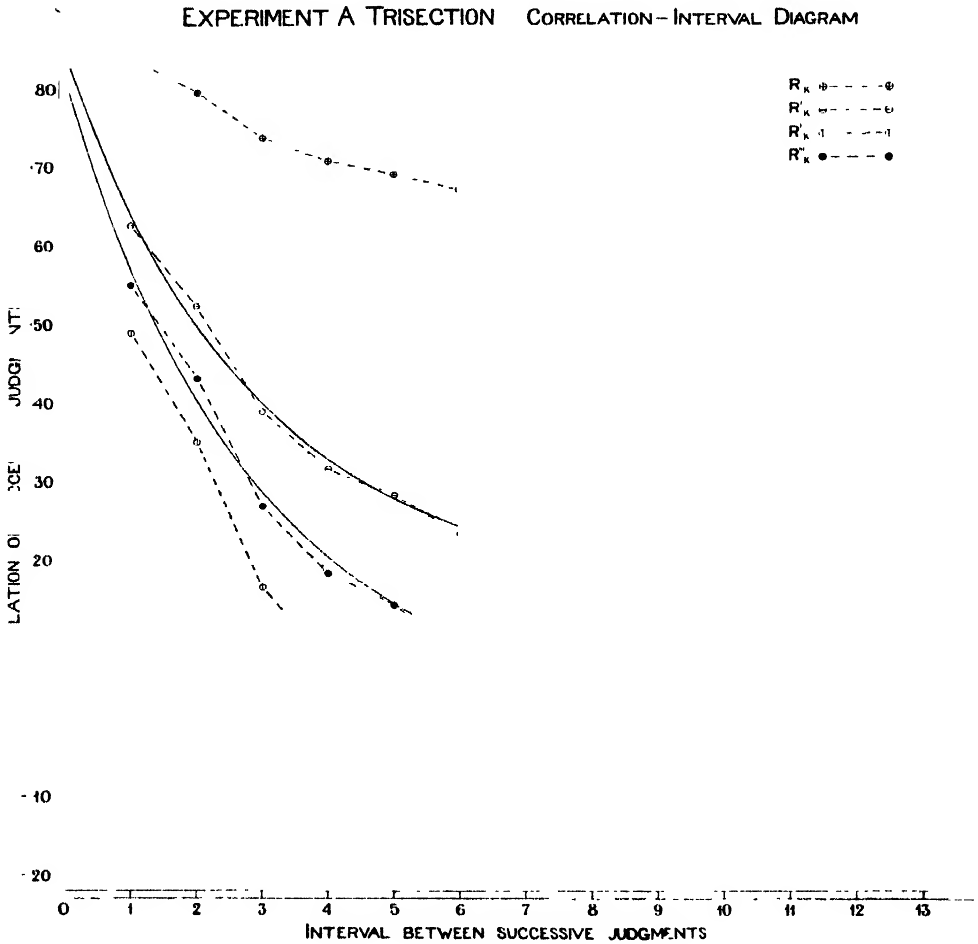


Fig. 8

In Figure 8 the values of $\mathbf{R}_k$, $\mathbf{R}_k'$, and $\mathbf{R}_k''$ (for linear sessional change) are plotted to k: the theoretical curves of the Equations (xlix) and (lvi) shown in the Figure will be discussed in Section XI, and also the points referred to as R_k'''.

VII. EXPERIMENT *B*, BISECTION. REDUCTION OF OBSERVATIONS.

(*a*) *The individual Series.*

In this Experiment the coefficients of correlation of successive judgments for the individual series were not all worked out, but only the values of ρ_1; these are tabled with σ_1 and d_1 in Table IX. The values of d_k for each series, $k = 1 \ldots 14$ are also given in Table X. It will be seen that there are not the same marked secular or sessional changes as characterised the Trisection Series. In Figure 9 the means of Groups 1 of each Series—or the d_1's—have been plotted to "order" and to "time," and again if

x is the personal equation or mean,

y the number of order of series,

z the number of days between 13th June and the date on which the series was carried out,

we have for the regression lines

$$x - 2{\cdot}8793 = - {\cdot}0010131\,(y - 10{\cdot}5) \quad \ldots\ldots\ldots\ldots\ldots(\text{xxxiv}),$$

$$x - 2{\cdot}8793 = - {\cdot}0005359\,(z - 32{\cdot}10) \quad \ldots\ldots\ \ldots\ldots(\text{xxxv}).$$

These lines have been drawn in the diagrams; the coefficients of correlation are

$$r_{xz} = - {\cdot}337 \pm {\cdot}134, \qquad r_{xy} = - {\cdot}156 \pm {\cdot}147, \qquad r_{yz} = + {\cdot}945 \pm {\cdot}016,$$

giving partial correlations

$$r_{xy.z} = + {\cdot}529 \pm {\cdot}109, \qquad r_{xz.y} = - {\cdot}589 \pm {\cdot}099.$$

TABLE IX. *Constants of Bisection Experiments.*

Series	d_1	σ_1	ρ_1	$\sigma_\delta = \sigma_1\sqrt{2(1-\rho_1)}$	Dates (1920) and time at start	Time taken for series
I	2·8648	·04997	+ ·4942	·0503	11 a.m. } 13 June	6^{m} 0^{s}
II	·8624	·05461	+ ·2609	·0661	2 45 p.m. }	5 20
III	·9262	·03821	+ ·0823	·0518	p.m. 15 ʺ	5 45
IV	·8642	·04690	+ ·4107	·0509	10 a.m. } 29 ʺ	5 30
V	·8290	·05158	+ ·5870	·0469	3 p.m. }	
VI	·9114	·01609	+ ·5768	·0124	a.m. 30 ʺ	
VII	·9178	·01415	+ ·2993	·0523	10 a.m. } 1 July	
VIII	·9218	·01766	+ ·4360	·0506	12.15 p.m. }	
IX	·8724	·01384	+ ·1389	·0575	10.30 a.m. } 2 ʺ	
X	·8990	·04579	+ ·1018	·0614	6.30 p.m. }	
XI	·9238	·03617	·0423	·0522	9.30 a.m. } 6 ʺ	
XII	·9298	·01810	+ ·5089	·0177	6 45 p.m. }	
XIII	·8806	·04407	+ ·2769	·0530	a.m. 7 ʺ	
XIV	·8312	·04955	+ ·4415	·0522	11 a.m. } 1 August	
XV	·8242	·03606	+ ·3331	·0416	2.30 p.m. }	
XVI	·7976	·04135	+ ·3190	·0483	p.m. 16 ʺ	
XVII	·8566	·03739	+ ·5531	·0353	a.m. 17 ʺ	
XVIII	·8808	·03497	+ ·2776	·0420	p.m. 18 ʺ	
XIX	·8890	·03986	+ ·5407	·0382	p.m. 19 ʺ	
XX	2·9030	·02610	+ ·1401	·0342	p.m. 20 ʺ	

Probable Errors of Coefficients of Correlation calculated from 50 pairs of the variates.

·80	± ·0343
·70	± ·0486
·60	± ·0610
·50	± ·0715
·40	± ·0801
·30	± ·0868
·20	± ·0916
·10	± ·0944
·00	± ·0945

Mean time taken for a series of 70 observations (including the 7 preliminary trials*) 5^{m} 58^{s}

Mean interval between records of judgment $5{\cdot}11^{s}$

* See p. 28, footnote.

TABLE X. Bisection.

Table of . . Groups (in inches

	G1	roup'	roup 3		Group	G Group			Group 10	G	Group 1	up 1	
I	2·8648	·8646	·8656	:8640	8652	·8644	2·8648	2·86·	2·8648	·8654	2·8658	674	8668
II	·8624	8656	·8676	·8692	·8704	·8736	·8756	87	8782	·8788	·8772	774	·8770
III	·9262	·9260	9256	·9238	·9228	·9232	·9234	92	·9242	·9244	·9234	224	9226
IV	·8642	8666	·8684	·8684	·8704	·8754	·8786	·88	·8834	·8856	·8868	886	·8890
V	·8290	·8304	8310	8320	·8334	·8344	8348	·83	·8362	·8376	·8378	380	8376
VI	·9114	·9136	·9154	·9172	9196	9214	9210	·92	·9196	·9202	·9218	212	9206
VII	·9178	·9192	·9196	·9200	9196	·9228	·9230	·92	9214	·9210	·9218	194	·9180
VIII	·9218	·9230	·9256	·9260	·9264	·9220	·9206	·92	·9188	·9170	9172	154	·9132
IX	·8724	·8736	8704	·8688	8686	·8682	·8680	8688	8676	·8676	8660	628	·8610
X	·8990	·8986	9004	·9030	9028	9024	·9024	9010	·9018	9016	·9028	024	·9002
XI	·9238	·9234	·9264	·9290	·9286	9264	·9262	9252	9252	·9286	9290	302	·9298
XII	·9298	9300	·9318	·9316	·9374	9416	·9426	·9412	·9400	9398	·9392	404	·9414
XIII	·8806	8804	8816	·8798	·8796	8818	8840	8876	·8876	8902	·8916	950	·8964
XIV	·8312	·8322	8340	8350	·8362	·8394	·840	·406	8406	·8416	8426	458	·8478
XV	·8242	·8254	8260	·8254	·8266	8294	·830	·306	·8324	·8342	·8364	386	·8408
XVI	·7976	·7978	·7974	·7970	·7966	·7996	800	020	8044	8060	·8068	086	·8094
XVII	·8566	·8548	·8522	·8512	·8514	8514	851	·518	·8520	8536	8544	528	·8530
XVIII	·8808	·8790	·8784	8770	·8740	8714	869	682	8676	8686	8674	664	·8664
XIX	·8890	·8886	8864	·8828	8816	·8796	877	778	·8778	·8786	·8788	786	·8784
XX	2·9030	9024	9018	·9022	·9014	9028	903	036	9050	9034	·9032	048	[illegible]
Means	2·87928		88028	·880	2 88		90		2 88243	:88319	2 8835(	2·8	

The dates on which the series were carried out—the z's—are given at the end of Table IX; the distribution was more satisfactory than that of the Trisections, and the significance of these two partial correlations will be referred to shortly.

The variation in the means of the series is much smaller than in the case of the Trisections; we have here a range from 2·93 to 2·80 ins. while in the other, from 2·70 to 2·34 ins.; in both cases the secular change is in the direction which lessens the measures, i.e. the marks on the forms in the later series were on the whole further to the observer's left hand than in the earlier series. Nor does experience appear to increase accuracy, for the true position of the half is at 2·97 inches (and of the third at 2·51 inches).

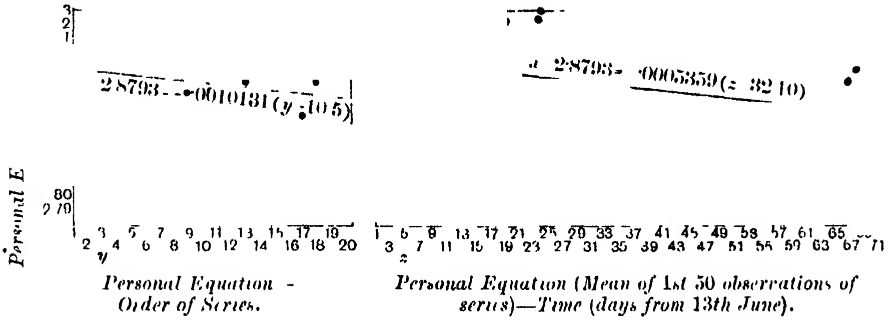


Fig. 9. Bisection. Means of Groups I of each series plotted with Order of Series and Date of Series

Next considering the sessional change, the values of y_t (defined on p. 47) have been plotted in Figure 10; the straight line "best" fitting these points is

$$y_t - 2{\cdot}8816 = + {\cdot}00035\,34\,(t - 32) \quad \text{.....................(xli)},$$

where t is the order of observation in a series, and the coefficient of correlation between y_t and t is $+ {\cdot}5294 \pm {\cdot}0137$*.

Using the relations of page 48, it is found that

$$\eta_{y_t} = {\cdot}271 \pm {\cdot}018, \quad \sqrt{1 - \eta^2{}_{y_t}} = {\cdot}963,$$

and on comparing this latter value with that for the Trisections (·815) we see that in the present case the mean sessional change is of less significance.

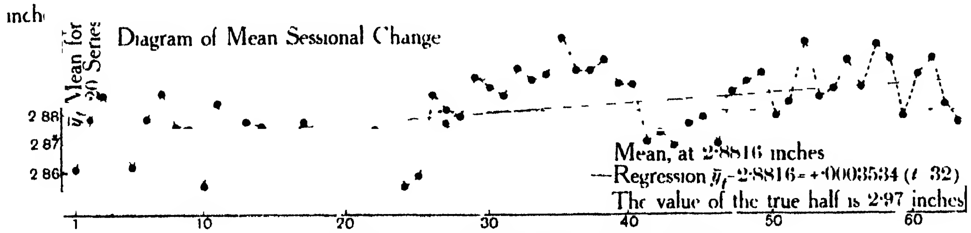
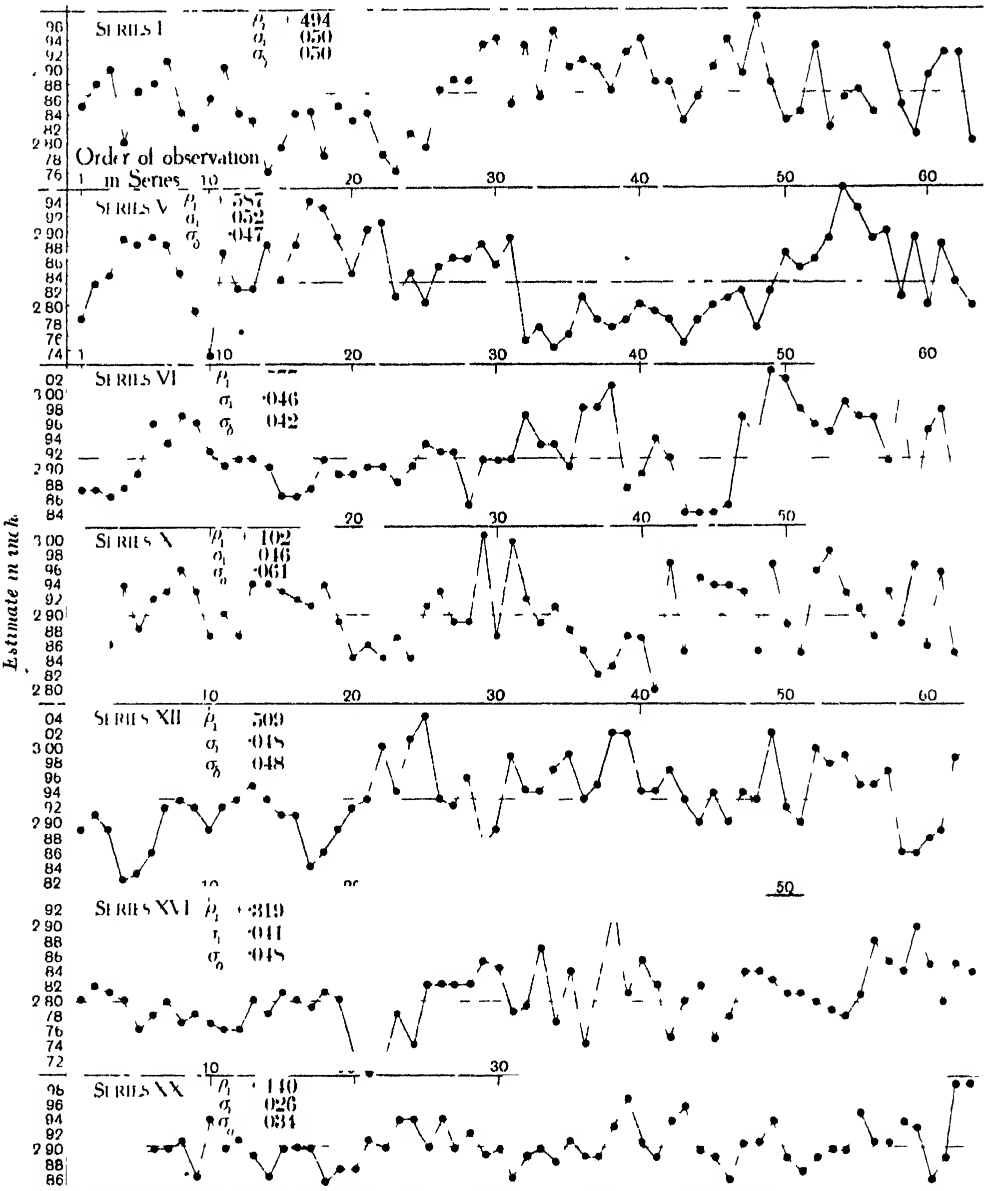


Fig 10. Bisection. t, Order of Observation in Series.

It will be noticed in looking at Figure 10 that the points (t, y_t) appear to be subject to a fairly consistent periodic variation about the regression line, the

* This correlation between the mean tth observation $(\bar{y}_t)$ and t must be distinguished from the correlation between the tth observation (y_t) and t, which is $+ {\cdot}143$, and as it should be, less than η

complete period covering from 20 to 22 observations. Without a detailed analysis of the separate series, it is not possible to say whether there is a period of this order underlying the variations in judgment in all series, or whether this periodicity in y_t results from large variations in one or two series; the diagrams of seven of the series, in Figure 11 do not certainly suggest any marked periodic variation, and it is possible that the drop at about the 44th and the peak near the



The horizontal line intersecting each graph gives the mean of the first 50 observations in that series.

Fig. 11. Bisections. Diagrams representing variations in judgment.

55th observations in Series V and VI, would go far to account for the similar features in the $\bar{y}_t$ diagram, the "y" scale of which is four times greater than that in Figure 11.

Using the method of Correlation of Ranks*, the correlation between σ_1 and ρ_1 has been calculated for the 20 Series; the result is

$$r_{\sigma_1\rho_1} = +\cdot420 \pm \cdot124.$$

Another coefficient which may be calculated, is that of the correlation between σ_δ, or the standard deviation of first differences of consecutive judgments, and ρ_1; using the same method as for $r_{\sigma_1\rho_1}$, it is found that

$$r_{\sigma_\delta\rho_1} = -\cdot416 \pm \cdot125 \text{ and again } r_{\sigma_\delta\sigma_1} = +\cdot465 \pm \cdot118.$$

Now ρ_1, σ_1 and σ_δ are not three independent quantities, as they are connected by the relation

$$\sigma_\delta = \sqrt{\sum_{t=1}^{n} \frac{(y_{t+1} - y_t)^2}{n}} = \sigma_1\sqrt{2(1-\rho_1)},$$

and it is open to question, which two are the most fundamental. In the ordinary theory of the Combination of Observations, where it is assumed that ρ_1 is zero, it is natural to consider σ_1 (or σ) as a fundamental constant, the measure of the accuracy of judgment; σ_δ appears to have no special significance and merely equals $\sqrt{2}\sigma$. If however there is a correlation of successive judgments, σ loses its importance; if we take a small number, p, of successive observations and calculate their standard deviation, s_p, we can no longer say that s_p, subject to its probable error $\pm \cdot6745 \dfrac{s_p}{\sqrt{2p}}$, will be equal to σ, the standard deviation of a long series of judgments. On the other hand there is every reason to expect that the σ_δ found from a few observations will give a fair approximation to the σ_δ found from a large number. σ is dependent to a high degree on the sessional change, for example it has been shown† that if this change can be represented by a straight line of the form $y = bt$, then σ', or the standard deviation of the observations freed from this change is given by

$$\sigma'^2 = \sigma^2 - \frac{b^2}{12}(n^2 - 1).$$

It is true that σ_δ is dependent to some extent on the sessional change, but far less so; for instance in the case of the linear sessional change, σ_δ', the standard deviation of the first differences of the successive residuals left after the removal of the line, is given approximately by the relation

$$\sigma_\delta'^2 = \sigma_\delta^2 - b^2.$$

And for any form of sessional change which is likely to occur in experiments of the type we are considering, the correction to the difference between two successive observations necessary to get the corresponding difference between the

* p. 52 and footnote.

† Section V (*b*) p. 43.

residuals after the removal of the sessional term, will be very small indeed compared with the standard deviation of this difference, or σ_δ. It is therefore suggested that in the combination of correlated observations, σ_δ, the average value of the jump in estimation between two successive judgments, is of more fundamental importance than σ. As an example, consider the diagrams of the observations of Series X and Series XX in Figure 11; the correlation, ρ_1, is very low in both cases, but it is suggested that the physiological significance of the difference in type between the two, lies in the fact that σ_δ for Series X is nearly twice as large as σ_δ for Series XX, rather than in the difference in the σ_1's. Or again in the diagrams of the Trisection Experiment, Figure 6, I would emphasise the same point in a comparison of the difference between the two highly correlated Series VIII and XVI.

Now returning to the coefficients of partial correlation

$$r_{xy.z} = + \cdot 529 \pm \cdot 109, \quad r_{xz.y} = - \cdot 589 \pm \cdot 099.$$

With the interpretation suggested on p. 54 for these coefficients, we are led to a rather suggestive conclusion. If we are dealing with a number of series carried out at equal intervals of time in the course of one, or even perhaps two days, but effectively at one epoch when comparison is made with the long range of nearly 70 days covered by the Bisection Series, then the correlation between x and y is positive, or the pencil mark in the later series tends to be made further to the observer's right than in the earlier series; this change is in the same direction as the sessional change within a series. There is indeed a curious coincidence, on which of course no stress must be laid,

$$r_{xy.z} = + \cdot 529 \pm \cdot 109, \quad r_{yt.t} = + \cdot 5294 \pm \cdot 0137.$$

That is to say the correlation between the mean of a series and the order of that series when a number of series are done in close succession, is of the same sign and magnitude as the correlation between the mean tth observation and its order, t, in the series. But if we are dealing with all the pth series of sets which have been carried out on different days with varying and perhaps many days' interval between, then the coefficient $r_{xz.y}$ is negative, or the bisection-marks on the later days have on the whole a tendency to move to the left of the observer; this is in the direction of the secular change.

The conclusion which it seems possible to draw is this; if a number of series are done at very short intervals, the interval of rest between the series will not be sufficient to break the effect of the sessional change; but if a considerable interval elapses between the carrying out of the series, then the sessional change in one series has no influence on the judgments in the succeeding series, but a quite distinct secular change may be noticeable. In the Bisection Experiment both secular and sessional changes are very small, but they are acting in opposite directions. If these two changes are due to different physiological factors, it seems possible that it is the fact that they are acting in opposite directions in the Bisection Experiment which causes them to be of so much smaller magnitude than in the Trisection Experiment, where they were acting in the same direction.

(b) The Combination of the Series.

For the combined series, the coefficients of correlation of successive judgments $\mathbf{R}_k$ for $k=1, 2 \ldots 13$ were calculated from 13 correlation tables each based on the 1000 combined observations; the results for D_k, S_k and $\mathbf{R}_k$ are tabled below (Table XI). The effect of the slight sessional change is noticeable in the increasing values of D_k.

Using the values of D_k, S_k and $\mathbf{R}_k$ and of ${}_p d_k$ from Table X, Equations (vi), (vii) and (viii) give $\sum_m (\rho_k \sigma_1 \sigma_{k+1})$ and $\sum_m (\sigma_k^2)$ for $k=1, 2 \ldots 14$. Equations (ix) and (x) then give the values of S_k' and $\mathbf{R}_k'$ contained in the 5th and 6th rows of Table XI. The value of $\mathbf{R}_1'$ found by this method should be compared with that found with the help of the ρ_1's, σ_1's and σ_2's of the individual series, namely

$$\mathbf{R}_1' = \frac{\sum_m (\rho_1 \sigma_1 \sigma_2)}{\sqrt{\sum_m (\sigma_1^2) \sum_m (\sigma_2^2)}} = + \cdot 3578 \pm \cdot 0186 \ldots\ldots\ldots\ldots\ldots \text{(x) bis.}$$

The difference which is well within the probable error arises from the fact that $\mathbf{R}_1$ has been found by grouping the observations in a correlation table, while the ρ_1's, σ_1's and σ_2's were found by direct multiplication of the crude values of the observations.

Another method of obtaining the $\mathbf{R}_k''$s is from the first difference correlation equations, or the method of Problem I, p. 41; the results are given in the 7th row of Table XI, while the constants ${}_1R_k$, the coefficients of correlation of successive first differences required in the solution, are in the 8th row of the Table. Comparing the values of $\mathbf{R}_k'$ found by the two methods, we find good agreement up to $k=6$, but beyond this point the $\mathbf{R}_k''$s of the second and approximative method assume much too large negative values*. It is however evident from the results of the first method that $\mathbf{R}_k'$ does become negative, and as it could not remain negative indefinitely as k increased, there seems here to be another indication that a periodic variation exists among the judgments at any rate in a certain number of the series. For a complete period covering from 20 to 22 observations suggested by the y_t diagram, $\mathbf{R}_k'$ should have a minimum value at $\mathbf{R}_{10}'$ or $\mathbf{R}_{11}'$; the figures suggest that the minimum occurs somewhat earlier, at about $\mathbf{R}_8'$, but the probable errors for these small coefficients are very large. When time is available it would be interesting to examine further the significance of this periodicity.

The points $(\mathbf{R}_k, k)$ and $(\mathbf{R}_k', k)$ have been plotted in Figure 12.

It will be noticed that the S_k''s in the later groups are larger than in the earlier, this suggesting again as in the case of the Trisections, that the observations become slightly more erratic towards the end of a series.

* This result tends to confirm the suggestion made on p. 60 that the difference correlation method gave too large negative values for $\mathbf{R}_k''$ in the Trisection Experiment.

TABLE XI.

Constants of Combined Series (Bisection).

	k=1	2	3	4	5	6	7	8	9	10	11	12	13	14
D_k	2·87936	2·87982	2·88038	2·88034	2·88068	2·88136	2·88154	2·88190	2·88236	2·88236	2·88314	2·88338	2·88366	2·88368
S_k	·05727 ± ·00086	·05706 ± ·00086	·05748 ± ·00087	·05770 ± ·00087	·05783 ± ·00087	·05770 ± ·00087	·05789 ± ·00087	·05814 ± ·00088	·05840 ± ·00088	·05792 ± ·00087	·05773 ± ·00087	·05794 ± ·00087	·05813 ± ·00088	·05850 ± ·00088
$\mathbf{R}_k$	+ ·6299 ± ·0129	·5599 ± ·0146	·5322 ± ·0153	·4755 ± ·0165	·4470 ± ·0171	·4243 ± ·0175	·3905 ± ·0181	·3727 ± ·0184	·3756 ± ·0183	·3817 ± ·0182	·3865 ± ·0181	·3621 ± ·0185	·3518 ± ·0187	
S_k'	·04364 ± ·00066	·04348 ± ·00066	·04360 ± ·00066	·04355 ± ·00066	·04373 ± ·00066	·04375 ± ·00066	·04437 ± ·00067	·04490 ± ·00068	·04541 ± ·00068	·04548 ± ·00069	·04566 ± ·00069	·04607 ± ·00069	·04658 ± ·00070	·04727 ± ·00071
$\mathbf{R}_k'$ 1st method	+ ·3519 ± ·0187	+ ·2306 ± ·0202	+ ·1828 ± ·0206	+ ·0849 ± ·0212	+ ·0389 ± ·0213	+ ·0119 ± ·0213	− ·0344 ± ·0213	− ·0542 ± ·0213	− ·0383 ± ·0213	− ·0204 ± ·0213	− ·0051 ± ·0213	− ·0302 ± ·0213	− ·0339 ± ·0213	
$\mathbf{R}_k'$ 2nd method	+ ·3578 ± ·0186	+ ·2364	+ ·1883	+ ·0900	+ ·0404	+ ·0009	− ·0577	− ·0886	− ·0835	− ·0730	− ·0647	− ·1070	− ·1250	
${}_1R_k$	− ·4055	− ·0571	+ ·0390	− ·0379	− ·0079	+ ·0149	− ·0215	− ·0280	− ·0042	+ ·0017	+ ·0394	− ·0189		

$S_0 = S_1'\sqrt{2(1-\bar{\mathbf{R}}_1)} = \cdot 04946.$

(*c*) *Comparison with Experiment A.*

The difference between the results of the two experiments is probably due to the fact that the estimation of a half is so much easier than the estimation of a third. The variations in the latter observations are all on a larger scale than in the former; the secular and sessional changes are very much greater, and if we compare the values of the fundamental constants, we find:

	S_1'	S_δ	$\mathbf{R}_1'$
Trisection	·0845	·0732	+ ·6246 ± ·0130
Bisection	·0436	·0495	+ ·3519 ± ·0187

EXPERIMENT B BISECTION CORRELATION - INTERVAL DIAGRAM

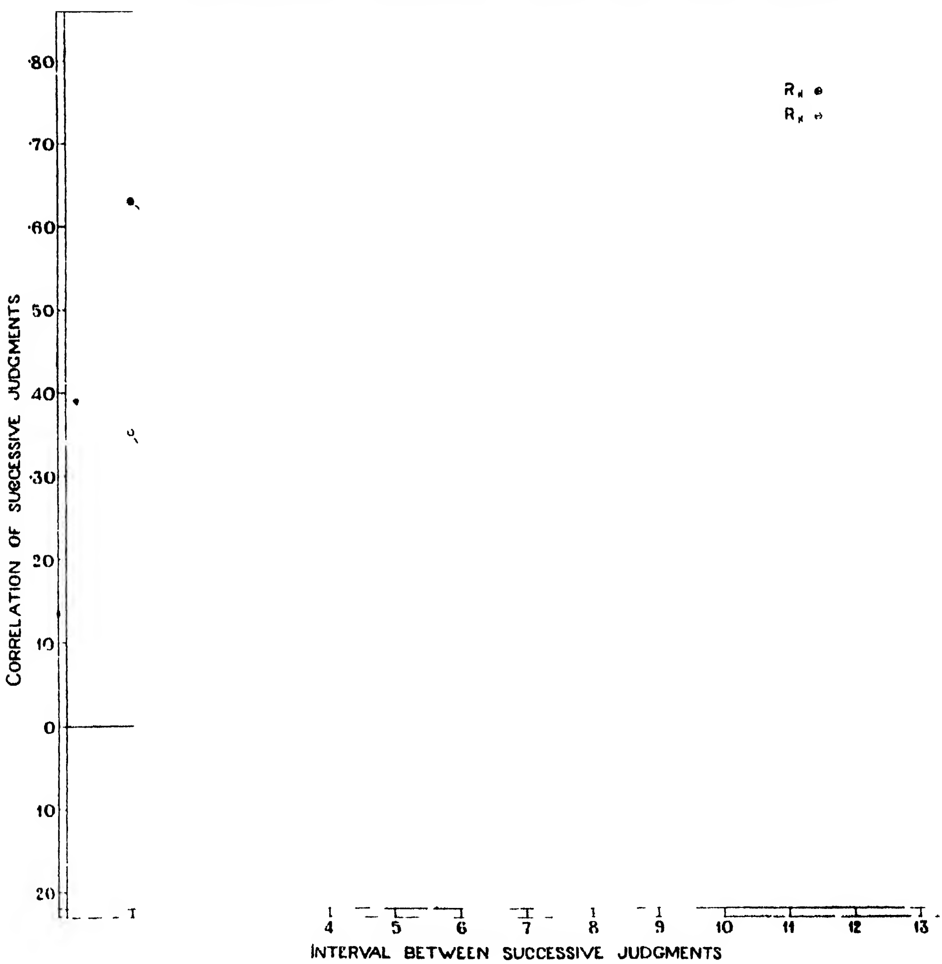


Fig. 12.

and even after the removal of the greater part of the sessional change—(the best fitting straight lines)—the coefficient $\mathbf{R}_1$ for the Trisections is + ·4892, or greater than $\mathbf{R}_1'$ for the Bisections. The ratio of the values of S_δ, or roughly 3 to 2, is a

measure of the relative uncertainty of the observer in making his estimate in the two different Experiments.

There is some evidence for a slight periodicity in the judgments in the Bisection Series; if there is any period in the Trisections it must cover at least 26 observations, for there is no indication of a significant increase in the values of $\mathbf{R}_k$ as far as calculated, i.e. up to $\mathbf{R}_{13}$.

VIII. Experiment C. Counting of 10 Seconds. Reduction of Observations.

(*a*) *The Individual Series.*

The values of d_1, σ_1 and ρ_1 for each of the 20 series are given in Table XII as well as the hour and date; the means (d_1) have been plotted to the order of series in Figure 13.

If x is the mean in the factor e/p for a series,

y the order of series,

z the time in hours and fractions of an hour between 2.0 p.m. on December 13, and the commencement of series

we have for the regression lines,

$$x - \cdot 9186 = - \cdot 006056\,(y - 10\cdot 5) \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{xxxvi}),$$

$$x - \cdot 9186 = - \cdot 001552\,(z - 38\cdot 24) \quad \ldots\ldots\ldots\ldots\ldots(\text{xxxvii}),$$

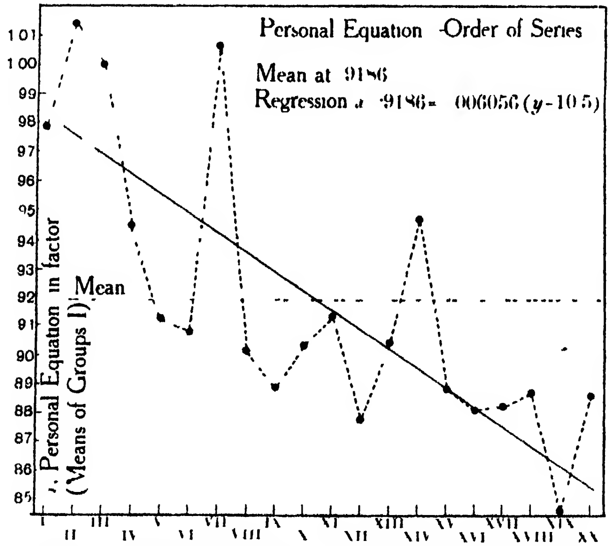


Fig. 13. 10 Second Counting. y, Order or Number of Series.

of which the first is represented in Figure 13.

The coefficients of correlation are,

$$r_{xy} = - \cdot 754 \pm \cdot 065, \quad r_{xz} = - \cdot 775 \pm \cdot 060, \quad r_{yz} = + \cdot 977 \pm \cdot 007,$$

giving partial correlation coefficients

$$r_{xy.z} = +\cdot 022 \pm \cdot 151, \quad r_{xz.y} = -\cdot 271 \pm \cdot 140.$$

The secular change corresponds to a gradual decrease in estimate throughout the course of the experiment; the value of the factor e/p for a true 10 second estimate would be $\frac{10\cdot 0}{10\cdot 2} = \cdot 98$, and this was closely approached by the means of the first three series, which were carried out on the first day, shortly after trial counts had been made with a watch. No further check with a watch was made during the succeeding days, and the length of estimation decreased and finally appears to have reached a fairly steady value at about ·88. The mean for the 20 Series was ·9186, or a count of 9·37 seconds.

TABLE XII.

Constants of Individual Series (Counting Seconds).

Series	d_1	σ_1	ρ_1	$\sigma_\delta - \sigma_1\sqrt{2(1-\rho_1)}$	Date (1920)	Time at Start
I	·9786	·04030	+ ·5283	·0391	13 December	2.30 p.m.
II	1·0140	·04331	+ ·4988	·0434	„	3.15 p.m.
III	·9998	·03844	+ ·0625	·0526	„	3.45 p.m.
IV	·9446	·03732	+ ·4027	·0408	14 December	10.15 a.m.
V	·9128	·03394	+ ·4378	·0360	„	11.20 a.m.
VI	·9090	·03015	+ ·5437	·0288	„	12.0 noon
VII	1·0070	·03981	+ ·3819	·0443	„	2.30 p.m.
VIII	·9012	·02488	+ ·4550	·0260	„	3.5 p.m.
IX	·8886	·03934	+ ·4326	·0419	„	3.35 p.m.
X	·9030	·02851	+ ·5439	·0272	15 December	10.0 a.m.
XI	·9130	·02982	+ ·5326	·0288	„	10.35 a.m.
XII	·8774	·01852	+ ·2850	·0221	„	11.10 a.m.
XIII	·9046	·02402	+ ·1894	·0243	„	11.50 a.m.
XIV	·9464	·02903	+ ·5085	·0288	„	2.30 p.m.
XV	·8880	·04162	+ ·7589	·0289	„	3.5 p.m.
XVI	·8812	·04947	+ ·8549	·0266	16 December	10.0 a.m.
XVII	·8828	·03945	+ ·6566	·0327	„	10.30 a.m.
XVIII	·8872	·02750	+ ·5406	·0264	„	11.5 a.m.
XIX	·8468	·02486	+ ·1266	·0329	„	11.35 a.m.
XX	·8864	·03345	+ ·6369	·0285	„	12.10 p.m.

Probable Errors of Coefficients of Correlation calculated from 50 pairs of the variates.

ρ	P.E.
·80	± ·0343
·70	± ·0486
·60	± ·0610
·50	± ·0715
·40	± ·0801
·30	± ·0868
·20	± ·0916
·10	± ·0941
·00	± ·0954

With the interpretation of p. 54, the insignificant value of the coefficient $r_{xy.z}$, suggests that for a number of series done in quick succession, there will be no change in personal equation: we shall therefore not expect to find any large general sessional change in the series. The diagram of mean sessional change is given in Figure 14, where y_t is plotted to t.

The equation of the straight line best fitting the points is

$$\bar{y}_t - \cdot 919 = +\cdot 0000731\,(t - 32) \quad \text{.........\ ....(xxxviii)},$$

and has been drawn in Figure 14.

Using the relations and interpretation of page 48, it is found that

$$\eta_{y_t} = \cdot 212 \pm \cdot 018 \text{ and } \sqrt{1 - \eta_{y_t}^2} = \cdot 977,$$

so that the mean sessional change is of even less significance than for the Bisections. In fact it is clear from the diagram that the regression line (xxxviii) very nearly coincides with the line of mean judgment, $y = \cdot 919$.

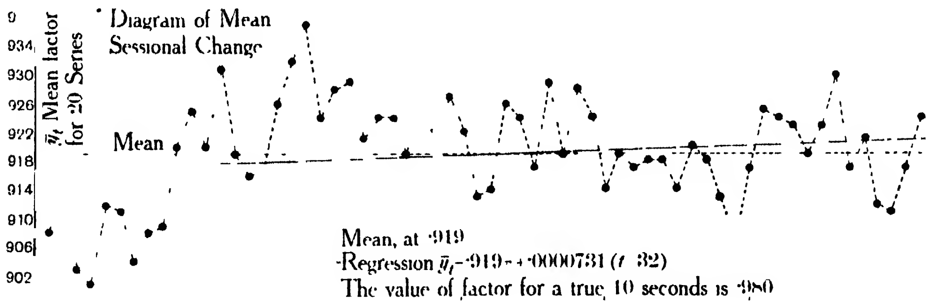


Fig. 14. 10 Second Counting. *t*, Order of Observation in Series.

The σ_δ's have been found for all the individual series, and using the values of S_1' and $\mathbf{R}_1'$ given below, we have for the combined series

$$S_\delta = S_1' \sqrt{2\,(1 - \mathbf{R}_1')} = \cdot 0338.$$

The method of correlation of Ranks gives

$$r_{\sigma_1 \rho_1} = + \cdot 329 \pm \cdot 135,$$

showing again that large variation is associated with high correlation.

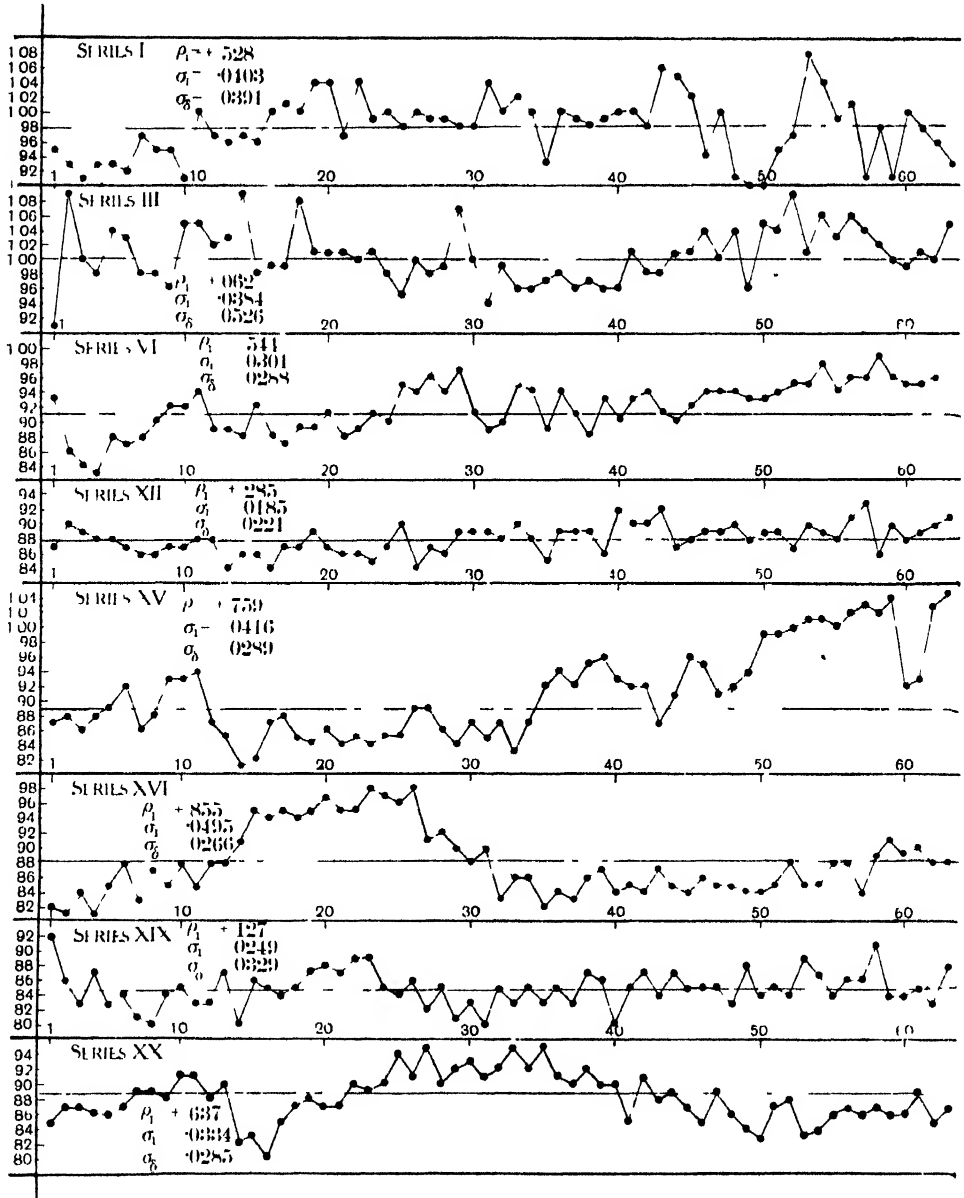
In Figure 15 are given eight representative series graphs which provide a good illustration of the variations in judgment. In the first two graphs (I and III), σ_δ is large and there are many sudden fluctuations, but in the later series σ_δ is lower and very constant in value. What may be described as the smoothness in the change of judgment is in some cases particularly noticeable; for example in the stretch between

$$\left.\begin{array}{l} y_{11} \text{ and } y_{33}, \text{ Series VI} \\ y_{2} \text{ and } y_{12}, \text{ Series XII} \\ y_{47} \text{ and } y_{50}, \text{ Series XV} \end{array}\right\} .$$

In making comparison with the similar diagrams for Trisections and Bisections allowance must be made for the differences in scale, but I think it is clear that this "smoothness" or gradual variation is a special feature of the 10 second counting; there is for instance no diagram of Trisections or Bisections which can compare with that of Series XVI of the counting, for high correlation combined with very gradual variation. But such a result is not unexpected, if the procedure of the experiment with the continuous counting be remembered.

A further point of interest is to examine how far a sudden "break" or discontinuity in the length of estimate influences the succeeding judgments. Among the 1000 observations forming the Groups I of the 20 series there are 61 "breaks" or differences between successive judgments of ·07 or over (in terms of the factor e/p),

i.e. of over twice S_δ, the standard deviation of first differences. In the diagram of Figure 16, ten observations are represented, the break between y_{t-1} and y_t or $y_t \sim y_{t-1}$, is supposed to be equal to, or greater than, ·07. If this large break influences the succeeding judgments, it is to be expected that the differences $y_{t+1} \sim y_t$, $y_{t+2} \sim y_t$, ... etc. will be smaller on the average than the differences $y_t \sim y_{t-1}$, $y_t \sim y_{t-2}$, ... etc.



The horizontal line intersecting each graph gives the mean of the first 50 observations in that series.

Fig. 15. 10 Second Counting. Diagrams representing variations in judgment.

In the first row of Table XIII are given the standard deviations of these differences taken from the 61 breaks; now in 14 of these cases there is what may be called a "double break," that is, after making one large variation to y_t, the judgment returns approximately to its previous state, both $y_t \sim y_{t-1}$ and $y_t \sim y_{t+1}$ being greater than or equal to ·07. While such cases may represent true variations in judgment, it is very possible that they result from some accidental error, a slowness in pressing the tapping key or in catching up the counting at the com-

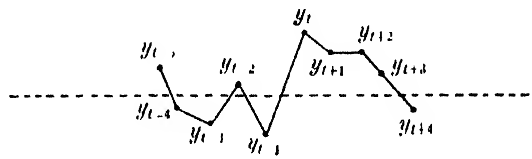


Fig. 16. Experiment C. Effect of a large break in judgment.

mencement of the observation, which was realised at the time and was not due to a real change in estimate. In the second row of the Table, therefore, are given the standard deviations taken from the 33 sets where there was no double break, and, in the third row, the standard deviations of 1st differences (taken from the whole 1000 judgments)

$$\text{between } y_t \text{ and } y_{t+1} = S_8 = S_1'\sqrt{2(1-\mathbf{R}_1')},$$

$$\text{,, } \quad y_t \text{ and } y_{t+2} \quad = S_1'\sqrt{2(1-\mathbf{R}_2')},$$

$$\text{,, } \quad y_t \text{ and } y_{t+3} \quad = S_1'\sqrt{2(1-\mathbf{R}_3')}, \text{ etc.}$$

TABLE XIII.

Standard Deviations of Differences between Judgment after "Break", y_t, and the Judgments y_{t-5} to y_{t+5}.

No. of Row	Number of Judgments	Previous Judgments					Succeeding Judgments				
		y_{t-5}	y_{t-4}	y_{t-3}	y_{t-2}	y_{t-1}	y_{t+1}	y_{t+2}	y_{t+3}	y_{t+4}	y_{t+5}
1	From 61 sets	·0692 ± ·0042	·0647 ± ·0040	·0624 ± ·0038	·0624 ± ·0038	·0851 ± ·0052	·0476 ± ·0029	·0541 ± ·0033	·0553 ± ·0034	·0549 ± ·0034	·0623 ± ·0038
2	From 33 sets	·0757 ± ·0063	·0682 ± ·0057	·0636 ± ·0053	·0667 ± ·0055	·0810 ± ·0067	·0346 ± ·0029	·0421 ± ·0035	·0508 ± ·0042	·0483 ± ·0040	·0635 ± ·0053
3	From total 1000	·0412	·0416	·0401	·0373	·0338	·0338	·0373	·0401	·0416	·0142

The probable errors are calculated from the usual expression, $\pm ·6715\, \sigma/\sqrt{2n}$.

If we consider the values of these standard deviations together with their probable errors, we may say definitely that the effect of a large break or discontinuity in judgment is quite significant, and that the influence appears to last for at least four or five judgments. It cannot of course be decided whether the

breaks were caused by some chance external factor, or were due to a conscious change in estimate made by the observer on deciding, whether rightly or wrongly, that his second count was too short or too long*.

It will be noticed that the standard deviations of differences between these special pairs of judgments are in all cases greater than the corresponding standard deviations from the total 1000 judgments; this is to be expected, for the judgments y_t from which all the differences are taken are not a random selection of 61 (or 33) judgments, but include many of the most erratic and therefore those furthest from the mean.

(*b*) *The Combination of the Series.*

In combining the twenty series, D_k, S_k and $\mathbf{R}_k$ were calculated from the thirteen correlation tables of the judgments, and the values of these constants are given in Table XIV below. A glance at any one of the correlation tables showed that the 1000 judgments in any group did not follow a normal distribution, and in order to get a measure of this, the coefficient of skewness for the 1000 judgments in the combined Groups 1 (i.e. for the judgments $y_1, y_2 \dots y_{50}$ of the twenty series) was calculated from the expression

$$\text{Skewness} = \frac{\sqrt{\beta_1}\,(\beta_2+3)}{2(5\beta_2-6\beta_1-9)},$$

where β_1 and β_2 are the fundamental ratios of the moments about the mean given by

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}, \quad \beta_2 = \frac{\mu_4}{\mu_2^2}.$$

The result was as follows:

$$\beta_1 = \cdot 2726, \quad \beta_2 = 2\cdot 9739, \quad \text{Skewness} = \cdot 3684 \pm \cdot 0339,$$

showing a very significant degree of skewness, and the frequency follows a Type I curve of limited range.

The distribution of these 1000 observations made within a period of four consecutive days, gives but another example of the frequent inapplicability of the Normal Error Law.

Using the values of ρ_1, σ_1 and σ_2, $\mathbf{R}_1'$ is obtained from

$$\mathbf{R}_1' = \frac{\sum\limits_m (\rho_1\sigma_1\sigma_2)}{\sqrt{\sum\limits_m (\sigma_1^2) \sum\limits_m (\sigma_2^2)}} = +\cdot 5200 \pm \cdot 0156 \quad \text{..............(x) bis},$$

and the remaining values of $\mathbf{R}_k'$, $k = 2, \dots 13$, by the approximate method of Problem I, p. 41. Perhaps the chief source of error in the method is variation in S_k, which has been assumed constant; in this experiment the range of S_k is only 1·8 % compared with 3·6 % for the Trisections and 2·5 % for the Bisections, and the results which are contained in the 6th row of Table XIV may be regarded, therefore, with reasonable confidence. As before, for the higher values of k, $\mathbf{R}_k'$ may be

* Eleven definite interruptions in the ordinary routine of counting, due to a mistap on the key or a miscount of the 10 seconds, were recorded at the time of observation, but only three of these resulted in breaks of judgment $\geqslant \cdot 07$, the limiting value taken in the above investigation.

TABLE X

f C bined S Co g Seco

D_k			198				11						
S_k	·05 ±·0C087	±·00087	·057·1 ±·00087	±·0008	·5789 ±·00087	±	·837 ·088			±·0008	±·0008		
$\mathbf{R}_k$	+·8317 ±·0C66	·7955 ±·0078	·7644 ±·0089	·7456 ±·0095	·7186 ±·0100	±	± ·29 ·11	±	±	·6619 ±·0120	·6586 ±·0121		
S_k'*)... ...	·0455	·03431											
S_k')... ...	·02426	—	—	—	--		·			—	—		
$\mathbf{R}_k'$	+·5200 ±·0156	+·4168 ±·0176	+ ·81 ± 90	+·2746 ±·0197	+·2263 ±·0202	+ ·818 ± ·206	+ ·248 ±·0210	+ ±	+·0636 ±·0212	+·0374 ±·0213	+·0282 ±·0213	−·0348 ±·0213	−·0598 ±·0212
$\mathbf{R}_k'$ from equation (lvii)	+·528	+·418	+	+·262	+·20	+ ·64	+·130	+ ·03	·082	+·065	+·051	+·040	+·032
Difference ...	−·008	−·001	−	+·013		+ ·018	−·005	+ ·10	·018		−·023		
${}_1R_k$	−·3925	−·0150	−	−·0054		·0130	−·0468	·036	·0239		+·0560	−·0396	

obtained $= \sqrt{\frac{1}{m}\Sigma\sigma_t^2}$ he valu σ_k bein obtained

obtained a of the correl ion tab ts p le error is

a little too low, and as a test of the amount of cumulative error which may be affecting $\mathbf{R}_{13}'$, I have worked out this constant directly from the relations

$$\mathbf{R}_{11} = \frac{\mathbf{R}_{13}' S_1' S_{14}' + \frac{1}{m}\sum_m (D_1 - d_1)(D_{14} - d_{14})}{S_1 \times S_{14}},$$

$$S_1^2 = S_1'^2 + \frac{1}{m}\sum_m (D_1 - d_1)^2, \quad S_{14}^2 = S_{14}'^2 + \frac{1}{m}\sum_m (D_{14} - d_{14})^2,$$

EXPERIMENT C 10 SECOND COUNTING CORRELATION INTERVAL DIAGRAM

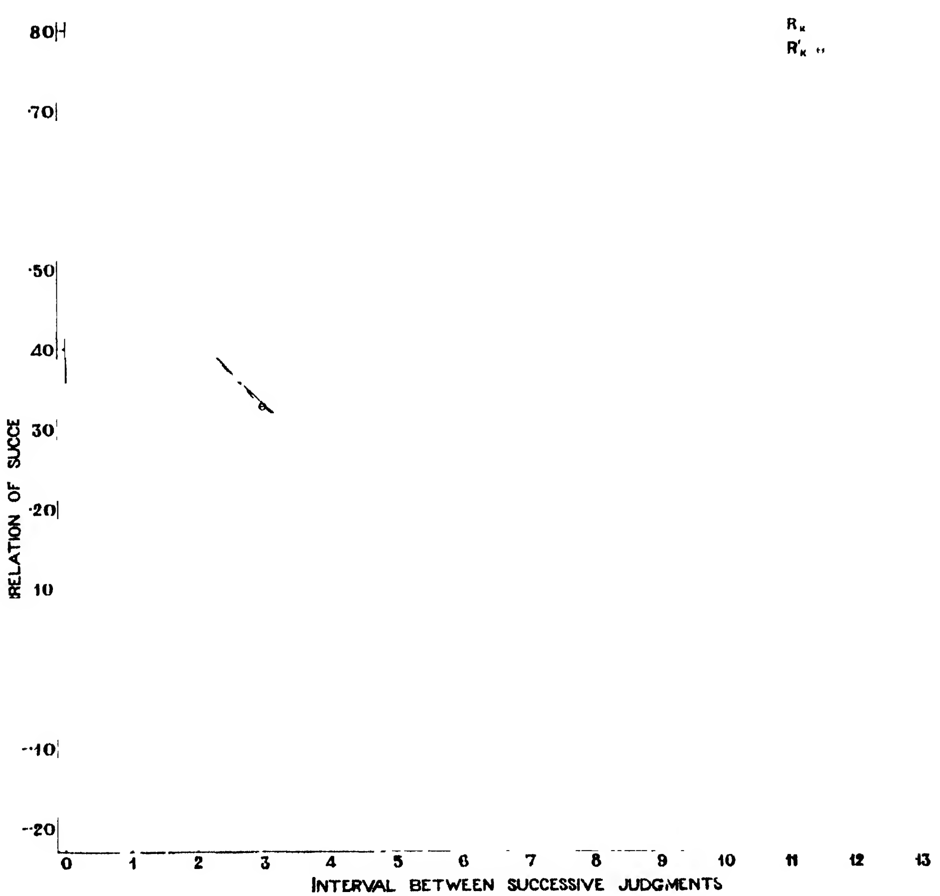


Fig. 17.

with the following results:

$$\mathbf{R}_{13}' = -\cdot 0124 \pm \cdot 0213, \quad L_{13} = +\cdot 632^*,$$
$$S_1' = \cdot 03426, \qquad S_{14}' = \cdot 03444.$$

$\mathbf{R}_{13}'$ and presumably $\mathbf{R}_{12}'$ are not therefore significantly negative, and it seems probable that $\mathbf{R}_k'$ tends to zero as k increases, without oscillating about that value. The points $(k, \mathbf{R}_k)$ and $(k, \mathbf{R}_k')$ are plotted in Figure 17; the theoretical curve drawn in the diagram will be referred to in Section XI.

* L_k, or the limit to which $\mathbf{R}_k$ approaches as $\mathbf{R}_k'$ tends to zero is discussed on p. 84.

IX. Experiment *D*. Estimation of 10 seconds. Reduction of Observations.

(*a*) *The Individual Series.*

In Table XV are given the values of d (the mean of the 63 observations of a series, *not* those of Group 1 only), and of σ_1 and ρ_1 for the individual series; the low values of ρ_1 will be noted at once, and also the high values of σ_1 compared with those in the Counting Experiment. In Figure 19 below the means have been plotted to order of series, and if

x is the mean in the factor e/p,

y the order of series,

z the time in hours and fractions of an hour between 10 a.m. on December 7th, and the commencement of series,

TABLE XV.

Constants of Individual Series (Estimate of Seconds)

Series	d	σ_1	ρ_1	Time of Start	Date (1920)
I	1·151	·1217	+·1518 ± ·0932	10.45 a.m.	7th December
II	1·111	·1254	+·2332 ± ·0902	11.30 a.m.	
III	1·109	·1330	−·0249 ± ·0953	12.10 p.m.	
IV	1·052	·1393	+·1803 ± ·0923	2.0 p.m.	
V	·973	·1292	+·2632 ± ·0888	3.0 p.m.	
VI	1·119	·1349	+·1300 ± ·0938	10.15 a.m.	8th December
VII	1·011	·1312	+·3673 ± ·0825	11.0 a.m.	
VIII	1·073	·1318	+·1631 ± ·0929	2.0 p.m.	
IX	1·003	·1108	+·1976 ± ·0917	2.30 p.m.	
X	1·089	·0989	+·0380 ± ·0953	3.15 p.m.	
XI	1·204	·1519	+·3405 ± ·0843	10.0 a.m.	9th December
XII	1·204	·1467	+·1415 ± ·0935	11.0 a.m	
XIII	1·091	·1166	+·3241 ± ·0854	12.0 midday	
XIV	1·036	·1059	+·0566 ± ·0951	2.0 p.m.	
XV	1·132	·1884	+·4814 ± ·0733	3.15 p.m.	
XVI	1·170	·1500	+·1036 ± ·0944	10.0 a.m.	10th December
XVII	1·421	·1520	−·0834 ± ·0947	11.0 a.m.	
XVIII	1·300	·1591	+·2314 ± ·0903	12.0 midday	
XIX	1·243	·1708	+·2260 ± ·0905	2.0 p.m.	
XX	1·170	·1833	+·1659 ± ·0928	2.45 p.m.	

Correlation between ρ_1 and σ_1, $r_{\sigma_1\rho_1} = +·176 \pm ·146$ (calculated from correlation of ranks).

we have for the regression lines

$$x - 1·1333 = +·01018\,(y - 10·5) \quad\text{(xxxix)},$$

$$x - 1·1333 = +·002493\,(z - 38·62) \quad\text{(xl)}.$$

The coefficients of correlation are

$$r_{xz} = +·638 \pm ·089, \quad r_{xy} = +·562 \pm ·103, \quad r_{yz} = +·983 \pm ·005,$$

giving partial correlation coefficients

$$r_{xz.y} = + \cdot 570 \pm \cdot 102, \quad r_{xy.z} = - \cdot 470 \pm \cdot 118.$$

These latter coefficients suggest that the secular change for observations spread over a number of days will be a lengthening in estimation, but that, if a number of series are done in rapid succession, the tendency will be for a shortening; in fact we should expect the sessional change to be in the opposite direction to the secular, as for the Bisections.

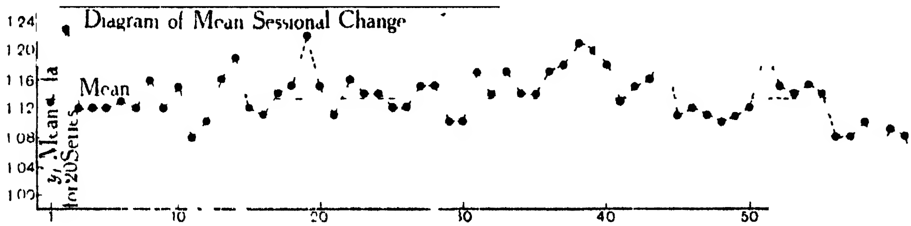


Fig. 18. 10 Second Estimation. *t*, Order of Observation in Series.

The values of y_t have been plotted in Figure 18; the best fitting line has not been calculated, but it would certainly correspond very closely with the mean, $y = 1 \cdot 1333$. There is in fact apparently no mean sessional change, though the drop in the last eight values of y_t may be significant, and a mark of the tendency suggested by the negative value of $r_{xy.z}$.

In Figure 19 the centres of the small circles represent the positions of the means of the 63 observations of each series; these points have been fitted with the cubic

$$x = 1 \cdot 093971 + \cdot 022116\,(y - 10 \cdot 5) + \cdot 001174\,(y - 10 \cdot 5)^2 - \cdot 0002002\,(y - 10 \cdot 5)^3 \ldots \text{(xli)},$$

which is the middle of the three curves. There is evidence of a slight secular change, the length of the estimation *increasing* towards the end of the experiment. If however it is remembered that the 20 series were carried out in 4 days, it will be seen that there is in general a *decrease* in estimation in the course of the 5 series done in any one day. It is this daily drop that the coefficient $r_{xy.z} (= - \cdot 470)$ is picking out. Now in addition to the secular change in personal equation, the figures in Table XV suggest that there is also a secular change in standard deviation. The vertical lines on each side of the series-means in Figure 19 equal in length the corresponding standard deviations, or σ_1's. These values of σ_1 have been fitted with the cubic

$$x' = \cdot 129006 + \cdot 001072\,(y - 10 \cdot 5) + \cdot 000302\,(y - 10 \cdot 5)^2 + \cdot 0000214\,(y - 10 \cdot 5)^3 \ldots \text{(xlii)},$$

and the other two curves in the diagram have ordinates equal to $x + x'$ and $x - x'$, so that the distance between the central curve and either of the outer curves, gives the smoothed value of the standard deviation at the point. The diagram provides a generalised representation of a secular change in personal equation and standard deviation.

The factor for a true 10 second interval would be $\frac{10 \cdot 0}{10 \cdot 2} = \cdot 98$, and was most nearly approached by the means of Series V, VII and IX, while in the case of XVII the mean estimation nearly reached the high value of 15 seconds.

DISTRIBUTION OF PERSONAL EQUATION IN ESTIMATING SECONDS

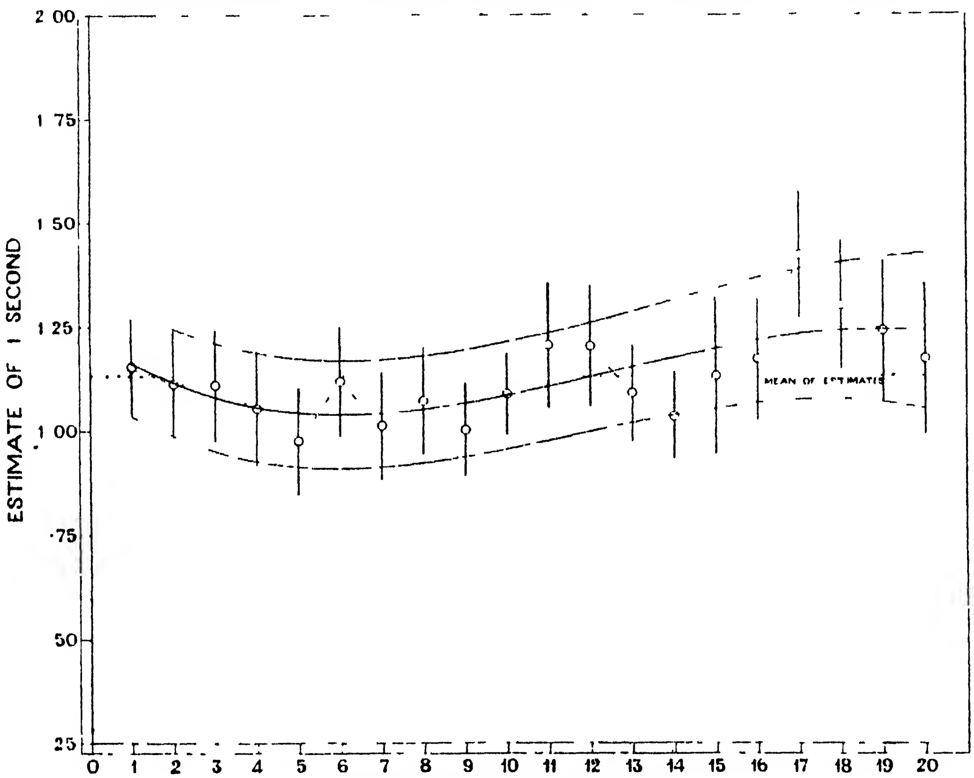


Fig. 19.

(*b*) *The Combination of the Series.*

In combining the twenty series, D_k, S_k and $\mathbf{R}_k$ were calculated from the thirteen correlation tables of the observations of the combined series. Using the correlations and standard deviations of the separate series, $\mathbf{R}_1'$ is obtained from

$$\mathbf{R}_1' = \frac{\sum\limits_m (\rho_1 \sigma_1 \sigma_2)}{\sqrt{\sum\limits_m (\sigma_1^2) \sum\limits_m (\sigma_2^2)}} = + \cdot 19841 \pm \cdot 02049 \text{ .... (x) bis,}$$

and $$S_1' = \cdot 14101, \quad S_2' = \cdot 14056.$$

Then using this value of $\mathbf{R}_1'$, and the first difference correlation equations (Problem 1, p. 41), $\mathbf{R}_k'$ can be calculated for $k = 2, \dots 12$. The values of these quantities are given in the Table XVI below.

The fall in $\mathbf{R}_k$ is small, and although there is considerable irregular variation from $\mathbf{R}_7$ onwards, it appears that $\mathbf{R}_k$ will not vanish as k increases, but approach a constant value in the neighbourhood of + ·35. This can be tested; we have from the equations (vi) to (x)

$$\mathbf{R}_k = \frac{\mathbf{R}_k' S_1' S'_{k+1} + \frac{1}{m}\underset{m}{\Sigma}(D_1 - d_1)(D_{k+1} - d_{k+1})}{\sqrt{\left\{S_1'^2 + \frac{1}{m}\underset{m}{\Sigma}(D_1 - d_1)^2\right\}\left\{S'^2_{k+1} + \frac{1}{m}\underset{m}{\Sigma}(D_{k+1} - d_{k+1})^2\right\}}} \quad \text{......(xliii)},$$

TABLE XVI.

Constants of Combined Series (Estimating Seconds).

	$k=1$	2	3	4	5	6	7	8	9	10	11	12	13
D_k	1·1421	1·1440	1·1415	1·1413	1·1421	1·1423	1·1416	1·1402	1·1395	1·1391	1·1382	1·1378	1·135
S_k	·1749 ± ·0026	·1749 ± ·0026	·1759 ± ·0027	·1761 ± ·0027	·1764 ± ·0027	·1760 ± ·0027	·1754 ± ·0026	·1757 ± ·0026	·1763 ± ·0027	·1756 ± ·0026	·1760 ± ·0027	·1767 ± ·0027	·177 ± ·002
$\mathbf{R}_k$	+ ·4825 ± ·0164	·4269 ± ·0174	·3965 ± ·0180	·3913 ± ·0181	·3761 ± ·0183	·3755 ± ·0183	·3983 ± ·0180	·4045 ± ·0178	·3488 ± ·0187	·3691 ± ·0184	·3521 ± ·0187	·3831 ± ·0182	
S_k'	·14101 ± ·00213	·14056 ± ·00212											
$\mathbf{R}_k'$	+ ·19841 ± ·02049	+ ·1123 ± ·0211	+ ·0652 ± ·0212	+ ·0570 ± ·0212	+ ·0338 ± ·0213	+ ·0332 ± ·0213	+ ·0693 ± ·0212	+ ·0798 ± ·0212	− ·0056 ± ·0213	+ ·0267 ± ·0213	+ ·0017 ± ·0213	+ ·0501 ± ·0213	
${}_1R_k$	− ·1463	·0243	·0243	+ ·0094	− ·0135	·0229	+ ·0160	+ ·0598	·0734	+ ·0357	− ·0158		

$$S_\delta = S_1' \sqrt{2(1 - \mathbf{R}_1')} = ·1785.$$

and as the sessional change for the series is very small, we may make the approximation

$$\underset{m}{\Sigma}(D_1 - d_1)(D_{k+1} - d_{k+1}) = \underset{m}{\Sigma}(D_1 - d_1)^2 = \underset{m}{\Sigma}(D_{k+1} - d_{k+1})^2 \text{ for all values of } k,$$

and in view of the constancy of S_k,

$$S_1' = S'_{k+1} \text{ for all values of } k.$$

Then on the assumption that there is no significant periodic variation in the observations,

$$\mathbf{R}_k' \rightarrow 0 \text{ as } k \text{ increases},$$

and from (xliii)
$$\mathbf{R}_k \rightarrow \frac{\frac{1}{m}\underset{m}{\Sigma}(D_1 - d_1)^2}{S_1'^2 + \frac{1}{m}\underset{m}{\Sigma}(D_1 - d_1)^2} = + ·354.$$

The correlations $\mathbf{R}_k'$ become rapidly insignificant; the values tabulated are of course subject to the errors of the method of approximation, but as in the case of the 10 second Counting Experiment, these should not be large owing to the constancy of S_k*. The points $(k, \mathbf{R}_k)$ and $(k, \mathbf{R}_k')$ are plotted in Figure 20; the two curves there drawn will be referred to in Section XI below.

* The difference between S_1 and S_{11} is one of 1·4 % only.

(c) *Comparison of Experiments C and D.*

It has been found that in both the Counting and the Estimating Experiments there is evidence of a secular change in personal equation, and that in both cases the tendency is for the estimates to depart further from the true value of 10 seconds in the later series; in the Counting Seconds there is a decrease, in the Estimating Seconds an increase in length of estimate. There is also very little evidence of regular sessional change in either experiment.

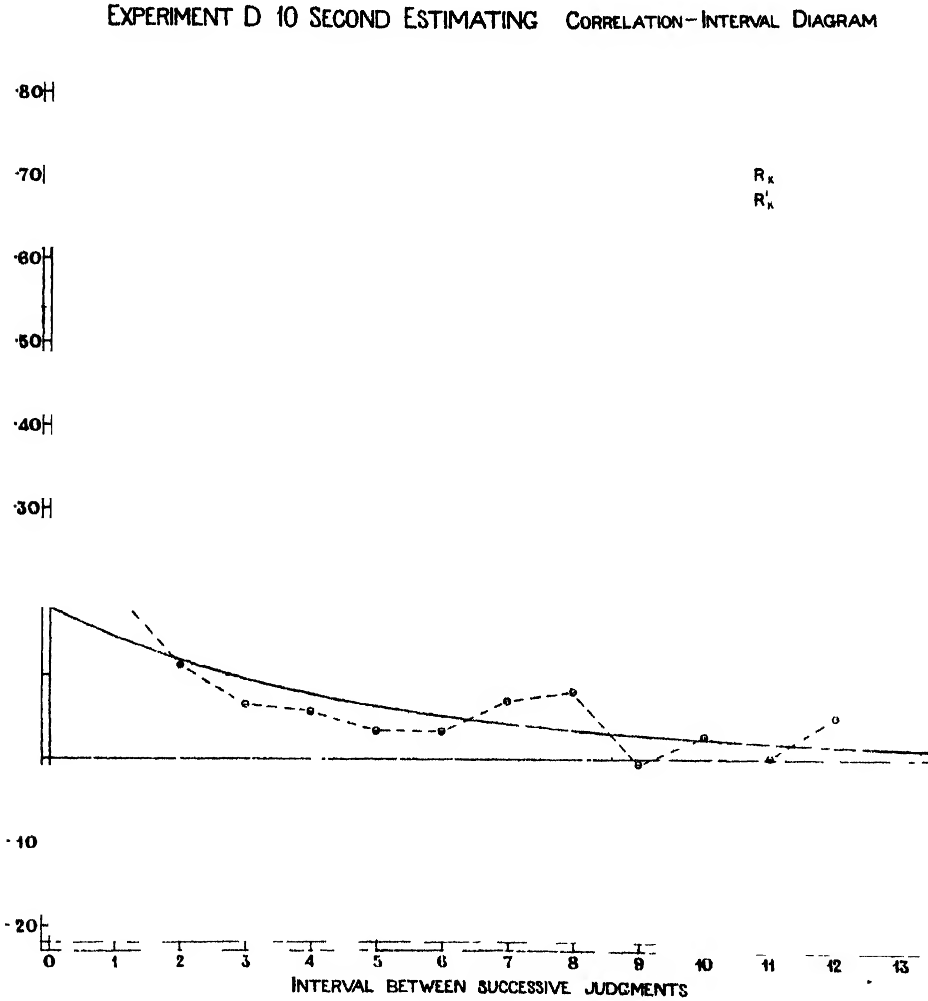


Fig. 20.

Beyond this the similarity ceases; it is only necessary to compare the values of the chief constants (defined on p. 36),

	S_1'	S_δ	$\mathbf{R}_1'$
Counting	·0343	·0338	+ ·5200 ± ·0156
Estimating	·1410	·1785	+ ·1984 ± ·0205

The variations in judgment in the Estimating Experiment are very large compared with those in the other, and at the same time there is low correlation between successive judgments, so that the observations will be found to be scattered far more nearly in accordance with the Normal Error Law than in the three preceding experiments. In the case of the Counting Experiment, the skew distribution of the 1000 observations has already been referred to. But for one or two exceptions (as III and XIX) the individual series in the Counting conform more closely to a general type than in the Trisections or Bisections, and this results in the very smooth values of the constants $\mathbf{R}_k$ and $\mathbf{R}_k'$.

X. EXPERIMENT *E*. PLATE MEASUREMENTS WITH ZEISS COMPARATOR.

The values of ρ_1 only have been calculated; these, with σ_1 and a brief description of the nature of the marking measured, are given in the Table XVII; σ_1 is in millimetres. Series I–VIII involved settings of both slide and micrometer, IX of micrometer only.

No great weight can be attached to the result of one series of 50 readings on a marking, but it is justifiable to draw certain conclusions from the results of the eight series. In the first place, there appears to be a significant correlation between the successive measures of the edge of a band (I and II), but in measuring the centres, i.e. in bisecting a bright maximum with the cross wire, there is on the whole no correlation. This perhaps might be expected; the edges of bands or maxima in photographic spectra are not quite sharply cut, so that some uncertainty must exist in the observer's mind as to where the real edge should be taken to be; his opinion on this point may vary throughout the course of the sitting, and consequently correlation will be found between the successive readings. On the other

TABLE XVII.

Series	ρ_1	σ_1	Description of marking	
I	$+ \cdot 384 \pm \cdot 081$	·0016	Sharp edge of bright band	Edge
II	$+ \cdot 467 \pm \cdot 075$	·0015	Slightly vaguer edge than I	Edge
III	$+ \cdot 117 \pm \cdot 094$	·0007	Clear and narrow maximum	Centre
IV	$+ \cdot 090 \pm \cdot 095$	·0012	,, ,, ,,	Centre
V	$+ \cdot 021 \pm \cdot 095$	·0016	,, ,, ,,	Centre
VI	$- \cdot 001 \pm \cdot 095$	·0019	Broad and obscure ,,	Centre
VII	$- \cdot 050 \pm \cdot 095$	·0022	,, ,, soft ,,	Centre
VIII	$+ \cdot 227 \pm \cdot 091$	·0011	,, ,, ,, ,,	Centre
IX	$+ \cdot 288 \pm \cdot 087$	·0004	Micrometer screw settings only	

hand, in the bisection of a narrow maximum, there will be little doubt as to the position of the centre; the real estimate of the observer will vary but slightly, and the variations in the reading will be due mainly to failure in breaking off the push

or pull of the slide at the right moment. It is possible that unconscious "over-pulls" or "under-pulls" may go in runs together, but the measures seem to show that this is not the case, and that the correlation of successive judgments is due rather to correlated changes of mental estimate than to those of a more physical character. If it were more difficult to bisect a maximum, if there were greater opportunity for variation, it is probable that there would be a correlation of successive judgments, and this is perhaps illustrated by the case of Series VIII, which has the largest standard deviation (·0041) and also a correlation ($\rho_1 = + ·227 \pm ·091$) possibly significant.

The result of IX suggests that there is a correlation between successive settings of the micrometer wires in the second eyepiece; this correlation would of course enter into the results of I – VIII, but the standard deviation of IX (·0004) is so small that the effect will be insignificant where the variations in slide settings are large.

As a matter of practical application these results serve to emphasise the importance of the routine of measurement usually adopted; if, for example, it is proposed to take four readings of each of a number of markings on a plate, the four readings should not be made in succession, but all the markings should be measured once, and then perhaps a short interval taken before the second measuring is made, and so on. This method should eliminate the error in the mean of several measurements of a marking, which may arise from a correlation of successive judgments, as well as errors due to change in temperature of instrument or plate, etc.

XI. Analysis of the Correlation between Successive Judgments.

(*a*) *The Theory of correlated Estimates and accidental Errors.*

It has been seen that in the case of the Bisection and Timing Experiments when the secular term was removed the coefficients of correlation of the successive judgments, or the constants $\mathbf{R}_k'$, diminished to approximately zero values as k, the interval between the judgments correlated, was increased. In the Trisection Experiment, owing to the marked sessional change which was repeated in practically all the series, $\mathbf{R}_k'$ appeared to approach a value of $+ ·16$ and not zero as k was increased; the sessional change in this case appeared to be of parabolic rather than linear form, and it seemed possible that if the ordinates of the "best" fitting parabola of each series were removed from the observations, the coefficients of correlation of the residuals, or the $\mathbf{R}_k''$'s, would tend to zero as k increased, as in the case of the other three experiments in which there was no large sessional change. The points representing the values of $\mathbf{R}_k'$ which have been plotted in Figures 8, 12, 17 and 20 appear on the whole to lie so nearly on a smooth curve, that it is of no little interest to inquire whether we can obtain equations to such curves based on some definite theory of the physiological factors underlying the variations in an observer's judgment.

In the first place we have seen that neither a secular change in personal equation—the variation in series means—nor a simple sessional change such as that represented by the straight line or by a second order parabola considered in the Trisection Experiment, will account for the whole of the correlation of successive judgments. We must therefore conclude that quite apart from the large scale variations in judgment which are due to the more gradual changes of state in the observer resulting, perhaps, from experience or fatigue, there is a definite relationship between the small scale variations in judgment; if judgment y_t is greater than the average of the five or six preceding judgments, then we shall on the whole expect that y_{t+1}, the next judgment, will also be greater. I propose therefore to consider what results will follow from the assumption that y_t has a correlation r with y_{t-1} and y_{t+1}, but that for y_{t+1} or y_{t-1} constant it has no partial correlation with y_{t-2} and y_{t+2} or judgments at greater intervals. In other words we will suppose that the observer's estimation at any moment is only influenced by the preceding estimation, and only through this, and not directly, by the earlier estimations.

Let us take the successive judgments $y_t, y_{t+1}, y_{t+2} \ldots y_{t+k} \ldots$ and suppose that the total correlation between y_t and y_{t+k} is ρ_k, where $k = 1, 2, 3, \ldots$, and $\rho_1 = r$. If there is no partial correlation between y_t and y_{t+2}, y_{t+1} being constant we must have

$$\rho_2 - \rho_1^2 = 0 \text{ or } \rho_2 = r^2.$$

In the same way if there is no partial correlation between y_t and y_{t+3} when y_{t+1} (or y_{t+2}) is constant,

$$\rho_3 - \rho_1\rho_2 = 0 \text{ or } \rho_3 = r^3,$$

and in general we find that

$$\rho_k = r^k \qquad \text{(xliv)}.$$

In reaching this simple result there is a point however that has been overlooked; it has been assumed that there is some physiological or psychological significance in the correlation of an estimate of a quantity and in the preceding estimate, but it must be remembered that the value which the observer records may not be exactly that which he wished to record, or in other words he may be unable to record his true estimate. Thus in bisecting a line it is likely that the pencil point will not strike the paper exactly at the spot intended, or in counting 10 seconds the tapping of the key may not be exactly synchronised with the beginning or end of the count, and there may be many other little external influences of which the observer is unaware, which will all combine to form what may be termed an accidental error superimposed upon the true correlated estimation. Let us examine how the relation (xliv) will be modified by introducing the idea of these accidental and uncorrelated errors, we must suppose that the observer's recorded judgment y_t is made up of two parts, α_t his actual estimate at the moment of record and β_t some complex of accidental errors affecting his record. Then

$$y_t = \alpha_t + \beta_t \qquad \text{(xlv)}$$

Now if we assume that the accidental errors β_t are as like to be positive as negative, and that they will not be correlated in any manner among themselves

nor with the fundamental part of the judgment α_t, we shall have the following approximate relations

$$\begin{aligned}
&\sum_1^N \beta_{t+k} = 0 \text{ for } k = 1, 2, 3, \ldots, && \text{where } N \text{ is large compared with } k \\
&\sum_1^N \beta_t \beta_{t+k} = 0 && \qquad\qquad \ldots\ldots\text{(xlvi)}. \\
&\sum_1^N \beta_{t+k}\alpha_{t+k'} = 0 \quad ,, \quad ,, && \text{where } k \text{ and } k' \text{ take any of the values } 1, 2, 3, \ldots \text{etc.}
\end{aligned}$$

But the correlation between successive values of the y's at intervals of k is

$$\rho_k = \frac{\sum_{t=1}^N (\alpha_t + \beta_t)(\alpha_{t+k} + \beta_{t+k}) - N \sum_{t=1}^N \frac{\alpha_t + \beta_t}{N} \sum_{t=1}^N \frac{\alpha_{t+k} + \beta_{t+k}}{N}}{\sqrt{\left\{\sum_{t=1}^N (\alpha_t + \beta_t)^2 - N\left(\sum_{t=1}^N \frac{\alpha_t + \beta_t}{N}\right)^2\right\}\left\{\sum_{t=1}^N (\alpha_{t+k} + \beta_{t+k})^2 - N\left(\sum_{t=1}^N \frac{\alpha_{t+k} + \beta_{t+k}}{N}\right)^2\right\}}}$$

$$= \frac{\sum_{t=1}^N (\alpha_t \alpha_{t+k}) - N \sum_{t=1}^N \frac{\alpha_t}{N} \sum_{t=1}^N \frac{\alpha_{t+k}}{N}}{\sqrt{\left\{\sum_{t=1}^N \alpha_t^2 - N\left(\sum_{t=1}^N \frac{\alpha_t}{N}\right)^2 + \sum_{t=1}^N \beta_t^2\right\}\left\{\sum_{t=1}^N \alpha_{t+k}^2 - N\left(\sum_{t=1}^N \frac{\alpha_{t+k}}{N}\right)^2 + \sum_{t=1}^N \beta_{t+k}^2\right\}}}$$

in view of the relations (xlvi)

$$= \frac{[\alpha_t \alpha_{t+k}]}{\sqrt{(\bar{\alpha}_1^2 + \bar{\beta}_1^2)(\bar{\alpha}_{k+1}^2 + \bar{\beta}_{k+1}^2)}},$$

where $[\alpha_t \alpha_{t+k}]$ is the first order product moment coefficient referred to mean of the successive α's at intervals of k,

and $\sqrt{\bar{\alpha}_k^2}$ is the standard deviation of $\alpha_k, \alpha_{k+1}, \ldots \alpha_{k+N}$,

$\sqrt{\bar{\beta}_k^2}$,, ,, ,, $\beta_k, \beta_{k+1}, \ldots \beta_{k+N}$,

and $\sqrt{\bar{\alpha}_k^2 + \bar{\beta}_k^2}$,, ,, ,, $y_k, y_{k+1}, \ldots y_{k+N}$.

Now unless there is a steady sessional change in the α's, we may assume that for large values of N

$$\bar{\alpha}_1^2 = \bar{\alpha}_2^2 = \ldots = \bar{\alpha}_k^2 = \ldots = \alpha^2, \text{ say,}$$

and similarly unless the accidental errors are steadily increasing or decreasing in magnitude

$$\bar{\beta}_1^2 = \bar{\beta}_2^2 = \ldots = \bar{\beta}^2,$$

and we have $$\rho_k = \frac{[\alpha_t \alpha_{t+k}]}{\bar{\alpha}^2 + \bar{\beta}^2} = \frac{\alpha^2}{\bar{\alpha}^2 + \bar{\beta}^2} \cdot \frac{[\alpha_t \alpha_{t+k}]}{\alpha^2} = \frac{\alpha^2}{\bar{\alpha}^2 + \bar{\beta}^2} \cdot r_{\alpha_t, \alpha_{t+k}}.$$

But on the assumption made above of zero partial correlation between two estimates which are not consecutive, we have found that $r_{\alpha_t, \alpha_{t+k}}$, the correlation between the observer's real estimates at intervals of k, can be expressed in the form r^k, and therefore

$$\rho_k = \frac{\bar{\alpha}^2}{\alpha^2 + \bar{\beta}^2} r^k = q r^k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xlvii)},$$

where q is a constant not depending on the interval k. With this expression for the correlation we shall of course find an apparent partial correlation between the judgments at intervals greater than one; for example the partial correlation between y_t and y_{t+2}, y_{t+1} being constant, is $\frac{q(1-q)r^2}{1-q^2r^2}$, and does not vanish unless $q=1$. According to the theory suggested this is however a spurious correlation due solely to the presence of the accidental errors.

The next problem is to inquire how far a relation of the type of (xlvii) will fit the correlation coefficients which have been calculated for the Experiments A, B, C and D. In the first place, in order to get as smooth values for the coefficients as possible we must combine the 20 series, which we may do if we remove the secular change as represented by the variation in the series means; this step is clearly necessary for we are considering the relationship between judgments made in close proximity and are not concerned for the moment with the variation in personal equation from day to day. We must therefore deal with the coefficients of correlation $\mathbf{R}_k'$ and endeavour to fit a curve $z=qr^x$ through the points $x=k$, $z=\mathbf{R}_k'$. I will consider the different experiments in turn.

(*b*) *Application of Theory to results of Experiments.*

Experiment A.

The curve represented by $z=qr^x$ is asymptotic to the x axis (as $r<1$), so that if it is to fit the points (k, $\mathbf{R}_k'$) it is necessary that $\mathbf{R}_k'$ should tend to zero as k increases. But the values of $\mathbf{R}_k'$ given in Table V, p. 58, appear to tend as k increases, to a limiting value between $+\cdot16$ and $+\cdot15$ rather than to zero. I think that this results from the marked sessional changes which have been represented in mean form by a second order parabola (see Equation (xxx) and Figure 4), and that if there is a physiological significance in the distinction between the sessional change and the residual variations of the observations when freed from this change, it will be of interest to find out how the coefficients of correlation of these successive residuals—what have been termed the $\mathbf{R}_k''$'s—fall off as the interval or k is increased. Should it be found that the $\mathbf{R}_k''$'s follow the law

$$\mathbf{R}_k''=qr^k,$$

the argument in favour of distinguishing the sessional change from the residual variations will be strengthened.

It was found that the values of $\mathbf{R}_k'$ given in Table V could be fitted closely by a curve of the form

$$z=p+qr^k \quad \text{(xlviii)},$$

where p, q and r are constants.

A rough trial gave the following approximate values:

$$p_0=\cdot157, \quad q_0=\cdot69, \quad r_0=\cdot73.$$

Now if

$$\begin{aligned} z &= f(p,q,r) \\ &= f(p_0,q_0,r_0)+\delta p\frac{\partial f_0}{\partial p_0}+\delta q\frac{\partial f_0}{\partial q_0}+\delta r\frac{\partial f_0}{\partial r_0} \quad \text{to first order,} \\ &= p_0+q_0r_0^k+\delta p+r_0^k\delta q+kq_0r_0^{k-1}\delta r, \end{aligned}$$

we have as equations of condition for a least square solution

$$\delta p + r_0^k \delta q + k q_0 r_0^{k-1} \delta r = \mathbf{R}_k' - p_0 - q_0 r_0^k, \text{ for } k = 1, 2, \ldots 13.$$

Using the values of p_0, q_0 and r_0 given above, the corrections δp, δq and δr were calculated and gave finally as the best fitting numerical equation,

$$\mathbf{R}_k' = \cdot 1524 + \cdot 6817\, (\cdot 7105)^k \text{ ....(xlix).}$$

TABLE XVIII.

Values of the $\mathbf{R}_k$'s for Trisection Experiments.

1	2 $\mathbf{R}_k'$ (direct calculation)	3 $\mathbf{R}_k'$ (from equation (xlix))	4 Difference Col. 2—Col. 3	5 Probable Error of $\mathbf{R}_k'$	6 Values obtained from (lii) on assumption of constancy of G_k S_k''	7 $\mathbf{R}_k''$	8 $\mathbf{R}_k''$ (from equation (lvi))
0		+ ·834	-				+ ·804
1	+ 625	·637	- ·012	± ·013	·0773	- ·550	·571
2	·523	·497	+ ·026	± ·016	·0776	·431	·406
3	·388	·397	·009	± ·018	·0778	·268	·288
4	·315	·326	− ·011	± ·019	·0781	·183	·205
5	·281	·276	+ ·005	± ·020	·0778	·142	·146
6	·232	·240	·008	± ·020	·0782	·084	·103
	·222	·215	+ ·007	± ·020	·0782	·071	·074
8	·191	·197	− ·006	± ·021	·0783	·035	·052
9	·165	·184	·019	± ·021	·0787	·006	·037
10	·183	·175	+ ·008	± ·021	·0802	·031	·026
11	·168	·168	·000	± ·021	·0823	·017	·019
12	·172	·164	+ ·008	± ·021	·0834	·023	·013
13	+ ·160	+ ·160	·000	± ·021	·0840	+ ·009	+ ·009
14					·0840		

In the second column of Table XVIII are given the values of $\mathbf{R}_k'$ taken from Table V and in the fifth column their probable errors; the values of $\mathbf{R}_k'$ given by equation (xlix) are in the third column, and in the fourth are the differences col. 2 − col. 3. It will be seen that the fit is a good one, the difference being only greater than the probable error in the case of $\mathbf{R}_2'$. The points (k, $\mathbf{R}_k'$) and the curve of (xlix) are shown in Figure 8 (p. 64).

The problem before us is therefore this; can we explain the constant p in equation (xlviii) in terms of the sessional changes? We have seen that the mean sessional change for the 20 series can be represented by a parabola of the second order, but we must allow for a different change in each series. Let us suppose that

$$y = f_p(t)$$

will represent the sessional change in the pth Series after the secular term represented by the series mean has been removed, so that instead of equation (xlv) of p. 89, we have

$$y_t' = f_p(t) + \alpha_t + \beta_t = f_p(t) + Y_t \text{ ....(l),}$$

where $Y_t = \alpha_t + \beta_t$.

Then if $\sum_m$ indicates summation for the m (or 20) series, $n = 50$, the number of observations in each group of a series, and k takes any of the group numbers 1, 2, ... 14, since $y = f_p(t)$ will be the "best" fitting curve of its type $\sum_{t=1}^{n} Y_{t+k-1} = 0$ approximately, and on combining the m series

$$\sum_m \sum_{t=1}^{n} (Y_{t+k-1}) = 0.$$

Again we have no reason to suppose that there will be any correlation between the sessional term $f_p(t)$ and the residual Y_t, so that

$$\sum_m \sum_{t=1}^{n} \{Y_{t+k-1} f_p(t+k'-1)\} = 0,$$

for all values of k and k' between 1 and 14.

As y_t' is freed from the secular term, using the relations above we have that

$$\mathbf{R}_k' = \frac{\sum_m \sum_{t=1}^{n} \{(f_p(t) + Y_t)(f_p(t+k) + Y_{t+k})\} - mn \sum_m \sum_{t=1}^{n} \left\{\frac{f_p(t)}{mn}\right\} \sum_m \sum_{t=1}^{n} \left\{\frac{f_p(t+k)}{mn}\right\}}{\sqrt{\left[\sum_m \sum_{t=1}^{n} (f_p(t) + Y_t)^2 - mn \left\{\sum_m \sum_{t=1}^{n} \frac{f_p(t)}{mn}\right\}^2\right] \left[\sum_m \sum_{t=1}^{n} (f_p(t+k) + Y_{t+k})^2 - mn \left\{\sum_m \sum_{t=1}^{n} \frac{f_p(t+k)}{mn}\right\}^2\right]}} \quad \text{(li)},$$

$$= \frac{\mathbf{R}_k'' S_1'' S_{k+1}'' + F_k}{\sqrt{(S_1''^2 + G_1^2)(S_{k+1}''^2 + G_{k+1}^2)}},$$

where $\mathbf{R}_k''$ is the coefficient of correlation between Y_t and Y_{t+k}, S_1'' and S_{k+1}'' are the standard deviations of the Y's of Groups 1 and $k+1$ (see (xi) and (xii) on page 35), and

$$\left.\begin{aligned} F_k &= \frac{1}{mn} \sum_m \sum_{t=1}^{n} f_p(t) f_p(t+k) - \left\{\sum_m \sum_{t=1}^{n} \frac{f_p(t)}{mn}\right\} \left\{\sum_m \sum_{t=1}^{n} \frac{f_p(t+k)}{mn}\right\} \\ G_k^2 &= \frac{1}{mn} \sum_m \sum_{t=1}^{n} \{f_p(t+k-1)\}^2 - \left\{\sum_m \sum_{t=1}^{n} \frac{f_p(t+k-1)}{mn}\right\}^2 \end{aligned}\right\}.$$

It will be seen that G_k is the standard deviation of the ordinates of the curves representing the sessional changes, $y = f_p(t)$, which correspond to the observations in the kth groups, while $\frac{F_k}{G_1 G_{k+1}}$ is the correlation of these successive ordinates at intervals of k. If the sessional changes were linear this correlation would be unity, and a little consideration will show that if the sessional change in each series can be represented by a curve of gradual bend, the correlation will not be far from this value. For example in the case of the parabola (Equation (xxx), p. 47) which was fitted to the *mean* sessional change and is drawn in Figure 4, it is found that

$$\frac{F_{13}}{G_1 G_{14}} = + \cdot 994.$$

We shall therefore make no great error in assuming that

$$F_k = G_1 G_{k+1},$$

and it follows that the relation for $\mathbf{R}_k'$ can be expressed in the form

$$\mathbf{R}_k' = \frac{1}{\sqrt{\left(1+\frac{S_1''^2}{G_1''^2}\right)\left(1+\frac{S_{k+1}''^2}{G_{k+1}^2}\right)}} + \frac{1}{\sqrt{\left(1+\frac{G_1^2}{S_1''^2}\right)\left(1+\frac{G_{k+1}^2}{S_{k+1}''^2}\right)}}\mathbf{R}_k'' \text{......(lii)}$$

$$= p_k + l_k\,\mathbf{R}_k'',$$

which must be compared with the relation

$$\mathbf{R}_k' = p + q\,.\,r^k \text{ ....(xlviii) bis,}$$

where $p = {\cdot}1524,\quad q = {\cdot}6817,\quad r = {\cdot}7105,$

that has been found empirically to fit the actual values of $\mathbf{R}_k'$.

If the expressions p_k and l_k were constant for $k = 1, 2 \ldots 14$ an interpretation of (lii) would be at once suggested. Namely that $\mathbf{R}_k''$, the coefficient of correlation of the successive residuals V_t and V_{t+k} left after the removal of the secular and sessional changes is expressible in the form

$$\mathbf{R}_k'' = q' r^k \text{....................................(liii),}$$

that is to say, making allowance for the presence of accidental errors, the law of relationship between the successive estimates suggested on p. 90 above, holds good. Now without finding the curve which represents the sessional change in each series we do not know the values of S_k'' and G_k. We have however that

$$S_k''^2 + G_k^2 = S_k'^2 \text{.................................(liv),}$$

where S_k' is the standard deviation of the observations in the kth groups after the removal of the secular term. The values of S_k' are given in Table V, p. 58; they are seen to increase as k increases and therefore p_k and l_k can only be constant for all values of k if

$$\frac{S_1''^2}{G_1^2} = \frac{S_2''^2}{G_2^2} = \ldots = \frac{S_{14}''^2}{G_{14}^2} \text{..............................(lv).}$$

That the relations (lv) should hold approximately is not at all improbable; for with a sessional change of the parabolic form of the curve (xxx) illustrated in Figure 4, the standard deviations of the ordinates in the later groups will increase owing to the increasing drop of the curve towards the end of the series while S_k'' may increase with k owing to greater variation towards the end of a session.

In fact for this particular mean series with its sessional curve represented by (xxx) it is found that

$$G_1 = {\cdot}0336 \text{ ins.},\quad G_{14} = {\cdot}0406 \text{ ins.},$$

while $S_1'' = {\cdot}0165 \text{ ins.},\quad S_{14}'' = {\cdot}0201 \text{ ins.},$

that is to say, the variations superimposed upon the main sessional change (the distances of the points plotted in Figure 4 from the parabola) become greater towards the end of the series when the observer's judgment perhaps became more erratic as he grew tired. These values give $\frac{S_1''}{G_1} = {\cdot}49$, $\frac{S_{14}''}{G_{14}} = {\cdot}50$ suggesting that the relations (lv) do hold very closely. What we find therefore in this typical mean

series represented by Figure 4 may well be expected to hold approximately in the individual series.

If then p_k is constant for $k = 1, 2, \ldots 14$ and equals p, we find readily from equations (lii) and (lv) that

$$l_k = 1 - p_k = 1 - p,$$

and hence $$(1-p)\,R_k'' = qr^k \quad \text{or} \quad \mathbf{R}_k'' = \frac{q}{1-p}\,r^k.$$

Making use of the numerical values $p = \cdot 1524$, $q = \cdot 6817$, $r = \cdot 7105$ we obtain finally

$$\mathbf{R}_k'' = \cdot 8043\,(\cdot 7105)^k \qquad \text{(lvi)},$$

as the theoretical expression for the correlation of the successive residuals after the observations have been freed from secular and sessional change. This curve is the lower of the two curves drawn in Figure 8. The points which are there plotted about this curve are the points $(k, \mathbf{R}_k'')$* obtained from equation (lii)

(a) On the assumption that $G_1^2 = G_2^2 = \ldots = G_{14}^2 = \text{constant}$,

(b) $$\frac{1}{\sqrt{\left(1 + \frac{S_1''^2}{G_1^2}\right)\left(1 + \frac{S_{14}''^2}{G_{14}^2}\right)}} = p_{14} = \cdot 1524,$$

(c) Making use of equations (liv) and the tabled values of S_k'.

The close fit of the curve to these points shows that the manner in which the values of $\mathbf{R}_k''$ fall off as k increases is not much affected by the different assumptions regarding the relations of the S_k'''s and the G_k's made in the two cases †.

Experiment B.

Reference has been made on p. 71 to evidence for a slight periodicity in the observations of this Experiment, which gives rise to small but apparently significant negative values to $\mathbf{R}_k'$, for $k > 7$. Further investigation might enable a correction for this periodicity to be made, but at present it is not possible to express $\mathbf{R}_k''$ with exactness in the form

$$\mathbf{R}_k' = qr^k.$$

For the purpose of comparison with the other experiments we can however obtain values of q and r which will give a rough fit for the first few values of $\mathbf{R}_k'$. Thus if we take

$$r = \cdot 72, \quad q = \cdot 47\ ‡,$$

we get the values

$$\mathbf{R}_1' = \cdot 34, \quad \mathbf{R}_2' = \cdot 24, \quad \mathbf{R}_3' = \cdot 18, \quad \mathbf{R}_4' = \cdot 13,$$

which agree roughly with the actual values given in Table XI, namely

$$\mathbf{R}_1' = \cdot 352, \quad \mathbf{R}_2' = \cdot 231, \quad \mathbf{R}_3' = \cdot 183, \quad \mathbf{R}_4' = \cdot 085.$$

* In Figure 8 these points have been indicated by $\mathbf{R}_k'''$ to distinguish them from the correlation coefficients of residuals after removal of linear sessional change, there denoted by $\mathbf{R}_k''$.

† The values of R_k'' calculated from equation (lvi) and of R_k'' and S_k'' calculated on the assumption of the constancy of G_k are given in the 8th, 7th and 6th columns of Table XVIII.

‡ $r \cdot 72$ is the value of the mean of the ratios $\frac{R_2'}{R_1'}$, $\frac{R_3'}{R_2'}$ and $\frac{R_4'}{R_3'}$, and using this value for r, q was taken as $\cdot 47$ by rough trial.

Experiment C.

At the end of the section dealing with the reduction of the observations for this experiment the conclusion reached was that $\mathbf{R}_{12}'$ and $\mathbf{R}_{13}'$ were not significantly negative; no difficulty therefore arises in fitting a curve of the form $y = qr^k$ to the values of $\mathbf{R}_k'$ given in the 6th row of Table XIV, p. 80. This was effected by the method of least squares, with the result

$$\mathbf{R}_k' = \cdot 6673 \times (\cdot 7917)^k \quad \text{.........................(lvii).}$$

In the 7th row of Table XIV are given the values of $\mathbf{R}_k'$ calculated from this equation, and in the 8th row the differences

$$(\mathbf{R}_k' \text{ from observations}) - (\mathbf{R}_k' \text{ from curve}).$$

If these differences are compared with the probable errors of $\mathbf{R}_k'$, it will be seen that the fit is very satisfactory, for the later calculated values of $\mathbf{R}_k'$ are in any case uncertain; $\mathbf{R}_{12}'$ and $\mathbf{R}_{13}'$ were indeed not used in the least square solution as they were known to have too high negative values.

Experiment D.

On p. 85 it was suggested that $\mathbf{R}_k$ would approach the value $+ \cdot 354$ as k increased. In this case a curve of the form

$$\mathbf{R}_k = \cdot 354 + qr^k,$$

was fitted to the calculated values of $\mathbf{R}_k$. The fitting was carried out by moments. Making $\mathbf{R}_k - \cdot 354 = z$, we have

$$\sum_1^s (z) \;= qr\frac{1 - r^s}{1 - r} = N, \text{ say, where } s \text{ is the number of ordinates, or } 12$$

$$\sum_1^s (zk) = q(r + 2r^2 + 3r^3 + \ldots + sr^s) = N \times \mu_1'$$

whence
$$\mu_1' = \frac{1}{1 - r} - \frac{sr^s}{1 - r^s} \quad \text{.............................(lviii),}$$

and is the distance of the mean from an origin at unit distance from the first ordinate qr,

$$N = qr\frac{1 - r^s}{1 - r} \quad \text{...................................(lix).}$$

The constants μ_1' and N are known; solving (lviii) by approximation we have r, and then (lix) gives q.

The values are
$$\left.\begin{array}{l} q = \cdot 1153 \\ r = \cdot 8121 \end{array}\right\},$$

and finally,
$$\mathbf{R}_k = \cdot 354 + \cdot 1153\,(\cdot 8121)^k \quad \text{..........................(lx).}$$

Then using the approximate relation

$$\mathbf{R}_k' = (\mathbf{R}_k - \cdot 354) \times \frac{S_1'^2 + \dfrac{1}{m}\sum_m (D_1 - d_1)^2}{S_1'^2}$$

—which is a modified form of Equation (xliii)—we obtain for $\mathbf{R}_k'$ the equation

$$\mathbf{R}_k' = \cdot 1785\,(\cdot 8121)^k \quad \text{.............................(lxi).}$$

Both of the curves, represented by equation (lx) and (lxi), have been drawn in Figure 20, and show a satisfactory fit, if the roughness of the data is taken into account.

The results of the Trisection and of the Ten-second Counting Experiments, and as far as the rough form of the data will allow, of the Ten-second Estimating Experiment, suggest therefore that there is some foundation for the theory of relationship between successive estimates put forward at the beginning of the present Section. To reach the expression qr^k for the correlation of successive judgments at intervals of k, it has been necessary in all cases to remove the secular change, and in one case a sessional change as well, but if these changes correspond in themselves to some definite mental or physical processes which can be separated in some degree from the causes underlying the residual variations, then we are justified in inquiring into the significance of the constants q and r. It has been suggested that

$$q = \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \bar{\beta}^2} \quad \text{.................................(lxii)},$$

so that q is dependent on the ratio between the correlated and the uncorrelated parts of the observer's judgment, that is between what I have considered as the true estimate and the accidental errors superimposed in the process of record. Now using (lxii) and the relation*

$$\sqrt{\bar{\alpha}^2 + \bar{\beta}^2} = S' \text{.................................(lxiii)},$$

(or S'' for the Trisections where it has been necessary to allow for a sessional change), we find that

$$\sqrt{\bar{\alpha}^2} = \sqrt{q}\, S', \quad \sqrt{\bar{\beta}^2} = \sqrt{(1-q)}\, S' \text{ ....(lxiv)},$$

and the values calculated in this way for $\sqrt{\bar{\alpha}^2}$ and $\sqrt{\bar{\beta}^2}$ are given in Table XIX.

TABLE XIX.

Experiment	q	S'	$\sqrt{\bar{\alpha}^2}$	$\sqrt{\bar{\beta}^2}$	r
Trisection	·80	·080 (= S'') in inches	·071	·036	·71
Bisection (approximate only) ..	·47	·045 in inches	·031	·033	·72
Ten-second Counting ...	·67	·034 in factor	·028	·020	·79
Ten-second Estimating	·18	·141 in factor	·060	·128	·81

If the Trisection and Bisection results are compared it will be seen that the standard deviations of the accidental errors ($\sqrt{\bar{\beta}^2}$) are nearly the same but that there is a large difference between the measures of the variations of the true

* It will be seen that owing to a sessional change in standard deviation, S_k'' for the Trisections (Table XVIII) and S_k' for the Bisections (Table XI) increase with k. To obtain an approximate value for the standard deviation of the whole 1200 observations as opposed to that for the 1000 observations of any particular Group k, I have used in equations (lxiii) and (lxiv) S' (or S'') given by

$$S'^2 = \tfrac{1}{14}(S_1'^2 + S_2'^2 + S_3'^2 + \dots + S_{14}'^2).$$

estimates ($\sqrt{\bar{\alpha}^2}$). This is a result which we should anticipate, for the method of recording the estimate was the same in each experiment, and accidental errors of the same magnitude would occur in both cases; on the other hand the observer was faced with a more difficult problem in estimating a third than in estimating a half, and this is shown by the greater variability of his estimate in the former case (·07 against ·03)*. For the Timing experiments, we find no correspondence between the $\sqrt{\bar{\beta}^2}$'s; the great difference between the counting of ten seconds and the attempted concentration of mind on the passing of an unbroken ten second interval has been emphasized in the description of the experiments above, and a correspondence was hardly to be expected. The standard deviations are in terms of the factors e/p and must be multiplied by 10·2 if required in seconds.

If now we turn to the values of r given in the last column of Table XIX, it will be seen that they lie near together, and although that for the Bisections is not an exact measure, there is a suggestion of close agreement between the r's in the pairs of similar experiments, for we have estimations of length with ·71 and ·72, and estimations of time with ·79 and ·81. This coefficient is a measure of the rate at which the correlation of successive judgments falls off or the influence of previous estimates vanishes from the observer's mind: on the theory of zero partial correlation it is simply the coefficient of correlation between a true estimate freed from accidental errors and the preceding estimate.

On any theory r would seem to be a fundamental constant not varying greatly for different types of observations, but perhaps varying considerably for different observers. The fact that it is so nearly the same for experiments with a five second interval between observations (Trisection and Bisection) and for others with an interval of ten seconds or more (Counting and Estimating) shows that the correlation of successive judgments is a function not only of the *time interval* between two judgments but also of the *number of intervening judgments*. For if it were purely a function of the time interval we should expect to find a greater difference between the values of r found for experiments with a five second interval and a ten second interval. Indeed if the experiments were exactly the same but for difference in interval, $\mathbf{R}_1'$ for that with ten seconds would equal $\mathbf{R}_2'$ for that with five seconds. Further experiments of the same type in which the interval between the recording of judgments was varied would undoubtedly throw much light on this point.

XII. Prediction.

If the values of "m" successive judgments are known and there is no correlation between them, the "most probable" value of the $(m+1)$th judgment, that is the most reasonable guess at its value that can be made, is the mean of the "m" judgments. If however the successive judgments are correlated, then it is possible to predict the value of the $(m+1)$th with much greater expectation of accuracy.

* This may be compared with the ratio of 3 to 2 given on p. 73 from a comparison of the S_δ's before making any allowance for the accidental errors.

In the Experiments B, C and D it has been found that the correlation between judgments at intervals of k, made in the same session, can be expressed approximately in the form

$$\mathbf{R}_k' = qr^k \qquad \text{(lxv)},$$

while for Experiment A, owing to the large sessional change, the expression was

$$\mathbf{R}_k' = p + qr^k \qquad \text{(lxvi)}.$$

The decrease of correlation in geometrical progression expressed by (lxv) follows precisely the law of ancestral heredity, for which the multiple regression equations required for prediction have already been worked out*. It is not therefore proposed to go further into the problem in the present Paper, nor to inquire whether the general multiple regression equations would reduce to as simple a form when the correlation is expressed by equation (lxvi) rather than (lxv).

XIII. SUMMARY AND CONCLUSIONS.

The secular change in personal equation is shown by the variation in the series' means, but it is only in Experiment A and perhaps Experiment C, where the general trend of the variations is markedly in one direction, that we find that type of change which is usually understood when a secular change is referred to. In the Bisection Experiment B the linear secular change is very small and its existence might well not be recognized, and yet the series' means are subject to fluctuations far exceeding those of random sampling. For the probable error of the mean of a series (or of the observations in Group 1) is

$$\pm \cdot 67449 \times \frac{S_1'}{\sqrt{50}} = \pm \cdot 00416,$$

but if we take the distribution consisting of the 20 series means d_1, we find that the standard deviation is ·037375, giving for the probable error of a mean d_1

$$\pm \cdot 02521,$$

which is more than six times as large as the probable error we have calculated by considering the variations within a series It is therefore clear that the 50 observations in a series are not random samples of the whole "universe" of observations, as they should be on the Gaussian hypothesis of normal errors.

It is again only in Experiment A that there is a fairly consistent sessional change from series to series which an observer might easily recognize and possibly allow for, and yet if we turn to any of the graphs for the Bisection or Seconds-counting which show the variations of judgment within a series (Figures 11 and 15), it will be seen how very often the mean of ten consecutive judgments will give but a poor approximation to the mean of the series; we cannot take the judgments within one series as scattered at random. When dealing with a sample of m

* The Galton-Pearson Law of Ancestral Heredity; the offspring and the mean of the kth grand parents have qr^k for their correlation.

correlated variates, the usual expression for the probable error of the mean is (1) $\pm \cdot 67449 \frac{(1-r^2)}{\sqrt{m}} \sigma_m$, as compared with (2) $\pm \cdot 67449 \frac{\sigma_m}{\sqrt{m}}$, when the variates are not correlated, but owing to the sessional variations to which a large part of the correlation is due, the expression (1) being the smaller, is in the present case a worse measure than (2), of the probable limits of divergence of the mean of the sample from the mean of the series. The graphs of Figures 6, 11 and 15 show that there is a tendency for the judgments to vary in waves, to be first on one side of the mean for the series, and then to change to the other, but with no definite period of variation. It is owing to these large correlated variations which cannot be expressed in any simple sessional term, that the coefficients of correlation, r_{ρ_1, σ_1}, between σ_1 and ρ_1 have been found to have positive values ranging from $+ \cdot 52 \pm \cdot 11$ in Experiment A to $+ \cdot 18 \pm \cdot 15$ in D, showing that greater variation is associated with higher correlation of successive judgments.

An analysis has suggested that the coefficients of correlation of the crude values of the observations at intervals of k can be expressed in the generalized form

$$\mathbf{R}_k = \frac{S_1'' S_{k+1}'' \mathbf{R}_k'' + F_k + \frac{1}{m} \sum_m (D_1 - d_1)(D_{k+1} - d_{k+1})}{\sqrt{\left\{S_1''^2 + G_1^2 + \frac{1}{m}\sum_m (D_1 - d_1)^2\right\}\left\{S_{k+1}''^2 + G_{k+1}^2 + \frac{1}{m}\sum_m (D_{k+1} - d_{k+1})^2\right\}}} \quad \text{...........(lxvii)},$$

where

$\sum_m (D_1 - d_1)(D_{k+1} - d_{k+1})$, $\sum_m (D_k - d_k)^2$ etc. are terms representing the secular change,

F_k and G_k are functions of the sessional change, and

$\mathbf{R}_k''$ and S_k'' are the correlation coefficients and standard deviations of the residuals left after secular and sessional changes have been removed.

In two experiments it has been found that $\mathbf{R}_1$ is greater than $+ \cdot 80$, which shows clearly that the estimates have not been distributed randomly in time.

The coefficients $\mathbf{R}_k''$ appear to fall off in geometrical progression, and to be closely represented by expressions of the form qr^k, in which q and r are constant for any experiment; it has been found that the introduction of the quantities F and G in equation (lxvii) in addition to the secular terms, is only necessary if there is a significant sessional change which repeats itself in series after series. Thus in Experiment C, where there was no such change, $\mathbf{R}_k$ could be expressed by the relation

$$\mathbf{R}_k = \frac{qr^k S_1' S'_{k+1} + \frac{1}{m} \Sigma (D_1 - d_1)(D_{k+1} - d_{k+1})}{\sqrt{\left\{S_1'^2 + \frac{1}{m}\sum_m (D_1 - d_1)^2\right\}\left\{S'^2_{k+1} + \frac{1}{m}\sum_m (D_{k+1} - d_{k+1})^2\right\}}}$$

A tentative interpretation has been given to the results of this analysis. The observations in Experiment A suggested that there was some physiological significance in the distinction between the secular and sessional changes, and this was

confirmed in Experiment *B*, where it was found that there was evidence of a linear sessional change acting in the opposite direction to the secular change. A discussion of the values of the partial correlation coefficients $r_{xy.z}$ (personal equation and order, time constant) and $r_{xz.y}$ (personal equation and time, order constant) suggested that if the interval between the successive series were made very short, it might not be sufficient to break the effect of the sessional change. The correlated variations which have been found to follow the law $\mathbf{R}_k' = qr^k$, have been considered as in some way separate from and superimposed upon the other more steady changes. Starting from the tentative assumption that there is little or no *partial* correlation between the observer's true estimates at intervals greater than one --that is to say that the observer's judgment at any moment is only influenced by the judgment immediately preceding, and only through this and not directly by the earlier judgments—it has been shown that the constant q in the relation

$$\mathbf{R}_k' = qr^k \quad \text{.................................(lxv) bis}$$

can be accounted for by the presence of uncorrelated accidental errors which are superimposed on the correlated variations in the observer's true estimate. Without further investigation it would be difficult to distinguish between what may perhaps be termed the physiological and the psychological factors; in the experiments that have been undertaken the variations in recorded judgment depend partly on the movements of the hand, so that the former factors are likely to have played some part as well as the latter. The successive recording motions of the hand may have been correlated as well as the variations in mental estimate.

The importance of the results of course depends on how far they may be considered as typical of any practical series of observations made by the astronomer or the physicist. Experiments were admittedly chosen in which it was expected that the variations in judgment would be large, and for the experienced observer working at the type of observation in which he has had much practice, the errors would no doubt be smaller, but it seems to me likely that the phenomena which have been discussed will be present in the judgments of other observers even if on a smaller scale. Experience and accuracy may be gained by practice, but it does not follow that the correlation between successive judgments will disappear. The secular and sessional changes may be small, but if rough comparisons of only the yearly mean personal equations of different observers are made, the finer changes, which may be of considerable importance in a combination of observations, cannot be recognized. The Law of Normal Errors requires but two constants to describe adequately any series of observations:

(1) the mean,

(2) the standard-deviation,

while the introduction of a third may be necessary if a gradual secular change in personal equation is noticed. But the more generalized Theory of Errors discussed in the preceding sections requires more detailed information and a greater number of constants to define the character of an observer's personal equation and variations in judgment. We shall require to know how the personal equation and the standard

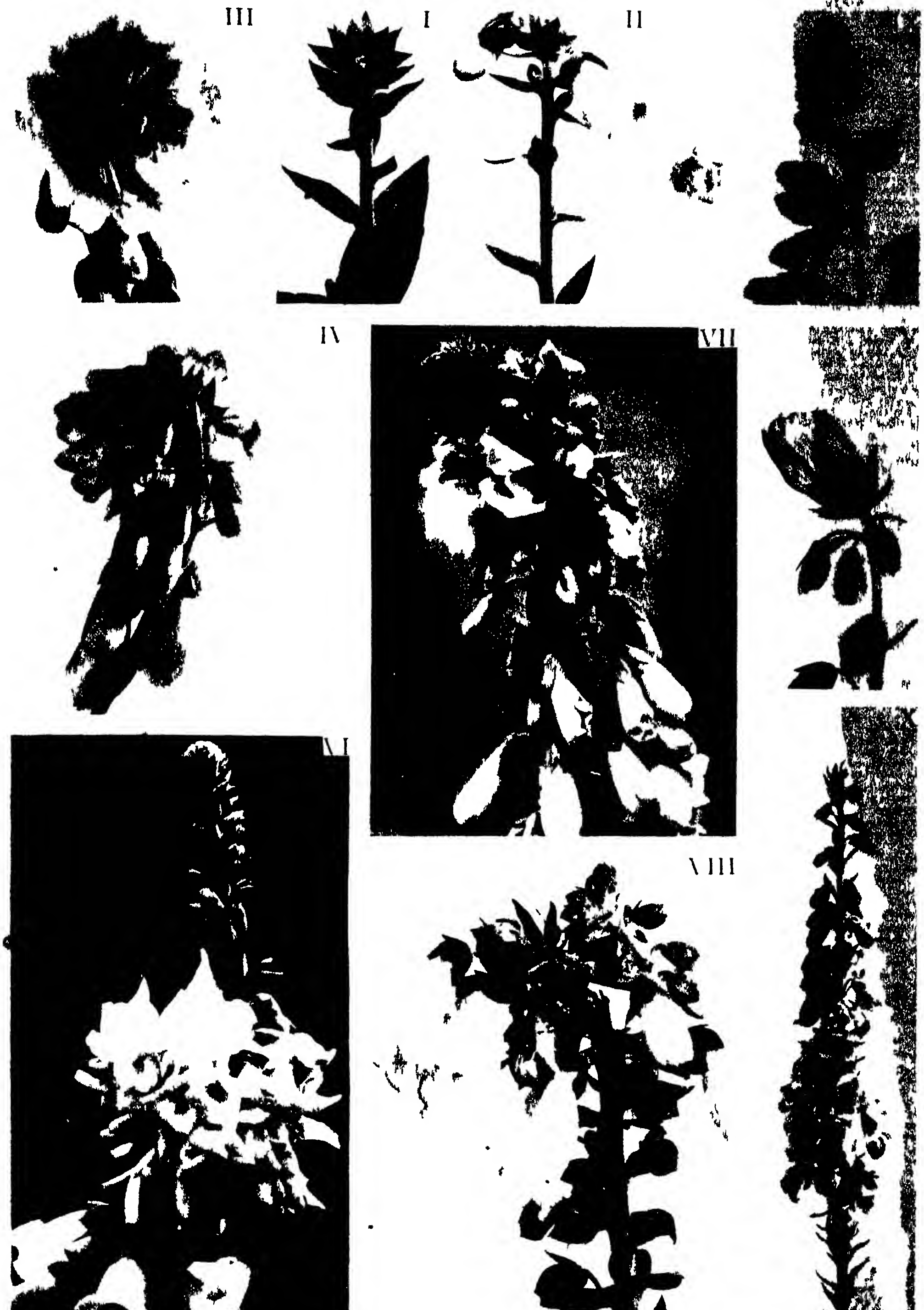
deviation vary both within a session and over long periods of time, and if there is any correlation between successive judgments, what is the form of the function ψ, which gives the value of the successive correlation coefficients in the relation

$$R_k' = \psi(k).$$

It is only by a detailed analysis of the observations themselves or of others carried out *ad hoc*, copying them as closely as possible, that full information on these points can be obtained; but if the possible complexities which may be present in the variations of judgment are fully realised, a great deal may be done in practical cases by the arrangement of the observations and the combination of the results, to eliminate the factors whose magnitude is unknown and to correct for others which are more easy to ascertain.

I have heartily to thank Miss I. McLearn for making the diagrams for Figs. 3, 4, 8, 12, 17, 19 and 20, and Miss M. Noel Karn for assistance in some of the computation.

Warren *Inheritance in Foxglove*



Figs I–IX Pelorism of various intensities Fig X Split corolla

INHERITANCE IN THE FOXGLOVE, AND THE RESULT OF SELECTIVE BREEDING.

By ERNEST WARREN, D.Sc. Lond.

In *Biometrika*, Vol. XI. pp. 302—327, 1917, the author published a preliminary report on the earlier results obtained in the breeding of foxgloves; and the present paper contains some account of the final results of the selection experiments.

In 1914 ten foxglove plants (*Digitalis gloxiniaeflora*), obtained from various sources and of different characteristics, were crossed among themselves and also self-fertilised. In subsequent years, 1915—19, new generations were obtained chiefly by the self-fertilisation of selected parents. The measurement, or when not possible the grading, of certain characters (pelorism, colour, size of flower, spotting of flower, etc.) was undertaken in all the generations in order to determine the effect of selection when selfing alone occurred in an apparently pure race.

1. Pelorism.

Mendelian inheritance occurred in a typical fashion. A peloric plant crossed with a non-peloric plant produced non-peloric offspring. On selfing these, or crossing them together, there resulted on the average one peloric to three non-pelorics.

Of the 10 parent plants two exhibited the peloric condition in a fully developed form, and the rest were non-peloric. The character was very perfectly recessive, and by breeding, it was found that three of the remaining plants were really heterozygous, while all the others were non-peloric and homozygous.

It was soon observed that the peloric condition was by no means a clearly defined and fixed character. Pelorism in the foxglove may be regarded as an abnormal lack of power to produce internodes between the flower-buds and consequently there may result considerable fusion of such buds with one another.

The maximum stage of pelorism is seen when the main-axis is short and abruptly ceases to grow in height. Only two or three normal flowers may be produced by the axis, and its blunt, sharply truncated end is surrounded by a whorl of bracts or sepals, petals being absent. Sometimes a ring of sessile anthers occurs (Pl. I, figs. I, II).

In typical pelorism the inability to produce internodes affects the terminal portions of all of the flower-axes of a plant, both central and side-axes. A variable number of flower-buds fuse and the corollas unite and may form a large symmetrical cup or saucer of some ornamental value, but the sepals mostly remain

separate (figs. III, IV). When numerous flower-buds fuse a dense rosette may be formed by the petals, and the result is not pleasing. The peloric or crown-flower opens early, often before any of the normal flowers. After the crown-flower has faded, the main-axis usually grows through the centre of it, and may even produce a second crown-flower (fig. VI); but in the case of the side-shoots the axis generally ends in an ovary and no further growth occurs (fig. V).

If the peloric tendency is not so well-marked, the main-axis may be only slightly affected by the suppression of several internodes, and by the partial fusion of flower-buds, at a variable distance above the lowest normal flower of the axis. Sometimes a considerable number of internodes may be unduly shortened, so as to produce excessive crowding of flowers which do not actually fuse (fig. VII), and frequently a strongly marked spiral bending of the axis occurs (fig. VIII).

At other times the suppression of the internodes may occur only high up on the flowering axis close to where it normally ceases to grow (fig. IX).

When the central axis is strongly peloric the side-axes are invariably so, and in all other cases the side-axes exhibit greater pelorism than the main-axis.

Finally, the main-axis may be quite normal and show no peloric tendency, but the side-axes may still be strongly peloric.

The last trace of pelorism in a plant is shown when only one or two of the weaker side-axes exhibit some slight sign of a peloric tendency.

It is unfortunate that it has not been found possible to devise any practical method of measuring the intensity of pelorism, and therefore the plants have been arranged in four grades.

0 grade = no peloric tendency.

1 — 25° grade = those in which the central axis is non-peloric, but the side-axes exhibit some peloric tendency.

26 — 50° grade = main-axis non-peloric, but side-axes may reach full pelorism.

51° – 75 grade = main-axis partially peloric, side-axes fully so.

76 —100 grade = plants ranging to complete pelorism in all axes.

In the generations produced from 1914—19 there were in all 128 fertilisations of different classes of individuals, recessive (peloric), homozygous dominant (non-peloric) and heterozygous dominant (non-peloric) plants, and families were raised. In the table on p. 105 the experimental and theoretical results are compared. The fertilisations of the classes $DD \times DD$, $RR \times RR$, and $DR \times DR$ include both selfing and crossing. The sum totals of the experimental and theoretical results are remarkably close; being, crowned, 1019 experimental and 1013 theoretical; non-crowned, 1169 experimental and 1175 theoretical.

It must be noted here that a plant was recorded as "peloric" or "crowned" if it exhibited the least tendency towards pelorism in any of the axes. Taking all the classes or groups together it may be said that the inheritance of the quality of pelorism is typically Mendelian. The group $RR \times RR$ should include no non-crowned offspring, and the 7 which occurred were obtained by gradual selection.

The group in which the experimental result diverged the most widely from the theoretical result was $DR \times RR$ (heterozygous plants crossed with recessives) and it would be interesting to know whether such is generally the case in Mendelian inheritance.

Gametic Nature of Pairings	Number of Families	Number of Offspring	Number of *Crowned* Offspring		Number of *Non Crowned* Offspring	
			Experimental	Theoretical	Experimental	Theoretical
$DD \times DD$	16	266	0	0	266	266
$RR \times RR$	43	741	731	741	7	0
$DR \times DR$	38	777	187	191	590	583
$DR \times DD$		93	0	0	93	93
$DR \times RR$		156	98	78	58	78
$DD \times RR$		155	0	0	155	155
Totals	128	2188			1169	1175

The Inheritance of the Degree or Intensity of Pelorism.

If a peloric plant be crossed with a non-peloric homozygous dominant, the offspring are heterozygous and non-peloric, and if these are self-fertilised or crossed together the peloric character re-appears in an apparently unchanged and undiluted condition. If, on the other hand, a strongly peloric plant is crossed with a weakly peloric one the offspring are more or less intermediate, and if the offspring are selfed or fertilised together the intermediate nature of the peloric character tends to be retained.

In the accompanying table *A*, *B*, *C*, *D*, *E* are plants of various gametic constitution. On selfing (*A*) the offspring were all fully peloric On selfing some 5 offspring, *A*, 2—9, the plants produced were all essentially fully crowned.

On crossing two recessive plants (*A* and *E*) of different peloric intensities (see bottom of table) the offspring tended to be intermediate.

On crossing (*A*) with an ordinary plant (*B*) the offspring were non-peloric and heterozygous. On selfing two of these plants, ($A \times B$) pls. 2 and 7, the offspring were either fully peloric, or non-peloric (heterozygous and homozygous). On selfing two recessives, ($A \times B$) 2, pls. 8 and 9 obtained from ($A \times B$) pl. 2, the offspring were all nearly completely peloric. Thus, there was no clearly marked dilution or apparent contamination by crossing a peloric plant with a non-peloric one. When, however, the same recessive plant (*A*) was crossed with a heterozygous plant (*C*) having in its gametes a weak peloric tendency of about 35° there was much variation in the offspring, and on selfing some of these plants, ($A \times C$) 1, 2, 7, 11, and raising a new generation it was obvious that considerable dilution of the peloric tendency had occurred. On crossing the same plant (*A*) with a heterozygous plant (*D*) having a stronger peloric tendency (75°) in its gametes it was clear that in the next generation raised ($A \times D$) 6, 5, 11 less dilution had taken place than in the former case.

Pelorism—Various Pairings.

Parentage	Peloric Offspring 100°	75°	50°	25°	Non-peloric	Offspring (selfed)	Peloric Offspring 100°	75°	50°	25°	Non-peloric
A (100° pelorism) Selfed = *RR* × *RR*	33	0	0	0	0	*A* pl. 2 (100° pelorism)	13	1	0	0	0
						A pl. 3 ,,	3	0	0	0	0
						A pl. 4 ,,	2	0	0	0	0
						A pl. 6 ,,	6	0	0	0	0
						A pl. 9 ,,	1	0	0	0	0
A ♀ (100° pelorism) × *B* ♂ (0 pelorism and homozygous) *A* × *B* = *RR* × *DD*	0	0	0	0	13	(*A* × *B*) pl 2 (non-peloric and heterozygous)	6	0	0	0	21
						(*A* × *B*) pl 7 (non-peloric and heterozygous)	5	0	0	0	23
						(*A* × *B*) 2 pl. 8 (100° pelorism)	12	1	0	0	0
						(*A* × *B*) 2 pl. 9 ,,	5	0	0	0	0
C (heterozygous) Selfed = *DR* × *DR*	1	0	2	3	13						
A ♀ (100° pelorism) × *C* ♂ (non crowned and heterozygous with, say, 35° pelorism in gamete) *A* × *C* = *RR* × *DR*	4	12	3	1	7	(*A* × *C*) pl 1 (75° pelorism)	20	4	9	2	0
						(*A* × *C*) pl 2 (50° pelorism)	6	7	1	1	0
						(*A* × *C*) pl 7 (heterozygous)			2	0	10
						(*A* × *C*) pl 11 ,,		2	3	1	7
D (heterozygous) selfed	1	1	1	0	8						
A ♀ (100° pelorism) × *D* ♂ (non-crowned and heterozygous with, say, 75° pelorism in gamete)	11	2	0	0	10	(*A* × *D*) pl 6 (100° pelorism)	17	2	2	0	0
						(*A* × *D*) pl 5 ,,	21	0	6	0	0
						(*A* × *D*) pl. 11 (75° pelorism)	27	5	1	0	0
A ♀ (100°) × *E* ♂ (50°)	4	1	3	0	0						

In the last generation it will be seen that there was no sharp separation of the plants into two groups attributable to the two grandparental factors. Thus, in the case of (*A* × *C*) pl. 2 (50°) the offspring are not clearly divisible into those of 100° resembling *A*, and those of 35° attributable to *C*; in other words there was no obvious segregation into two degrees of pelorism.

On the factorial and chromosome hypotheses we must suppose that the factor or factors governing the peloric character tend to become mutually changed and intermediate in nature when the male and female chromosomes containing the factors for the two degrees of pelorism lie alongside each other in the zygote.

It will be of interest to obtain a general measure of the strength of inheritance between mid-parent and offspring with respect to the transmission of the degree or intensity of pelorism. For this purpose only recessives were used, involving

30 mid-parents. Employing Prof. Karl Pearson's method the accompanying table gives the correlation surface.

Pelorism—Correlation Table—Recessives. Mid-parent and Offspring.

Offspring. Grade of Pelorism.

Mid-parents. Grade of Pelorism	76°—100°	51°—75°	26°—50	1°—25°	Totals
1°—25°	6	2	23	18	49
26°—50	58	61	68	11	198
51°—75°	64	31	15	–	110
76—100	143	14	11	5	173
Totals	271	108	117	34	530

The coefficient of correlation, calculated from the table, between mid-parent and offspring is ·52. The result can be regarded as only a very rough approximation, since a satisfactory method of measuring pelorism has yet to be found. The figure obtained is somewhat low, but it would seem to indicate that the inheritance of the degree of pelorism is of the nature of ordinary blended inheritance.

The point of interest to notice is that the union of two peloric plants of different peloric intensities influences the gametes, while the union of a peloric plant with a homozygous non-peloric plant does not very readily affect the purity of the gametes with respect to pelorism.

Pelorism. Effect of Selection in a homogeneous race.

A peloric plant (*C*) with pelorism of about 85° intensity was self-fertilised, and the offspring, 16 in number, were as follows: 7 with 100°, 4 with 75° and 5 with 50° of pelorism.

Parentage (Self-fertilisation)	Crowned Offspring 100	75	50	25°	Not crowned
C (85°) ...	7	4	5	0	0
C 2 (50°) ...	7	7	10	0	0
C 7 (50°) ...	17	11	18	2	0
C 7, 10 (25°) ...	0	2	18	2	5
C 7, 10, 20 (25°)	0	0	1	2	7

Parentage (Self-fertilisation)	Crowned Offspring 100°	75	50	25°	Not crowned
C 2, 11 (75°)	6	13	5	0	0
C 2, 2 (50°)	2	16	10	0	0
C 2, 8 (50°)	1	13	11	0	0
C 7, 10, 20, 4 (0°)	0	0	0	0	6

Two of these plants of 50° (*C* 2 and 7) were selfed, and the generation raised exhibited a lowered pelorism. The various selections made and the results obtained are shown in the accompanying table. It will be seen that finally on the selfing of plant *C* 7, 10, 20, 4 (0°) only non-peloric offspring were obtained.

2. General Coloration of the Corolla.

As described in the previous report (*loc. cit.*) the intensity of the purple coloration was measured by comparing it with a colour-scale founded on the intensity of colour by transmitted light of varying depths of a standard colour-solution.

Purple and white foxgloves exhibit the ordinary Mendelian relationship, purple being dominant. A confusing aspect of the problem is introduced by the fact that "white" foxgloves are not necessarily entirely white, since they may exhibit a faint purple coloration which on the colour-scale adopted may amount to about 5. On crossing such a plant with an ordinary purple plant segregation occurs when the heterozygous offspring are self-fertilised. Any higher coloration, say 10—15, does not exhibit segregation, but gives a blended inheritance, and such a plant is to be regarded as a very pale purple one and not "white." From certain observations that have been made it is probable that a similar condition occurs in the Blue Agapanthus lily, since some of the "white" plants have flowers faintly tinged with blue. It is quite likely that the phenomenon is general, and it may throw an important light on the physical theory of heredity. Possibly it may be surmised that a factor for a coloration of less than 5 units is unable to blend with, or influence, the factor controlling a higher coloration, in that we have reached the lowest dynamic unit.

Of the ten original plants, five were purple and homozygous, four were purple and heterozygous and one was white or recessive. These were very variously crossed in all manner of ways. In the accompanying table the experimental results are compared with the Mendelian expectation for the different gametic pairings.

General Coloration of Corolla—Breeding Results.

Gametic Nature of Pairings	Number of Families	Number of Offspring	White		Purple	
			Experimental	Expectation	Experimental	Expectation
$DD \times DD$	120	1620	2+3	0	1615	1620
$RR \times RR$	17	336	330	336	6	0
$DR \times DR$	50	785	190	196	595	589
$DR \times DD$	11	103	0	0	103	103
$DR \times RR$	8	76	24	38	52	38
$DD \times RR$	8	87	0	0	87	87
Totals	214	3007	549	570	2458	2437

In the gametic group $DD \times DD$ (homozygous purple × homozygous purple) there were 1620 offspring. These should have been all purple, but there were two white plants which occurred in two deeply coloured families and three white plants which occurred in one pale-coloured family. I do not believe that there was contamination, and it is probable that the two former plants were sports, while the three latter plants were produced by selection.

In the group $RR \times RR$ (white × white) there were 336 offspring, and these should have been all white, but there were six pale-coloured plants. The difficulty in distinguishing a tinged "white" plant from a pale-coloured plant may account for this result, but I favour the view that we are here witnessing the beginning of a coloured race.

The result given by $DR \times DR$ (heterozygous purple × heterozygous purple) is very closely Mendelian. Out of 785 offspring there were 190 white plants while the expectation was 196.

Heterozygous plants crossed with dominants ($DR \times DD$) gave nothing but coloured plants, and this was also the case with dominants crossed with recessives ($DD \times RR$).

The gametic group $DR \times RR$ (heterozygous plants × recessives) gave a result which diverged rather widely from the expectation: there were insufficient whites, there being 24 whites and 52 purples instead of 38 of each. The numbers are somewhat small for drawing conclusions, but it is important to notice that in the character of pelorism it was the same gametic group which diverged the most widely of all the classes from the theoretical expectation. On the chromosome hypothesis it may be conjectured that possibly preferential pairing of the male and female chromosomes may explain the discrepancy.

The Inheritance of the Intensity of Coloration.

On crossing a purple homozygous plant with a white plant the offspring were all heterozygous and all coloured, but the intensity of the coloration was mostly reduced very considerably. On selfing these offspring the next generation yielded some homozygous dominants in which the original colour-intensity of the grand-parent was regained; thus, at first sight it appeared that there had been no real dilution of the colour by crossing with the white. This was my first impression from the earlier results, but with more extended experience I found that there was certain evidence that the crossing with the white did have some deleterious action on the intensity of the coloration of the dominant grandchildren, although the coloration which appeared was much greater than a half and half blend with white.

If two homozygous dominants of marked difference in colour-intensity were crossed, the offspring tended to be intermediate. On selfing these offspring the next generation was similarly intermediate, and there was no segregation into the two different intensities of the grandparents. Thus a true blend of the two intensities had taken place.

In the accompanying table the results of some instructive crossings and self-fertilisations are given. In Series I two dominants (*E* and *F*) of different colour-intensities were selfed and the families raised showed that the parents were homozygous. On crossing (*E*) and (*F*) a family of intermediate offspring was obtained.

Colour-Intensity—Various Pairings.

No. of Series	Gametic Constitution	Parentage	Mid-Parental Colour	Colour-Scale—Offspring: Coloured plants: 125°—139°	110°—124	95°—109	80°—94°	65°—79°	50—64°	35°—49°	20°—34°	5°—19°	"White": 0°—4
I	*DD*×*DD*	*E* (selfed)	90	—	2	0	5	8	—	—	—	-	—
	DD×*DD*	*F* (selfed)	68	-	-	—	—	6	3	—	—	—	-
	DD×*DD*	♀*E*×♂*F*	79	—	—		4	3	5	—	—	—	—
	DD×*DD*	(*E*×*F*) pl. 16 (selfed) ...	82	—	—	2	6	2	—	—	—	—	—
	DD×*DD*	(*E*×*F*) pl. 9 (selfed) ...	61	—	—	—	—	1	10	1	-	—	
II	*DD*×*RR*	♀*E*×♂(White) ...	45	—	—	—	—	11	1	4	2	—	-
	DR×*DR*	*E*×(White) pl. 18 (selfed)	71	—		1	1	5	1	1	1	-	2
III	*DD*×*DD*	*B* (selfed)	95	3	0	2	3	6	1	—	—	—	—
	DD×*RR*	♀*B*×♂(White) ..	17		—	-	1	1	4	1	1		—
	DR×*DR*	(*B*×White) pl. 1 (selfed)	80	—	—	—	—	1	2	—	—	-	—
	DR×*DR*	(*B*×White) pl. 5 (selfed)	32	—	-	—		—	—	2	3	—	5
IV	*DD*×*DD*	*B* (selfed)	95	3	0	2	3	6	1	—			-
	DR×*DR*	*A* (selfed)	70	—	—	—	4	12	1	1		-	7
	DD×*DR*	♀*B*×♂*A* ...	82	2	0	1	4	4	2	—		·	-
	DD×*DD*	(*B*×*A*) pl. 7 (selfed) ...	130	1	1	3	12	13	—	—	—	—	-
	DD×*DD*	(*B*×*A*) pl. 2 (selfed) .	65	-	—	—	2	13	12	—		—	—
V	*DD*×*DD*	*B* (selfed)	95	3	0	2	3	6	1	—	—	—	
	DR×*DR*	*C* (selfed)	34	—		—	—	—	1	9	2	—	4
	DD×*DR*	♀*B*×♂*C*	65	-		—	—	2	4	2	—	—	—
	DD×*DD*	(*B*×*C*) pl. 8 (selfed)	50			—	—	—	2	1	—	—	-
	DD×*DD*	pl. 4	50	—	—	—	—	4	17	3		—	
	DD×*DD*	pl. 7	58	—	—	—	—	5	4	1	1	—	-
	DD×*DD*	pl. 6	68		—	—	—	6	14	9	—	—	—
	DD×*DD*	pl. 1	70		—	2	4	20	4	—			-

Two of these offspring were selected, (*E*×*F*) pls. 16 and 9, as widely divergent from each other as possible, and selfed. In the families obtained there was no tendency for the occurrence of segregation into the two colour-intensities of (*E*) and (*F*) respectively. There was thus a definite blend, and the means of the two families approached the respective colour-intensities of the two self-fertilised plants.

In Series II the same homozygous dominant plant (*E*), with colour-intensity of 90°, was crossed with a white plant and all the offspring were heterozygous and intermediate. On selfing one of the darker coloured offspring, no. 18, the dominant plants raised tended to be of about the same colour-intensity as the grandparent

(E). In Series III a dark-coloured homozygous dominant plant (B) was also crossed with a white plant. One of the darkest heterozygous offspring ($B \times$ White) pl. 1 was selfed and the coloured plants raised tended to be paler than the grandparent, but the family was small.

In Series IV the dark-coloured homozygous plant (B) was crossed with a dark heterozygous plant (A). From the offspring raised, two were selected and selfed, one very dark and the other moderately dark. The two families included only coloured plants, and consequently the parents may be supposed to have been homozygous. The moderately dark parent ($B \times A$) pl. 2 failed to produce any offspring as dark as the grandparent (B).

In Series V the same plant (B) was crossed with a light heterozygous plant (C). From the offspring produced five homozygous dominants were selfed, and in the five families raised only two plants reached the colour-intensity of the grandparent (B).

On taking all these results together it may be said that there is evidence for the view that crossing a dark race of foxgloves with white plants tends to dull the colour-intensity of homozygous dominants of subsequent generations.

General Coloration Strength of Inheritance and Effect of Selection.

In 1914 a dark-coloured homozygous plant (B_4 ♀) was crossed with a somewhat pale-coloured heterozygous plant (C_1 ♂) $= DD \times DR =$ III. The offspring would consist theoretically of approximately equal numbers of dominants and heterozygous individuals. The reciprocal cross (C_1 ♀ $\times B_4$ ♂) was also made = II. Several dominants were selfed and families were raised. Out of these families certain plants were selected and selfed and new families were obtained. This procedure was continued until 1917, and the results are given in the accompanying table. The families of the different years are arranged in ascending order of the colour-intensities of the parents. On comparing the means of the families with the colour-grade of the parents (shown in brackets) it will be at once seen that small variations in the colour-intensity of the parents tended to be transmitted to the offspring. It is obvious that the table exhibits the effect of selection in self-fertilised homozygous generations.

For example we may take the following:

Homozygous plant,	II. 1	had a colour of	70	and a mean of offspring	74
An offspring of above,	II. 1, 4	„	„	71	„	„	„	82
An offspring of above,	II. 1, 4, 17	„	„	110	„	„	„	95
Homozygous plant,	III. 2	„	„	75	„	„	„	66
An offspring of above,	III. 2, 1	„	„	66	„	„	„	55
An offspring of above,	III. 2, 1, 18	„	„	80	„	„	„	85
An offspring of above,	III. 2, 1, 18, 28	„	„	95	„	„	„	100

Reverse selection is shown also:

Homozygous plant,	III. 2	„	„	75	„	„	„	66
An offspring of above,	III. 2, 5	„	„	52	„	„	„	57
An offspring of above,	III. 2, 5, 5	„	„	40	„	„	„	41
An offspring of above,	III. 2, 5, 5, 12	„	„	30	„	„	„	32

Inheritance of Colour-Intensity among Dominants.

Grades of Colour-Scale (Offspring)	*DR* × *DD*: II = C_1,34 × B_4 (80)	Dominant Generations (Self-fertilisation): Parents II 4 (50)	II 6 (68)	II 1 (70)	II 4, 8 (45)	II 4, 6 (49)	II 6, 3 (49)	II 6, 4 (54)	II 4, 2 (61)	II 6, 11 (61)	II 6, 1 (68)	II 4, 12 (69)	II 1, 4 (74)	II 1, 2 (91)	II 1, 1 (120)	II 1, 2, 20 (53)	II 6, 11, 6 (59)	II 1, 4, 3 (64)	II 1, 1, 2 (77)	II 1, 4, 17 (110)
30–39	2	-	1	—	4	—		—		—	—		—		—	—	-	—	—	
40–49	0	3	8	—	15	—	2	—	—	—	—	—	—	—	—	—	—	—		—
50–59	4	8	13	1	6	1	15	5	1	1	2	-	—	5	1	3	—	1	—	
60–69	1	12	4	9	—	7	—	7	7	7	6	3	4	2	2	4	3	3	1	—
70–79	1	1	3	11	—	2	—	-	2	8	1	—	7	4	5		1	3	0	—
80–89	—	—	—	1			—	—	—	-	—	—	6	3	1	—	—	2	6	1
90–99	-	-		4	—	-	—	—	—	—			3	1	—	—	—	—	—	0
100–109				1	—	—		-	—	—	—	—	1	1	—			—	—	2
110–119	—	—	—	—	-	-		—	—	—	-	—	2	0	—	-	—	—		—
120–129	—			-		—	—	—	—	—	—	—		1	—	—	—	—	—	—
Means	53	59	54	74	51	66	54	62	66	69	61	64	82	78	70	59	65	71	81	95

Grades of Colour Scale (Offspring)	III = B_4 (40) × C_1 (34)	Parents III 2 (75)	III 2, 5 (52)	III 2, 7 (52)	III 2, 13 (63)	III 2, 1 (66)	III 2, 2 (71)	III 2, 8 (72)	III 2, 5, 5 (40)	III 2, 5, 18 (52)	III 2, 7, 9 (62)	III 2, 5, 10 (72)	III 2, 1, 18 (80)	III 2, 5, 5, 12 (30)	III 2, 5, 10, 17 (37)	III 2, 5, 5, 11 (42)	III 2, 1, 18, 10 (53)	III 2, 5, 10, 22 (70)	III 2, 1, 18, 12 (85)	III 2, 1, 18, 21 (95)
20–29			—	—	—	—	—	—		—	—	—	—	3	—	—	—	—	—	—
30–39	1		—	—	—	—	—	—	1	2	—	1	—	7	2	2	2		—	—
40–49	0		3	3	—	5	—	—	7	8	—	0		—	4	5	6	--	—	—
50–59	1	4	9	6	3	5	—	1	—	5	5	2	1		1	1	2	1	—	—
60–69	0	17	3	2	1	2	6	11	—	—	4	2	1	—	—	—	—	0	1	3
70–79	2	9	2	—	—	0	11	9	—	—	1	4	2	—	—	—	—	0	1	0
80–89	1	1			—	1	1	0	—	—	—	2	3	—	—	—	—	2	1	1
90–99	—		—	—	—	—	—	1	—	—	—	—	5	—	-		-	—	—	2
100–109	—	—	—	—	-	—	—			—	—	—	2	—		-	—	—	-	2
110–119		—	—	—		—	—	—	—		—	—	—	—	—	—	—	—	—	1
120–129	—	—	—	—	—	—		—	—				—	—	—	—		—	—	0
130–139	--	—	-	—		—	-	—	—	—	—	—	—		—	—	—	--	—	1
Means	67	66	57	54	60	55	71	69	41	47	62	68	85	32	43	47	44	74	75	91

Thus, starting with a plant of about 70 colour-intensity we arrive by selection of self-fertilised plants at mean family intensities of 100 in one direction and 32 in the reverse direction.

In another series, starting with a homozygous dominant plant of colour-intensity of about 11, I have by selection obtained plants in which the corolla exhibited no general tint. On selfing the pale plant no white plants occurred,

and the offspring were all pale-coloured; but when the intensity was decreased by selection to about 4, the "white" plants showed Mendelian segregation, for the offspring arising from the plants produced from a cross with a dark-coloured plant were sharply divisible into strongly coloured and "white" individuals.

As a further example of selection, I started with a homozygous medium-coloured (48) plant (G). This was selfed and a family of 31 coloured plants was raised, there were no whites. Thus, the parent plant may be regarded as homozygous. A plant (G 3) in this family, not far removed in colour (55) from the average, was selfed and the resulting family had a mean colour approximating to the colour of the parent. A light-coloured (27) plant (G 3, 20) and a dark (81) plant (G 3, 13) in this last family were selfed also, and the two families raised tended to resemble their respective parents. In a succeeding generation further progress was obtained in securing a dark race and a pale race. The necessary details are given in the accompanying diagrammatic table. The families printed in heavy type are those leading to a dark race, while those in ordinary type are passing into a light race.

Formation of Light and Dark Races from a Dominant (homozygous) G

Parents	Number	Offspring—Scale of Colour 95—104	85—94	75—84	65—74	55—64	45—54	35—44	25—34	15—24
G (selfed)		—		--	1	9	15	6	-	—
G pl 3 ...	*55*	—	—	2	1	0	4	5	3	1
G 3, pl. 20 ...	*38*	—	—	—	—	—	—	1	1	—
G 3, pl. 13 ...	*81*	--	**2**	**3**	**2**	**2**	**2**	**3**	-	—
G 3, pl. 20 ...	*27*		—	—	—	—	—	4	5	2
G 3, 13, pl. 2 ..	*80*	**2**	**6**	**2**	-	—	—	—	—	—

Correlation Table—Colour-Intensity—Dominants (homozygous). Series II and III.

Parents. Grade of Colour-Intensity	130—139	120—129	110—119	100—109	90—99	80—89	70—79	60—69	50—59	40—49	30—39	20—29	Offspring
30—39	—	—	—	—	—		-	-	1	4	9	3	17
40—49	-	—	—	—	—	—	2	7	26	29	10	-	74
50—59		—	—	-	—	-	4	31	38	23	4	—	100
60—69	—	—	—	—	—	3	18	37	31	13	1	-	103
70—79	—	-	2	2	8	19	54	50	9	0	1	-	145
80—89	—	—	—	2	5	4	3	2	1	—	—	—	17
90—99	2	5	9	7	15	8	5	5	5	—	—	—	61
100—109	—	—	—	-		—	--		--	—	—	—	0
110—119	—	--	-	2	0	1	—	--	—	—	—	—	3
120—129	—	—	—	-	-	1	5	2	1	—	—	-	9
Totals	2	5	11	13	28	36	91	131	112	69	25	3	529

In the last table, p. 113, a correlation surface is shown between parents and offspring. It is formed from the series of families given in the table preceding the last, and arising by self-fertilisation.

The constants calculated from the table are: standard deviation of weighted parents 1·7805 units, and of offspring 1·8962 units, coefficient of correlation ·707.

In this table 39 families were involved, as detailed in the previous table. The starting points were four homozygous dominant plants occurring in the two families raised from the reciprocal crosses ($C_1 \times B_4$) and ($B_4 \times C_1$).

3. Brown Spots.

The amount of spotting on the inside of the corolla is not closely correlated to the intensity of the general purple coloration of the flower, for even in white plants the spots may be numerous and of a deep purple colour. In coloured plants the spots were almost always dark purple. As a very rare exception in the coloured plants (4 plants in about 2500) some of the spots were russet brown, and in the case of the larger spots there was a middle area of brown bordered by a margin of purple. In white flowers the spots were fairly frequently brownish-green or brown. In such brown spotted white flowers I could never detect the slightest tinge of purple on the general surface of the corolla, while in purple-spotted white flowers a faint tinge of purple could often be seen. The brown spots of white flowers might not become visible until the flowers were on the point of fading, and in the case of any given white plant it was wholly impossible to affirm that brown spots were, or would be, entirely absent from all of the flowers.

With the exception of the four plants mentioned above there was a sharp discontinuity to the naked eye between purple spots and brown spots, intermediate conditions being absent. The brown colouring matter may be regarded as altered or decomposed anthocyanin. In purple spots a microscopic examination often showed a certain amount of decomposition; but, with the exception of the four plants, the amount was not enough to alter the colour of the spots sufficiently for detection by the naked eye. Thus, the discontinuity lies between a normal small amount of decomposition, and an abnormal entire decomposition. It may be stated that under ordinary circumstances brown or greenish spots (as seen by the naked eye) are linked to a perfectly white corolla, but purple spots occur in both purple and "white" flowers, and an apparently perfectly white corolla may also bear purple spots.

If a brown spotted plant is crossed with a purple spotted one the offspring are all purple spotted and heterozygous. The brown spotted condition is inherited in Mendelian fashion, and is recessive to purple spots.

No special crossings have been made to investigate the matter, and the results which are given below are merely picked out from the records of the numerous families which have been raised for other purposes.

In the accompanying table it is useless to include families in which there was no taint of whiteness, since all the individuals (except 4 plants out of 2500) had purple spots.

Brown Spots—Families White or Some Taint of Whiteness.

Gametic Nature of Pairings	Number of Families	Number of Offspring	Purple Spotted		Brown Spotted	
			Experimental	Mendelian Expectation	Experimental	Mendelian Expectation
$DD \times DD$	13	344	344	344	0	0
$RR \times RR$	11	169	0	0	169	169
$DR \times DR$	13	213	166	160	47	53
$DR \times DD$	15	137	137	137	0	0
$DR \times RR$	1	8	3	4	5	4
$DD \times RR$	6	70	70	70	0	0
Totals	59	941	720	715	221	226

It is obvious from the table that the brown spotted condition exhibits Mendelian inheritance.

4. INHERITANCE OF CERTAIN SPORT ABNORMALITIES.

Crenate Margin.—In a homogeneous family of 29 plants there appeared one plant in which the free edge of the mouth of the flower exhibited a well-marked serrated condition. All the flowers of a main-axis of considerable size were similarly affected, and later, lateral flowering axes were formed, and the flowers were also serrate. The character was sufficiently marked to be noticeable at a casual glance of the plant, and since all the numerous flowers were alike in this particular, the character was clearly inherent in the plant, and was not due to a chance environmental disturbance influencing a young growing axis or certain flower-buds. The plant was self-fertilised, and it was confidently expected that the character would reappear in the offspring. Out of a family of some 20 plants 12 flowered and no sign of the peculiar serrated condition could be detected in any one of the plants. Here we have a conspicuous character in a large healthy plant affecting every flower of all the flowering axes, and yet apparently it was incapable of being transmitted to the offspring.

Split Corolla.—In a homogeneous family (XXXIV) of 27 plants there appeared one plant in which in the great majority of the numerous flowers the corolla was symmetrically divided into an upper, a lower and two lateral pieces by four lateral splits extending down to the base of the flower. The plant was a large, healthy one and produced a number of similar lateral axes. At least 90 °/₀ of the flowers were completely split (Pl. I, fig. 10).

In a family (VIII 7) unrelated to the above there were 16 plants, and of these, four plants were similarly affected. In one of these plants practically all

(99 %) of the flowers were entirely split into four pieces, while in the remaining three plants some 50—60 % of the flowers were split. All the plants were large and vigorous. It was thought that very probably the character would exhibit Mendelian inheritance. The results of crossing and selfing are shown in the accompanying table.

Inheritance of Split Corolla.

Parents: Registered Number	Mid-parental degree of Splitting	Offspring, Percentage of Splitting: No Splitting	1–14	15–29	30–44	45–59	60–74	75–99
XXXIV (selfed)	0	26	0	0	0	0	0	1
VIII 7 (selfed)	0	12	0	0	0	1	2	1
XXXIV 4 (90) × VIII 7, 9 (99) = S. J.	94	8	6	3	0	0	0	0
S. J. pl. 9 (selfed)	0	7	3	0	0	1	1	2
S. J. pl. 18 (selfed)	13	2	1	0	1	1	1	5
S. J. pl. 6 (selfed)	18	3	2	0	1	1	3	4
S. J. 18 pl. 11 (selfed)	0	1	1	0	2	0	3	1
S. J. 18 pl. 4 (selfed)	4	0	2	0	2	0	2	1
S. J. 18 pl. 10 (selfed)	99	0	0	0	0	0	1	3
II 6, 1 (selfed)	0	9	0	0	0	0	0	0
XXXIV (selfed)	0	26	0	0	0	0	0	1
II 6, 1 (0) × XXXIV 4 (90) = R. J.	45	12	0	0	0	0	0	0
R. J. pl. 9 (selfed)	0	15	0	0	0	0	0	0
R. J. pl. 16 (selfed)	0	10	0	0	0	0	0	0
R. J. 16 pl. 14 (selfed)	0	10	1	0	2	0	0	0

The first mentioned plant (XXXIV 4) with 90 % of the flowers split was crossed with an unrelated plant with some 99 % of the flowers split (5th vertical column of table). Of the 17 offspring 8 plants were wholly unsplit, while the remainder exhibited the character in a very greatly weakened condition. Three of these offspring, S. J. nos. 9, 18 and 6 having 0 %, 13 % and 18 % of the flowers split respectively, were selfed, and the families raised all contained some plants very conspicuously split, but the character was more marked in the two families raised from parents 18 and 6 which showed some degree of splitting. In a subsequent generation (S. J. 18 pl. 4 and S. J. 18 pl. 10) raised by selfing, the character became very strongly pronounced.

An unrelated non-split plant (II 6, 1) was crossed with the first mentioned plant having at least 90 % of the flowers split (XXXIV 4). In the family of

12 plants raised none of the plants exhibited splitting. Two of these offspring (R. J. nos. 9, 16) were selfed and no splitting occurred in the two families. Another generation was raised from R. J. 16, plant 14 and some re-appearance of splitting was detected. The table includes all the split plants which have occurred among some 3000 plants which have been under observation.

The results obtained indicate that heredity has some influence, but the data are insufficient for determining the nature of the transmission which does not bear a Mendelian aspect.

Creased Upper Lip.—In a certain plant in the majority of the flowers the upper surface and lip exhibited a conspicuous pucker or crease. This plant was crossed with an unrelated normal plant with no crease. Most of the seedlings were killed by the violent elements, but four plants were raised, and in one, a number of flowers exhibited a crease, which, however, was much less developed than in the paternal parent. The data are scanty, but the hereditary transmission does not seem to be Mendelian.

Spontaneous Appearance of White plants.—Among the numerous homozygous dominant coloured families that have been raised a white plant appeared spontaneously on two occasions in two unrelated families. These plants, of course, bred true, and as there was no evidence of contamination of the seed the plants must be regarded as new sports.

5. INHERITANCE OF SEED-LENGTH.

The mean length of the seed varied considerably in different plants. No discontinuous variation could be detected, and inheritance was of the blended type. Ten seeds were taken at random from one or more capsules of a number of plants of certain series and the means determined. The seeds of a capsule exhibited a moderate amount of variation, but they were monomorphic in varietal crossings, and not dimorphic as was noticed in an interspecific crossing. The distribution was more or less normal. Unfortunately there was very considerable variation in the mean size of the seeds in different capsules of the same plant, and consequently no very accurate determination of the strength of inheritance was possible with this character without an excessive number of measurements. As it was, the investigation entailed the measurement of about 1000 seeds.

A plant, C_1 (mean seed-length 639 units), was crossed with B_4 (mean seed-length 628 units) and a family was raised, $C_1 \times B_4 = \text{II}$. In family II twelve plants were selfed, namely II 1, II 2 ... II 12, the seeds were measured and twelve families were obtained. In family II 1 three plants were selfed and the seed-length determined, namely (II 1) 1, (II 1) 2 and (II 1) 4. The means of the seed-lengths of these three plants were compared with the seed-length of the parent II 1. Similarly, for example, in family II 1, 2 two plants were selfed, namely (II 1, 2) 5 and (II 1, 2) 20, and the means of the seed-lengths of these two plants were compared with the seed-length of the parent II 1 2. The data are given in the accompanying table

Mean Seed-length, Parents and Offspring.

Parent (selfed) Designation	Parent (selfed) Mean Seed-length	Offspring (selfed) Designation	Offspring (selfed) Mean Seed-length
II 1	606	II 1, 1	572
		II 1, 2	668
		II 1, 4	649
II 1, 2	668	II 1, 2, 5	668
		II 1, 2, 20	642
II 1, 4	649	II 1, 4, 3	655
		II 1, 4, 17	674
II 2	528	II 2, 1	624
		II 2, 3	582
		II 2, 5	637
		II 2, 16	566
II 3	629	II 3, 1	686
		II 3, 4	686
		II 3, 15	672
II 4	592	II 4, 2	668
		II 4, 6	657

Parent (selfed) Designation	Parent (selfed) Mean Seed-length	Offspring (selfed) Designation	Offspring (selfed) Mean Seed-length
II 4	592	II 4, 8	628
		II 4, 12	598
II 6	620	II 6, 1	621
		II 6, 3	641
		II 6, 4	670
		II 6, 11	695
II 6, 11	695	II 6, 11, 6	665
II 7	547	II 7, 1	671
		II 7, 12	570
		II 7, 14	624
II 7, 1	671	II 7, 1, 7	649
II 8	620	II 8, 2	629
		II 8, 3	624
II 9	653	II 9, 2	633
		II 9, 11	620
		II 9, 10	630

Parent (selfed) Designation	Parent (selfed) Mean Seed-length	Offspring (selfed) Designation	Offspring (selfed) Mean Seed-length
II 9	653	II 9, 3	629
II 10	646	II 10, 1	660
		II 10, 2	669
		II 10, 5	649
		II 10, 7	660
		II 10, 8	713
		II 10, 13	681
II 10, 1	660	II 10, 1, 18	642
II 10, 5	649	II 10, 5, 5	598
		II 10, 5, 10	629
		II 10, 5, 18	649
II 10, 7	660	II 10, 7, 9	653
II 11	615	II 11, 8	617
II 12	679	II 12, 9	642

C_1 (self-pollen) seed-length = 639
B_4 (self-pollen) „ = 628
C_1 (B_4 pollen) „ = 642, these last seeds produced fam. II

The coefficient of correlation, calculated from the above numbers, between parents (selfed) and offspring (selfed) is ·378. This is low for mid-parental correlation; but as all the generations arose by self-fertilisation we ought to have practically no correlation at all according to the pure-line hypothesis, for the two original parents (C_1 and B_4) were closely similar to each other in the character under investigation.

6. Purple Spotting of the Corolla.

The purple spotting of the lower surface of the corolla-tube and lower lip varied greatly in the original parent plants, and the character was obviously inherited. The amount of spotting had little relationship to the intensity of the general coloration of the corolla, and "white" flowers were sometimes richly spotted with purple.

The percentage area of the lower surface covered with spots was estimated by comparing the flowers with a series of diagrams each covered with a definitely known percentage of spotting. With practice it was found that sufficiently uniform results could be obtained by this method.

In plants which had lost completely the power of producing any purple coloration whatever, the spots were brown* and usually small and scanty, and among such plants an almost entire absence of spots of any kind occasionally occurred. We have already seen that with regard to the colour of the spots (brown and purple) Mendelian segregation takes place.

In the inheritance of the *amount* of purple spotting no Mendelian relationship could be detected. The smallest amount of purple spotting met with in coloured foxgloves equalled about 1 %, and the maximum about 70 %. It will be remembered that on crossing a dark purple plant with a plant bearing flowers very faintly tinged with purple (say, colour 4 of standard), definite segregation into "white" and purple plants occurred in the second generation following; but on crossing a plant possessing an abundance of purple spots (say, 50 %) with a plant bearing very few purple spots (say, 2 % or 3 %) no such segregation was found, and the spotting tended to remain intermediate in amount.

In the numerous crosses that have been made for various purposes the condition of the spotting was observed, and it is undoubtedly true that the means of the spotting of the families resulting from the crosses tended on the average to approximate to the spotting of the mid-parent, $\frac{1}{2}(♂ + ♀)$ No difference could be detected between the reciprocal crosses of two plants.

Influence of Selection and Strength of Inheritance in Self-fertilised Generations.

In this connection details of Series II and III may be given (see p. 120). Plant C_1 with 11 % spotting was crossed with pollen of plant B_4 with spotting 48 % = II. Seven of the offspring were selfed and the spotting of the resulting families was determined. Subsequently two other generations were raised by selfing. Plant B_4 was crossed with pollen of C_1 = III. Four of the offspring were selfed and subsequently three other generations were raised by self-fertilisation.

The distributions of the spotting in the families of the different generations are shown in the accompanying table. In each generation the families are arranged in the ascending order of the parental spotting (see the top and middle horizontal lines). A casual inspection indicates at once that the general trend of the family-distributions follows the gradual increase in the spotting of the parents.

As an example of selection we may take:

III 2 (9 %) selfed	produced with others	a plant		III 2, 5 (15 %)
III 2, 5 (15 %) selfed	"	"	"	III 2, 5, 10 (22 %)
III 2, 5, 10 (22 %) selfed	"	"	"	III 2, 5, 10, 17 (27 %)

III 2, 5, 10, 17 (27 %) selfed produced a family with mean spotting of 39 %

Thus, we have passed from a plant with 9 % spotting to a plant with 27 %, which on selfing produced a family with a mean spotting of 39 %.

With reference to the strength of inheritance two tables are given on p 121, one for parents and offspring, and one for grandparents and grandchildren. The respective coefficients of correlation are ·560 and ·395. This correlation does not arise by the mixture of two races which have been sorted out by segregation

Purple Spotting—Families from Self-fertilised Parents. Series II and III

Offspring. Percentage Grade of Spotting	C_1 (11) selfed	B_1 (48) selfed	II = C_1 (11) × B_1 (48)	II 3	II 6	II 1	II 8	II 2	II 2	II 4	II 1, 2	II 1, 1	II 6, 11	II 7, 1	II 7, 2
Spotting of Parents				13	14	16	16	17	17	28	10	13	13	14	15
0—3	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—
4—7	1	—	—	—	—	—	—	—	—	—	1	—	—	—	—
8—11	5	—	—	3	1	3	—	2	—	2	6	—	1	3	—
12—15	2	—	3	7	9	5	1	8	3	2	6	6	6	3	1
16—19	—	—	3	5	4	9	0	0	3	8	1	1	4	5	1
20—23	—	2	1	2	10	10	0	3	1	3	—	2	2	1	—
24—27	—	1	0	2	1	2	1	1	3	3	—	—	1	1	—
28—31	—	2	1	1	—	—	0	—	0	4	—	—	—	—	—
32—35	—	2	—	—	—	—	0	—	1	0	—	—	—	—	—
36—39	—	3	—	—	—	—	0	—	—	1	—	—	—	—	—
40—43	—	1	—	—	—	—	0	—	—	—	—	—	—	—	—
44—47	—	3	—	—	—	—	1	—	—	—	—	—	—	—	—
48—51	—	0	—	—	—	—	—	—	—	—	—	—	—	—	—
52—55	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—
56—59	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—
Means	10	37	18	17	18	18	28	16	20	21	11	18	16	16	16

Offspring. Percentage Grade of Spotting	II 6, 3	II 6, 1	II 1, 4	II 6, 4	II 4, 6	II 7, 11	II 4, 2	II 7, 12	II 5, 3	II 4, 12	II 4, 8	II 8, 3	II 6, 11, 6	II 1, 2, 20	II 1, 4, 17	II 1, 1, 3	II 1, 1, 2
Spotting of Parents	19	20	21	21	22	24	25	25	26	28	30	45	13	17	17	18	21
0—3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
4—7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
8—11	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	—	—
12—15	1	—	4	—	—	—	1	2	—	—	—	—	9	6	1	3	—
16—19	4	—	6	3	4	—	3	2	1	—	1	—	0	—	1	6	1
20—23	3	—	1	1	1	—	4	4	2	—	1	—	2	—	—	—	3
24—27	2	2	1	4	2	—	2	4	—	1	3	—	—	—	—	—	2
28—31	0	1	1	1	3	—	—	0	—	1	3	—	—	—	—	—	1
32—35	1	2	1	1	0	—	—	0	—	0	4	—	—	—	—	—	—
36—39	—	2	—	—	1	1	—	0	—	0	0	1	—	—	—	—	—
40—43	—	1	—	—	1	—	—	1	—	1	1	0	—	—	—	—	—
44—47	—	1	—	—	1	—	—	—	—	—	1	2	—	—	—	—	—
48—51	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
52—55	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
56—59	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Means	21	34	19	24	27	37	20	22	20	32	30	43	16	14	14	16	23

Offspring. Percentage Grade of Spotting	III = B_1 (48) × C_1 (11)	III 2	III 1	III 3	III 4	III 2, 1	III 2, 2	III 2, 5	III 2, 7	III 4, y	III 2, 13	III 2, 8	III 1, 11	III 1, [illegible]
Spotting of Parents		9	20	20	20	12	14	15	15	15	17	19	23	21
0—3	—	—	—	—	—	—	—	2	—	—	—	—	—	—
4—7	—	5	—	—	—	3	1	0	1	—	—	—	1	—
8—11	1	9	2	—	—	1	4	2	3	—	—	8	1	1
12—15	0	7	3	1	2	2	7	4	5	5	1	8	4	0
16—19	1	10	7	0	1	4	1	7	1	3	2	0	2	3
20—23	3	—	9	3	1	1	1	2	—	3	—	1	2	4
24—27	—	—	3	—	5	—	—	—	—	2	—	0	1	3
28—31	—	—	4	—	3	—	—	—	—	—	—	0	—	3
32—35	—	—	0	—	1	—	—	—	—	—	—	1	—	—
36—39	—	—	1	—	0	—	—	—	—	—	—	—	—	—
40—43	—	—	1	—	1	—	—	—	—	—	—	—	—	—
44—47	—	—	0	—	1	—	—	—	—	—	—	—	—	—
48—51	—	—	2	—	—	—	—	—	—	—	—	—	—	—
Means	18	12	23	20	27	13	13	14	12	18	16	13	16	23

Offspring. Percentage Grade of Spotting	III 2, 5, 18	III 2, 7, 9	III 2, 5, 5	III 4, y, 30	III 4, r, 27	III 2, 1, 18	III 2, 5, 10	III 4, r, 23	III 4, r, 5	III 4, r, 22	III 2, 5, 5, 12	III 2, 1, 18, 12	III 2, 1, 18, 10	III 2, 1, 18, 28	III 2, 5, 5, 11	III 2, 1, 18, 21	III 2, 5, 10, 22	III 2, 5, 10, 17
Spotting of Parents	2	6	10	13	18	19	22	25	27	29	11	16	17	19	19	20	26	27
0—3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
4—7	5	—	4	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—
8—11	6	1	2	—	—	—	—	—	—	—	—	2	—	3	—	—	—	—
12—15	1	7	3	4	—	5	2	—	—	—	2	1	—	6	—	1	—	—
16—19	—	2	3	4	2	0	1	—	1	1	4	1	—	5	—	0	—	—
20—23	—	—	—	2	0	6	2	—	0	0	2	4	1	4	1	3	—	—
24—27	—	—	—	1	1	1	4	1	4	2	2	4	3	6	4	1	1	—
28—31	—	—	—	—	1	—	1	2	2	2	—	3	1	3	2	1	0	1
32—35	—	—	—	—	—	—	1	2	0	2	—	3	4	1	3	3	0	1
36—39	—	—	—	—	—	—	—	2	2	2	—	2	1	1	1	0	0	1
40—43	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	1	1	3
44—47	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
48—51	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Means	8	14	11	18	23	19	25	32	28	31	19	25	30	21	29	28	38	39

during the different self-fertilised generations. Inspection of the tables shows that the distributions of the various families give no indication whatever of the occurrence of segregation into little spotted and much spotted plants. The gradual rise in the degree of spotting of the different parents is followed by a gradual increase in the spotting of the respective families obtained by self-fertilisation. The fact that the correlation between the grandparents and grandchildren is less than that between the parents and offspring is further evidence that the small, apparently fortuitous, variations in spotting occurring among self-fertilised generations are inherited. This result is opposed to the pure-line hypothesis, according to which such small variations are regarded as slightly different expressions of the same identical character which remains unchanged in its essence from one self-fertilised generation to another. If such were the case

Correlation Table—Spotting—Parents and Offspring. Series II and III.

Offspring. Grades of Spotting

Parents Grades of Spotting	48—51	44—47	40—43	36—39	32—35	28—31	24—27	20—23	16—19	12—15	8—11	4—7	0—3	Totals
0—3	—	—	—	—	—		—	—	—	1	6	8	—	15
4—7	—		—	—		—	—	—	2	7	1		—	10
8—11	—		—	—		—	2	2	18	18	17	10	1	68
12—15	—	—	—	—	—	1	8	28	40	68	19	5	2	171
16—19	—	1	1	5	14	10	28	34	33	49	20	1		196
20—23	2	3	5	4	9	16	25	26	25	17	3	1	—	136
24—27	—	2	5	6	3	8	15	14	10	3	1	—	—	67
28—31	—	1	3	3	6	10	9	1	10	2	2	—	—	50
32—35			—	—			—	—	—			—	—	0
36—39			—	—	—	—	—	—	—	—			—	0
40—43	—	—	—		—		—	—	—	—	—	—		0
44—47		2	—	1	—	—	—	—	—	—	—	—	—	3
Totals	2	9	14	19	32	45	87	108	138	165	69	25	3	716

Correlation Table—Spotting—Grandparents and Grandchildren. Series II and III.

Grandchildren. Grades of Spotting.

Grand parents. Grades of Spotting	44—47	40—43	36—39	32—35	28—31	24—27	20—23	16—19	12—15	8—11	4—7	0—3	Totals
8—11	—	—	1	1	2	6	8	19	35	19	5	2	101
12—15	1	1	2	5	4	17	21	20	31	10	12	—	124
16—19	2	3	6	12	9	20	22	25	37	15	2	1	151
20—23	2	4	1	1	4	7	9	15	13	3	1	—	60
24—27		1	6	4	7	8	0	4	—	—	—	—	30
28—31	2	3	1	4	7	8	6	8	1	—	—	—	40
Totals	7	12	17	30	33	66	66	91	117	47	20	3	509

the small variations would be fluctuating, non-inheritable variations; but the results in the present case are definitely against a supposition of this kind.

It might be urged by some that the result is really due to the existence of genotypes, and that variations within the limits of each genotype are not inheritable. The distributions of the families in the table do not indicate the occurrence of genotypes of any considerable magnitude. If the genotypes are supposed to be very small the practical result would become indistinguishable from the inheritance of continuous variations.

7. Ratio of Breadth to Length of Corolla.

The breadth was measured as the maximum horizontal width across the mouth of the corolla of a fully expanded flower in which the anthers had opened; the length was the maximum distance measured along the mid-adcauline surface with the lower lip stretched out straight in the long axis of the flower. It is convenient to express the ratio in the form, $\frac{\text{Breadth}}{\text{Length}}$ 1000. The mean of the ratios of the four lowest flowers of an axis was taken as the mean of the plant.

The original parent plants varied widely in this ratio, and the families raised by selfing tended to have the same ratio as their parents.

A plant bearing wide flowers was crossed with one having narrow flowers, and the offspring tended to be intermediate. On selfing these offspring the new generation exhibited, of course, considerable variation, but taken as a whole the intermediate condition was retained, and there was clearly no segregation into wide flowers and narrow flowers. Thus, the different degrees of this character blend readily on crossing, and the mode of inheritance is very similar to that of the spotted condition.

The results of a multitude of crossings of plants bearing variously shaped flowers have been carefully determined and tabulated, and there is no question about the general accuracy of the statement made above. In the present place we may confine our attention to the self-fertilised generations of Series II and III (p. 123).

A plant (♀ C_1) with relatively wide flowers (ratio 608) was crossed with a plant (♂ B_4) having relatively narrow flowers (ratio 487). The family (= II) had flowers approximately intermediate. The reciprocal cross = III. The distributions of the families of the various generations raised by selfing are shown in the accompanying table. The families of each generation are given in an ascending order of the ratios of the parents. As in the case of the character of spotting it will be seen that there is a clearly marked tendency for the mean ratios of the families to approximate to the ratios of the respective parents. In none of the families do we find any definite segregation into plants with wide flowers and plants with narrow flowers resembling those of the two progenitors of the series.

Wide and narrow races could be raised by selection using only self-fertilisation.

Thus in family III with a mean ratio of 531 there was a single plant (III 2) with as high a ratio as 575. This was selfed and the mean ratio of the offspring

Ratio of Flower—Families from selfed plants. Series II and III

Offspring. Grades of Ratios Width/Length × 1000	B_4 (selfed) 487	C_1 (selfed) 608	II = $C_1 \times B_4$	II 6	II 3	II 4	II 7	II 1	II 8	II 2	II 6, 4	II 6, 3	II 4, 2	II 4, 6	II 4, 12	II 7, 2	II 1, 2	II 6, 1	II 4, 8	II 7, 1	II 7, 12	II 7, 14	II 1, 4	II 1, 1	II 6, 11	II 8, 3	II 8, 2	II 1, 4, 3	II 1, 4, 17	II 1, 1, 2	II 1, 2, 20	II 6, 11, 6
Ratio				475	497	518	529	571	578	580	443	476	482	491	496	499	509	527	529	529	529	540	541	542	553	554	601	487	510	534	567	585
350—379	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—
380—409	—	—	—	1	1	—	—	—	—	—	1	—	—	1	—	—	—	0	—	—	1	—	—	—	—	—	—	—	—	—	—	—
410—439	—	—	—	1	1	—	—	—	—	—	2	—	1	1	3	—	—	1	—	1	0	—	2	—	—	—	—	—	—	—	—	—
440—469	2	—	—	2	2	4	—	—	—	2	5	1	5	1	—	1	2	3	1	5	0	—	0	—	2	—	—	—	—	—	—	—
470—499	5	—	2	9	4	10	1	4	—	2	3	3	1	2	—	0	7	3	4	4	4	—	4	5	6	1	—	3	—	—	—	2
500—529	2	2	2	3	7	5	4	5	—	5	—	7	2	6	—	1	1	1	6	1	6	1	4	3	2	0	—	5	—	2	3	1
530—559	1	2	1	7	2	1	3	9	2	1	—	—	1	2	—	—	1	—	3	0	0	—	3	1	3	2	1	1	—	1	4	0
560—589	—	2	3	1	3	2	2	8	0	4	—	—	—	—	—	—	3	—	—	1	2	—	1	—	2	—	1	—	2	3	—	1
590—619	—	1	—	—	—	—	1	1	1	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1	—	1	1	—	—
620—649	—	1	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
650—679	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Means	494	561	535	495	506	497	538	546	567	522	451	502	477	493	427	485	511	459	505	477	507	522	505	502	508	522	582	502	577	556	524	511

Offspring. Grades of Ratios Width/Length × 1000	III = $B_4 \times C_1$	III 3	III 4	III 1	III 2	III 4, x	III 4, y	III 2, 8	III 2, 1	III 2, 5	III 2, 2	III 2, 13	III 1, 11	III 2, 7	III 4, x, 27	III 4, x, 5	III 4, x, 22	III 4, x, 23	III 2, 5, 10	III 4, y, 30	III 2, 7, 9	III 2, 1, 18	III 2, 5, 5	III 2, 5, 18	III 2, 5, 10, 22	III 2, 5, 5, 11	III 2, 5, 10, 17	III 2, 1, 18, 12	III 2, 1, 18, 10	III 2, 5, 5, 12	III 2, 1, 18, 21	III 2, 1, 18, 28
Ratio		490	528	535	575	451	518	518	533	549	574	587	588	674	421	434	483	490	505	524	[illegible]	551	559	611	432	488	497	531	537	556	569	598
350—379	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—		—	—	—	—	—	—
380—409	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1		—	—	—	—	—		—	—	—	—	—	—
410—439	—	—	1	1	—	4	—	—	—	—	—	—	—	—	—	4	1	2	2	1	—	—	—	—	—	—	1	—	—	—	—	—
440—469	—	1	4	4	—	3	3	1	—	—	—	—	—	—	2	2	4	1	5	0	—	—	1	—	1	3	2	—	—	—	—	—
470—499	1	2	1	10	3	5	1	2	—	—	2	—	—	—	1	1	4	3	2	1	—	1	1	—	2	3	1	4	—	—	—	—
500—529	2	1	7	10	2	1	4	6	1	3	2	—	—	1	—	1	1	1	1	3	—	0	4	—	—	3	2	5	1	2	—	—
530—559	1	—	1	5	7	1	5	8	7	2	8	—	2	0	—	0	—	—	—	5	1	5	3	2	—	2	1	7	4	4	1	2
560—589	1	—	1	3	8	—	—	0	1	6	2	2	2	2	—	1	—	—	—	1	1	3	2	7	—	—	—	3	1	4	3	6
590—619	—	—	1	—	3	—	—	1	0	5	—	0	5	2	—	—	—	—	—	—	3	4	—	4	—	—	—	2	3	—	5	13
620—649	—	—	—	—	6	—	—	—	2	2	—	1	2	3		—	—	—	—	—	3	—	—	2	—	—	—	1	1	—	1	7
650—679	—	—	—	—	2	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	2	—	—	—	—	—	—	1	—	—	—	2
680—709	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Means	531	481	506	507	574	465	509	526	563	579	538	592	592	614	464	469	474	468	440	522	618	561	526	587	477	499	488	550	573	547	592	606

was 574. In this family there was a plant (III 2, 1) with a ratio of 533 and the mean of offspring = 563. III 2, 1, 18 (ratio 551) produced a family with mean 561, and III 2, 1, 18, 28 (ratio 598) produced a family with a mean of 606.

In the reverse direction, through III 2, III 2, 5, III 2, 5, 10 and III 2, 5, 10, 22 we pass from a parent of ratio 575 to a family having a mean ratio of 477.

With the data given in the preceding table, correlation tables have been prepared for parents and offspring, and grandparents and grandchildren.

Correlation Table—Ratios of Corolla—Parents and Offspring.
Series II and III.

Offspring.

Parents. Grades of Ratios $\frac{B}{L}$ 1000	680—709	650—679	620—649	590—619	560—589	530—559	500—529	470—499	440—469	410—439	380—409	350—379	Totals
410—439	–	–	–	–	1	0	1	4	5	4	–	—	15
440—469		—	-	—	—	1	1	8	8	6	1	–	25
470—499		—	-		4	16	39	35	23	11	3	1	132
500—529	—	2	3	8	15	28	45	40	28	6	2	1	178
530—559		1	6	15	29	47	38	32	7	3	—	—	178
560—589	—	2	12	15	30	34	18	13	2		—	—	126
590—619	1	2	9	18	14	5	—	—			—	—	49
620—649	—	—	—	—			–	—	—	—	—	—	0
650—679		2	3	2	2	0	1	--	—	—	—		10
Totals	1	9	33	58	95	131	113	132	73	30	6	2	713

Correlation Table—Ratios of Corolla—Grandparents and Grandchildren.
Series II and III.

Grandchildren.

Grandparents. Grades of Ratios $\frac{B}{L}$ 1000	680—709	650—679	620—649	590—619	560—589	530—559	500—529	470—499	440—469	410—439	380—409	350—379	Totals
440—469	—	—	-	—	1	0	3	9	9	7		--	29
470—499	—	—		—	2	3	10	15	11	3	1	1	46
500—529	-	—			4	22	36	25	22	12	2	—	123
530—559	1	3	11	38	37	34	24	16	9	2	1	—	179
560—589	—	2	8	10	18	33	21	21	3	2	-	—	118
590—619	—	—	—	—	—	—	—	—	—	—	—	—	0
620—649	—		—		—		—	—	—	—	—		0
650—679	—	2	3	3	1	1	—	—	—	—	—	—	10
Totals	1	7	25	51	63	93	94	86	54	26	4	1	505

The coefficients of correlation are ·601 for parents and offspring and ·492 for grandparents and grandchildren. The latter figure is somewhat high; but taking the results altogether they are incompatible with any notion of pure-lines.

8. GENERAL CONCLUSIONS.

In the various characters that have been dealt with in the crossing of different strains of the garden foxglove we have seen that in pelorism, colour of corolla and colour of spots, the mode of inheritance is Mendelian with reference to the qualities: peloric and non-peloric, purple and white corolla, purple spots and brown spots. If, however, there are any marked differences in the intensities of these qualities, the mode of inheritance of the intensity of the quality was found to be of the blended type.

The other characters examined were quantitative in nature, such as degree of the development of purple spots and the ratio of breadth to length of corolla, and these characters blended completely.

When the intensity of a quality is very slight and approaching zero the difficulty arises as to which category the individual should be referred. When Mendelian inheritance is in evidence the critical point may apparently be determined by the occurrence of segregation. Thus, if a homozygous plant with a very faint tinge of purple (say an intensity of about 4) is crossed with a homozygous strongly coloured plant, segregation occurs in the so-called F_2 generation, and we obtain on the average 1 faintly tinged plant to 3 much more darkly coloured plants. When, however, the pale plant has a somewhat greater intensity (say about 10), the F_2 and subsequent generations are intermediate, and definite segregation does not occur. In accordance with this procedure a plant with flowers having an intensity of general coloration which did not reach 5 of the scale was classed as "white." Without employing such a line of demarcation the results obtained were wholly unintelligible.

From the strict Mendelian standpoint, in the example given above, it would probably be affirmed that the faint tinge of purple on "white" flowers is not really a fractional part of the general purple coloration of coloured plants, but is a distinct character governed by a different factor or set of factors in the chromosomes. To one who has grown the plants this view appears an artificial one. In my previous account I stated that there appeared to be a distinct gap among my plants between "white" plants and coloured plants, and that colorations of about 8–25 of the scale were extremely rare or almost absent, but I have subsequently obtained a number of plants having such intensities of coloration, passing imperceptibly down to absolute whiteness. Consequently it is quite unlikely that the faint tinge of purple on "white" flowers is anything else than the last remnant of a general purple coloration.

It is quite similar in the character of pelorism, but the difficulty in finding a suitable method of measuring this character renders the matter less obvious. Thus, it would appear that if a character is not present beyond a certain minimum or unit quantity it may be unable to blend on crossing with a plant possessing the character in a well-marked degree.

With reference to the characters which blend, the accompanying table summarizes the results obtained for parental correlation. Mid-parents and self-fertilised parents are regarded as comparable.

Character	Number of Offspring	Coefficient of Correlation. Parents and Offspring
Intensity of pelorism (homozygous recessive, mid-parents and self-fertilised parents)	530	·520
Intensity of general purple coloration (homozygous dominants, self-fertilised parents)	529	·707
Seed-length (self-fertilised parents)	46	·378
Spotting (self-fertilised parents)	716	·560
Ratio of Corolla (self-fertilised parents) ...	713	·601

The probable errors of these results are reasonably small and the average coefficient for the 5 characters is ·553 which is not far removed from the average coefficient found by Professor Karl Pearson for a large number of characters in a variety of different organisms.

It must be again emphasized that these results are based on self-fertilised generations of pedigree plants of known gametic constitution, and on Johannesen's theory of pure-lines these parental coefficients should be zero, or at least very small.

The evidence of the present investigation is therefore definitely against any general application of the theory of pure-lines and of genotypes of any appreciable magnitude, and further it indicates that selective breeding within self-fertilised generations of a homogeneous race is capable of modifying that race to a marked degree

EXPLANATION OF PLATE I.

Figs. 1 and 2.—Pelorism of maximum intensity; grade 100°. Corollas absent, sessile anthers.

Figs. 3 and 4.—Perfect pelorism, grade 100°. Corollas joined along their split edges forming a complete saucer. Stamens with filaments.

Fig. 5.—Peloric flower of side-axis; the axis terminates in an ovary.

Fig. 6. -Pelorism of grade 100°. Numerous flowers fused irregularly forming a rosette, the axis has grown through the crown.

Figs 7 and 8.—Incomplete pelorism of main axes, grade 75°. A spiral bending often occurs.

Fig. 9.—Faintly defined pelorism. When such occurred on the lateral axes the plant was said to possess a grade of 25°. Side view, and view from above.

Fig. 10.—Flowering axis of a conspicuous sport in which practically all the corollas are completely split longitudinally into four elongated blades. Nature of inheritance obscure.

The photographs were kindly taken by Dr Conrad Akerman.

ON POLYCHORIC COEFFICIENTS OF CORRELATION.

By KARL PEARSON, F.R.S. and EGON S. PEARSON.

(1) One of the difficulties which are constantly recurring in statistical practice is that of the correlation or contingency table in which the two variates are classified in broad categories. We may indeed proceed by the method of mean square contingency and correct for the grouping of both variates by the class index corrections on the assumption that the marginal totals for both variates may be assumed to follow approximately normal distributions. Such a procedure gives reasonable satisfactory results*, provided the marginal totals are not in very unequal groupings and the correlation is not intense (say, ·85 and above). The polychoric table has been discussed by Ritchie-Scott and he has described a method of reaching a polychoric coefficient of correlation from the weighted mean of the possible tetrachoric values†. Such a process is, however, so laborious that it can hardly establish itself in practice. From the theoretical standpoint, however, Ritchie-Scott's paper was of great interest (i) as guiding us by the size of the probable errors to discriminate between the valuable and worthless dichotomies in tetrachoric determinations of the correlation‡, (ii) as providing standard values by which those obtained by other procedures could be directly tested.

We shall endeavour to reach in this paper another form of polychoric coefficient,—that is a correlation coefficient which does use all the information given in a polychoric table,—but which requires less analysis than Ritchie-Scott's weighted mean coefficient. Thus what may be lost in exactness will possibly be repaid by practical efficiency. There is another point also of very considerable illustrative importance; we desire wherever the data are suitable actually to exhibit in the form of a graph the relation between the two variates. This should be possible in the case of a polychoric table, and in the past has frequently been done by approximate methods of more or less validity.

We can indeed take such methods as our present starting point as they will directly indicate to the reader our line of approach.

We start with the hypothesis that the marginal totals of our polychoric table can be represented on a normal scale. This is no great assumption in itself. If a true quantitative scale ever becomes available it can be attached at once and with little trouble to the normal scale. To exhibit a variate on a normal scale makes

* By "reasonably satisfactory results," we mean that in cases which can be directly checked by the product moment method the difference is within the range of practical insignificance as judged by probable error.

† *Biometrika*, Vol. XII. pp. 93—133.

‡ Thus in a 3×3 table it is possible for two of the corner dichotomies, i.e. those unassociated with the diagonal in the sense of the correlation, to have even *negative* weights, so that they should be omitted in finding the mean.

no greater assumption than when we exhibit a pressure-volume curve as a straight line by using a logarithmic scale.

Now let the polychoric table be such that in the population N under discussion, the sth category of the first variate A contains $n_{s\cdot}$ individuals and the s'th category of the second variate B contains $n_{\cdot s'}$ individuals, while the number of individuals who combine in the population N the sth category of A and the s'th category of B is $n_{ss'}$.

Now when we proceed to exhibit the categories of the A-variate on a normal scale, the process will give us two important quantities:

(*a*) We shall have the ratio of abscissa to standard deviation at the dichotomy between each pair of broad categories.

If $n_{1\cdot}, n_{2\cdot}, n_{3\cdot}, \dots n_{s\cdot}, \dots$ be the frequencies of the A-variate for the several categories the values of the ratios of abscissae to standard deviation will be specified as

$$-\infty, \quad h_1, \quad h_2, \quad h_3, \quad h_4, \dots h_{s-1}, \quad h_s, \dots +\infty.$$

Here h_{s-1}, h_s are the values on either side of the category $n_{s\cdot}$ and if there be q categories, $n_{\cdot 1}$ is bounded by h_0 or $-\infty$ and h_1, while $n_{\cdot q}$ is bounded by h_{q-1} and h_q or $+\infty$. The lower h's will have negative and the upper positive signs and the greatest care must be taken to see that the proper signs are given to the values of h. Similarly if the frequencies of the various categories of the B-variate be

$$n_{\cdot 1}, \quad n_{\cdot 2}, \quad n_{\cdot 3}, \dots n_{\cdot s'}, \dots,$$

the values of the ratios of ordinates to standard deviation will be represented by

$$-\infty, \quad k_1, \quad k_2, \quad k_3, \dots k_{s'-1}, \quad k_s, \dots k_{q'}, +\infty,$$

where $k_{s'-1}$ and $k_{s'}$ give the dichotomies on either side of $n_{\cdot s'}$.

We may consider the coordinate at the back of the variate A when represented on a normal scale to be x', the origin being taken at the mean on the normal scale. Hence if the standard deviation be σ_x, we shall find it convenient to write the absolute normal abscissae

$$x' = \sigma_x x, \quad h_s' = \sigma_x h_s.$$

Similarly we take y' for the coordinate at the back of the variate B, measured from the mean, and write:

$$y' = \sigma_y y, \quad k_s' = \sigma_y k_s,$$

where σ_y is the standard deviation of B. Clearly until a quantitative scale has been determined we shall know h, k, x, y but not h', k', x', y', σ_x and σ_y.

(*b*) We shall determine the ratio of abscissa to standard deviation, or the ratio of ordinate to standard deviation of the centroids or means of the groups $n_{s\cdot}$ and $n_{\cdot s'}$.

Let $$H_s = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h_s^2}, \qquad K_{s'} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k_{s'}^2},$$

then the means of the categories $n_{\cdot s}$ and $n_{s\cdot}$ are determined by

$$\bar{h}_s = (H_{s-1} - H_s)\Big/\frac{n_{s\cdot}}{N}, \qquad \bar{k}_{s'} = (K_{s-1} - K_s)\Big/\frac{n_{\cdot s'}}{N} \quad \dots\dots\dots\dots\dots(\text{i})$$

respectively. The numerical values of $\bar{h}_s$ and $\bar{k}_{s'}$ can be easily ascertained from the table published recently of ordinates of normal curve to permilles of area*. Care must be taken in every case to give the correct sign to $\bar{h}_s$ and $\bar{k}_{s'}$.

Now if there were no correlation, $\bar{h}_s$ and $\bar{k}_{s'}$ combined would give the mean of the group $n_{ss'}$, and they give a fair approximation to the result if there are numerous categories, that is if the range of the categories be small.

The correlation found from these marginal centroids would then be

$$r_c = S\,(n_{ss'}\bar{h}_s\bar{k}_{s'})/N \text{ ....(ii)},$$

but as Ritchie-Scott has shown† this r_c diverges much more than r_ϕ the mean square contingency value from the true correlation, and considerably more than the tetrachoric or polychoric coefficients do. The reason for this is clear and was pointed out by one of us in 1913‡. Namely $\bar{h}_s$ and $\bar{k}_{s'}$ do not give the coordinates of the mean of $n_{ss'}$. In fact $n_{ss'}\bar{h}_s\bar{k}_{s'}$ is not the contribution of $n_{ss'}$ to the product-moment.

We propose in the present paper to give first the actual contributions of $n_{ss'}$ to the means and product-moments of the two variates and then to apply these results in order to obtain (*a*) a polychoric coefficient, and (*b*) a graph of the relation of the two variates.

The essential assumptions that will be made are the following:

(i) The marginal totals having been reduced to a normal scale, and the correlation being supposed to be r, we shall calculate what the contents of the sth-s'th cell would be on the assumption that the frequency surface is the normal surface represented by the given correlation and the marginal totals reduced to normal scales. We shall further calculate the x-moment, the y-moment and the xy product-moment of the sth-s'th cell on the same hypothesis.

(ii) From these data we shall determine the most suitable value to give to r, so that the actually observed frequencies differ least from those that would be given by such a correlation surface. We shall also obtain a formula for calculating the mean value of y for the array of B-variates, $n_{s.}$ in number, which corresponds to the sth category of A. We shall thus be in a position to plot the regression line of B on A and test at the same time the closeness with which it fits the thus calculated array means, both variates being represented on a normal scale.

We shall write the real coefficient of correlation of the population r, the coefficient as found from a single sth-s'th cell, as $r_{ss'}$, and those found from the $n_{s.}$ and $n_{.s'}$ arrays as $r_{s.}$ and $r_{.s'}$ respectively.

$\bar{h}_{ss'}$, $\bar{k}_{ss'}$ will be the A- and B-variate means of the sth-s'th cell and $\pi_{ss'}$ the product-moment, per unit of the population, of the frequency in the sth-s'th cell about the mean axes as determined from the marginal totals on the normal scale.

* See *Biometrika*, Vol. XIII. pp. 426–8.
† *Biometrika*, Vol. XII. p. 122.
‡ *Biometrika*, Vol. IX. p. 138.

(2) The developments we require involve the use of the tetrachoric functions. The tetrachoric function of the order t is given by*

$$\tau_t = \frac{1}{\sqrt{t\,!}}\left(-\frac{d}{dx}\right)^{t-1} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} \quad \text{..........................(iii).}$$

The tetrachoric functions τ_1 to τ_6 are tabled for *positive* values of x in *Tables for Statisticians and Biometricians*† to five decimal places. For *negative* values of x tetrachoric functions of an odd order remain unchanged, but those of an even order must have their signs given in the tables reversed.

It will frequently be needful to take the difference of the tetrachoric functions at the boundaries of a marginal category. Thus if $\tau_t(h)$ denotes the value of the tetrachoric function for $x = h$, we shall need for the sth marginal total

$$\tau_t(h_s) - \tau_t(h_{s-1}).$$

This difference we shall write, for brevity,

$$\vartheta_s \tau_t,$$

and in obtaining its numerical value from tables of the tetrachoric functions it is essential to remember that s (or s') is supposed to increase in the positive direction of the axis of x (or y), and that when h (or k) is negative attention must be paid to changing the sign of the tabled value of τ_t, if t be even.

The formula for determining the successive tetrachoric functions for a given value of x is

$$\tau_t = xp_t\tau_{t-1} - q_t\tau_{t-2} \quad \text{........................(iv),}$$

where p_t and q_t are given by the following table:

t	p_t	q_t	t	p_t	q_t
2	·707,1068	·000,0000	14	·267,2612	·889,4990
3	·577,3503	·408,2483	15	·258,1989	·897,0851
4	·500,0000	·577,3503	16	·250,0000	·903,6962
5	·447,2136	·670,8204	17	·242,5356	·909,5085
6	·408,2483	·730,2968	18	·235,7023	·914,6592
7	·377,9645	·771,5168	19	·229,4157	·919,2547
8	·353,5534	·801,7838	20	·223,6068	·923,3804
9	·333,3333	·824,9578	21	·218,2179	·927,1051
10	·316,2278	·843,2740	22	·213,2007	·930,4842
11	·301,5113	·858,1163	23	·208,5144	·933,5637
12	·288,6751	·870,3880	24	·204,1241	·936,3819
13	·277,3501	·880,7047	25	·200,0000	·938,9709

Since $\tau_1 = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}$, it can be found at once from the tables for the ordinates of the normal curve, and will indeed have been computed at each division in order

* The reasons why the tetrachoric functions are tabled with the factor $1/\sqrt{t\,!}$ are: (*a*) because this factor greatly simplifies our formulae and (*b*) because a factor of some such order is essential, if we are to have manageable tabulated values. As a matter of fact the factor chosen reduces all tetrachoric functions to numerical values lying between 0 and 1.

† Cambridge University Press, pp. 42–51.

to determine $\bar{h}_{s\cdot}$ and $h_{\cdot s'}$. It is then often simpler to work directly with (iv) rather than interpolate into the tabled values of the functions.

In an earlier paper* dealing with the tetrachoric functions one of us has shown that if

$$z = \frac{N}{2\pi\sqrt{(1-r^2)}} e^{-\frac{1}{2}\frac{x^2-2rxy+y^2}{1-r^2}}$$

be the equation to a normal correlation surface the variates being measured in the standard deviations as units, then

$$z/N = \tau_1\tau_1' + 2r\tau_2\tau_2' + 3r^2\tau_3\tau_3' + \ldots + (t+1)\,r^t\tau_{t+1}\tau'_{t+1} + \ldots \quad \ldots\ldots(\text{v}),$$

where $\tau_t = \tau_t(x)$ and $\tau_t' = \tau_t(y)$.

Now in order to proceed further it is needful to determine the following integrals:

$$\int_{h_{s-1}}^{h_s} \tau_t\,dx, \qquad \int_{h_{s-1}}^{h_s} x\tau_t\,dx.$$

We can determine these by using (iii) after in the second case integrating by parts. We have:

$$\int_{h_{s-1}}^{h_s} \tau_t\,dx = \frac{1}{\sqrt{t\,!}}\int_{h_{s-1}}^{h_s}\left(-\frac{d}{dx}\right)^{t-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx$$

$$= \left[-\frac{1}{\sqrt{t\,!}}\left(-\frac{d}{dx}\right)^{t-2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right]_{h_{s-1}}^{h_s}$$

$$= -\frac{1}{\sqrt{t}}\,\vartheta_s\tau_{t-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{vi}).$$

Again:

$$\int_{h_{s-1}}^{h_s} x\tau_t\,dx = \int_{h_{s-1}}^{h_s}\frac{1}{\sqrt{t\,!}}x\left(-\frac{d}{dx}\right)^{t-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx$$

$$= \left[\frac{1}{\sqrt{t\,!}}(-x)\left(-\frac{d}{dx}\right)^{t-2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right]_{h_{s-1}}^{h_s}$$

$$+ \frac{1}{\sqrt{t\,!}}\int_{h_{s-1}}^{h_s}\left(-\frac{d}{dx}\right)^{t-2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx$$

$$= -\left[\frac{1}{\sqrt{t}}\tau_{t-1}x\right]_{h_{s-1}}^{h_s} + \frac{1}{\sqrt{t}}\int_{h_{s-1}}^{h_s}\tau_{t-1}\,dx$$

$$= \left[-\frac{1}{\sqrt{t}}\tau_{t-1}x - \frac{1}{\sqrt{t(t-1)}}\tau_{t-2}\right]_{h_{s-1}}^{h_s}$$

$$= -\frac{1}{\sqrt{t}}\left[\tau_{t-1}x + \frac{1}{\sqrt{t-1}}\tau_{t-2}\right]_{h_{s-1}}^{h_s} \quad \ldots\ldots\ldots\ldots\ldots(\text{vi})\text{ bis}.$$

But by (iv):

$$\tau_{t-1}x = \frac{\tau_t + q_t\tau_{t-2}}{p_t},$$

where $\quad p_t = 1/\sqrt{t}. \quad q_t = (t-2)/\sqrt{t(t-1)}.$

* *Phil. Trans.* Vol. 195 A, p. 4, Equation (XIV), with a slight change of notation. In that paper, $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}v_n\frac{1}{\sqrt{(n+1)\,!}}$ is written for τ_{n+1} and $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\frac{w_n}{\sqrt{(n+1)\,!}}$ for τ'_{n+1}.

Thus: $$\tau_{t-1}x + \frac{1}{\sqrt{t-1}}\tau_{t-2} = \sqrt{t}\,\tau_t + \sqrt{t-1}\,\tau_{t-2}.$$

Accordingly
$$\int_{h_{s-1}}^{h_s} x\tau_t\,dx = -\frac{1}{\sqrt{t}}\left[\sqrt{t}\,\tau_t + \sqrt{t-1}\,\tau_{t-2}\right]_{h_{s-1}}^{h_s}$$
$$= -\frac{1}{\sqrt{t}}\left(\sqrt{t}\,\vartheta_s\tau_t + \sqrt{t-1}\,\vartheta_s\tau_{t-2}\right) \quad\text{..............(vii).}$$

The latter form throws us back on $\vartheta_s\tau_t$ which will have to be calculated to determine the integral in (vi) for the successive values of t and s.

On the other hand a table of
$$T_{t-1} = \sqrt{t}\,\tau_t + \sqrt{t-1}\,\tau_{t-2} \quad\text{..........................(viii)}$$
would be a convenient method of determining the integral and tables of T might be easily formed, say up to T_6.

In this case we may write (vii):
$$\int_{h_{s-1}}^{h_s} x\tau_t\,dx = -\frac{1}{\sqrt{t}}\vartheta_s T_{t-1} \quad\text{..........................(ix).}$$

We are now in the position to compute all the requisite integrals we need; if we write $\bar{n}_{ss'}$ for the contents of the sth-s'th cell, then on the supposition that the surface is normal, has correlation r and follows the actual marginal frequencies, we have:
$$\frac{\bar{n}_{ss'}}{N} = \int_{h_{s-1}}^{h_s}\int_{k_{s-1}}^{k_s}\frac{z}{N}\,dx\,dy$$
$$= \vartheta_s\tau_0\vartheta_{s'}\tau_0' + r\vartheta_s\tau_1\vartheta_{s'}\tau_1' + r^2\vartheta_s\tau_2\vartheta_{s'}\tau_2' + \dots + r^p\vartheta_s\tau_p\vartheta_{s'}\tau_p' + \dots \quad\text{(x),}$$
$$\frac{\bar{n}_{ss'}}{N}\bar{h}_{ss'} = \int_{h_{s-1}}^{h_s}\int_{k_{s-1}}^{k_s}\frac{xz\,dx\,dy}{N} = \vartheta_s T_0\vartheta_{s'}\tau_0' + r\vartheta_s T_1\vartheta_{s'}\tau_1' + r^2\vartheta_s T_2\vartheta_{s'}\tau_2'$$
$$+ \dots + r^p\vartheta_s T_p\vartheta_{s'}\tau_p' + \dots \quad\text{.......... (xi),}$$
$$\frac{\bar{n}_{ss'}}{N}\bar{k}_{ss'} = \int_{h_{s-1}}^{h_s}\int_{k_{s-1}}^{k_s}\frac{yz\,dx\,dy}{N} = \vartheta_s\tau_0\vartheta_{s'}T_0' + r\vartheta_s\tau_1\vartheta_{s'}T_1' + r^2\vartheta_s\tau_2\vartheta_{s'}T_2'$$
$$+ \dots + r^p\vartheta_s\tau_p\vartheta_{s'}T_p' + \dots \quad\text{.......... (xii),}$$
$$\frac{\bar{n}_{ss'}}{N}\pi_{ss'} = \int_{h_{s-1}}^{h_s}\int_{k_{s-1}}^{k_s}\frac{xyz\,dx\,dy}{N} = \vartheta_s T_0\vartheta_{s'}T_0' + r\vartheta_s T_1\vartheta_{s'}T_1' + r^2\vartheta_s T_2\vartheta_{s'}T_2'$$
$$+ \dots + r^p\vartheta_s T_p\vartheta_{s'}T_p' + \dots \quad\text{.........(xiii).}$$

It is desirable to say a few words about the functions τ_0 and T_0 which may at first present difficulties to the reader. $-\tau_0$ clearly stands for the integral
$$\int_{h_{s-1}}^{h_s}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,dx, \text{ i.e. } \int_{h_{s-1}}^{h_s}\tau_1\,dx,$$
and is therefore simply $n_{s\cdot}/N$.

Similarly $-\tau_0' = n_{\cdot s'}/N$.

Next clearly $-\vartheta_s T_0$ stands for

$$\int_{h_{s-1}}^{h_s} \tau_1 x\,dx = \int_{h_{s-1}}^{h_s} \frac{1}{\sqrt{2\pi}} x e^{-\frac{1}{2}x^2} dx$$

$$= -\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\right]_{h_{s-1}}^{h_s}$$

$$= -\vartheta_s \tau_1,$$

or

$$\vartheta_s T_0 = \vartheta_s \tau_1$$

which is precisely the value given by (viii).

Thus (viii) is shown to be correct even for this special case although a form like (vi) bis through which it is reached shows difficulties.

Similarly $\vartheta_s T_0' = \vartheta_s \tau_1'$ *.

The remainder of the τ's knowing τ_0 and τ_1 come directly from (iv) and the T's are always given by (viii).

Now it is clear that (x) to (xiii) provide a large number of ways of determining r. We might find r, i.e. $r_{ss'}$, from the single cell by writing in (x) $n_{ss'}$ for $\bar{n}_{ss'}$. Or we may find

$$\bar{h}_{s\cdot} = \frac{1}{n_{s\cdot}} \underset{s'}{S} (n_{ss'} \bar{h}_{ss'})$$

$$= \frac{N}{n_{s\cdot}} \underset{s'}{S} \left(\frac{n_{ss'}}{\bar{n}_{ss'}} \{\vartheta_s T_0 \vartheta_{s'} \tau_0' + r \vartheta_{s'} T_1 \vartheta_{s'} \tau_1' + \ldots + r^p \vartheta_s T_p \vartheta_{s'} \tau_p' + \ldots\}\right) \quad \text{(xiv)},$$

where $\bar{n}_{ss'}$ is given by (x). But $\bar{h}_{s\cdot}$ is the known centroid of the $n_{s\cdot}$ marginal total, and accordingly the above is an equation to find r, i.e. $r_{s'}$, from a given column of the table.

If we use this value of $r_{s\cdot}$ in (x) and (xii) to find $\bar{n}_{ss'}$, and $\bar{k}_{ss'}$, we obtain the theoretical cell frequency and y-mean of the cell as found from a column.

Now sum $\bar{k}_{ss'}$ for every value of s' and we find $\bar{k}_{s\cdot}$ the y mean of a column depending on the data as found from the column, i.e.

$$\bar{k}_{s\cdot} = \frac{N}{n_{s\cdot}} \underset{s'}{S} \left(\frac{n_{ss'}}{\bar{n}_{ss'}} \{\vartheta_s \tau_0 \vartheta_{s'} T_0' + r \vartheta_s \tau_1 \vartheta_{s'} T_1' + \ldots + r^p \vartheta_s \tau_p \vartheta_s T_p' + \ldots\}\right) \quad \text{(xv)},$$

where $n_{ss'}$ is the observed cell frequency and $\bar{n}_{ss}$ the frequency found by (x) when we insert the value of r as found from (xiv). We are thus in a position theoretically to determine on a normal scale the mean of a column from the correlation actually determined from that column. This would be the ideal method of determining the mean of a row or column; but it would involve a great deal of hard work, as with the two regression curves we should need to find r for every row and column by an equation of a high order. Hence in most cases we are likely to content ourselves by finding r for the whole table and then use this value in (x) to determine $\bar{n}_{ss'}$ and in (xv) to find the mean of the array. $\bar{k}_{s\cdot}$ plotted to the known $\bar{h}_{s\cdot}$ on the normal scale will give the regression curve.

* We can thus take $T_0 = \tau_1$ and $T_0' = \tau_1'$.

The question now arises as to the manner in which we can find r for the whole table most effectively.

Clearly we might assume the product-moment components from (xiii) and sum for all cells. We should have

$$\sum_{s, s} \left(\frac{n_{ss'} \pi_{ss}}{N}\right) = r,$$

since the coordinates are measured from the means in terms of the standard deviations as units.

Hence substituting from (xiii) we have:

$$r = \underset{s, s}{S} \left(\frac{n_{ss'}}{\bar{n}_{ss}} \{\vartheta_s T_0 \vartheta_{s'} T_0' + r \vartheta_s T_1 \vartheta_{s'} T_1' + \ldots + r^p \vartheta_s T_p \vartheta_{s'} T_p' + \ldots\}\right) \quad \text{(xvi)}.$$

Here $\bar{n}_{ss}$ must be substituted from (x) and we have finally

$$r = \underset{s, s}{S} \left(\frac{n_{ss'}}{N} \left\{\frac{\vartheta_s T_0 \vartheta_s T_0' + r \vartheta_s T_1 \vartheta_s T_1' + \ldots + r^p \vartheta_s T_p \vartheta_s T_p' + \ldots}{\vartheta_s \tau_0 \vartheta_{s'} \tau_0' + r \vartheta_s \tau_1 \vartheta_s \tau_1' + \ldots + r^p \vartheta_s \tau_p \vartheta_{s'} \tau_p' + \ldots}\right\}\right) \quad \text{(xvi) bis}.$$

This equation based upon the product-moment method of finding r is clearly likely to be very complicated, and although it can be proved that the product-moment method is the "best" method of finding r when we are dealing with a series of quantitatively measured individuals, we have no certainty that it is the best method in the present case of broad categories. It may indeed be questioned whether another method now to be considered cannot be shown to be better or at least equally efficacious.

Let us consider for a moment what we have in view. We observe n_{ss} as the frequency of the sth-s'th cell; we find that with a given correlation r the frequency of this cell would be $\bar{n}_{ss'}$ on the assumption that the frequency surface is the normal frequency surface corresponding to the observed marginal totals. Accordingly the most probable value to give to r would be that which made

$$\chi^2 = \underset{s}{S} \frac{(\bar{n}_{ss'} - n_{ss'})^2}{\bar{n}_{ss'}} = \text{minimum},$$

or, what is the same thing,

$$S\left(\frac{n^2_{ss'}}{\bar{n}_s}\right) = \text{minimum}.$$

This leads us, differentiating with regard to r, to

$$\underset{s, s'}{S} \left\{\left(\frac{n_{ss}}{\bar{n}_{ss'}}\right)^2 \frac{d\bar{n}_{ss'}}{dr}\right\} = 0,$$

or, writing at length, our equation for r is:

$$\underset{s, s}{S} \left\{\left(\frac{n_{ss}}{N}\right)^2 \frac{\vartheta_s \tau_1 \vartheta_{s'} \tau_1' + 2r \vartheta_s \tau_2 \vartheta_s \tau_2' + \ldots + pr^{p-1} \vartheta_s \tau_p \vartheta_{s'} \tau_p' + \ldots}{(\vartheta_s \tau_0 \vartheta_{s'} \tau_0' + r \vartheta_s \tau_1 \vartheta_{s'} \tau_1' + r^2 \vartheta_s \tau_2 \vartheta_{s'} \tau_2' + \ldots + r^p \vartheta_s \tau_p \vartheta_{s'} \tau_p' + \ldots)^2}\right\} = 0 \quad \text{(xvii)}.$$

Neither (xvi) nor (xvii) are very readily solved. Probably the easiest way will be to obtain an approximate value of r by existing methods either from a good fourfold table, or from contingency, and then evaluate (xvi) or (xvii) for values of r, one well above and one well below this result, so that the real value of r lies

between the two. A linear interpolation will probably suffice in most cases to determine r with sufficient accuracy.

It will be observed that what we are trying to do is to fit a normal correlation surface to a series of cell frequencies. We may do this by equating product-moments, or actual cell frequencies properly weighted. The factors $\frac{n_{ss'}}{\bar{n}_{ss'}}$ and $\left(\frac{n_{ss'}}{\bar{n}_{ss'}}\right)^2$ come into our equations as a form of weights. When $n_{ss'}$ is small as compared with $\bar{n}_{ss'}$ that cell will contribute less to the general equations for r, and when $n_{ss'}$ is large as compared with $\bar{n}_{ss'}$, the contribution will be considerable. If the observed results were closely normal then $n_{ss'}$ would be nearly $\bar{n}_{ss'}$. If we might assume the differences of $n_{ss'}$ and $\bar{n}_{ss'}$ so small as to be negligible we should have:

$$r = S\,(\vartheta_s T_0 \vartheta_{s'} T_0' + r \vartheta_s T_1 \vartheta_{s'} T_1' + \ldots + r^p \vartheta_s T_p \vartheta_{s'} T_p' + \ldots \qquad \text{(xvi) ter,}$$

and

$$0 = \underset{s,s'}{S}\,(\vartheta_s \tau_1 \vartheta_{s'} \tau_1' + 2r \vartheta_s \tau_2 \vartheta_{s'} \tau_2' + \ldots + p r^{p-1} \vartheta_s \tau_p \vartheta_{s'} \tau_p' + \ldots \qquad \text{(xvii) bis,}$$

instead of (xvi) bis and (xvii). These equations it will be found are identically satisfied. Hence our values for r from (xvi) and (xvii) depend on $\bar{n}_{ss'}$ differing from $n_{ss'}$.

(3) We now proceed to illustrate the application of these results.

Stature of Father and Son.

The following table gives a correlation table for the inheritance of stature in Father and Son made up in broad categories corresponding to eye-colour groups*. Upon this material we shall be able to test our correlations and our graph against those found by definite numerical groupings

Stature of Father (Broad Categories).

Stature of Son Broad Categories	*1*	*2*	*3*	*4*	*5*	*6*	*7*	Totals
1'	4	22	7	—	1	—	—	34
2'	23	154	84	26	8	6	–	301
3'	8	87	75	66	22	24	2	284
4'	1	29	36	37	14	14	6	137
5'		18	27	26	11	18	5	105
6'	—	9	26	19	7	29	8	98
7'		3	9	6	6	10	7	41
Totals	36	322	264	180	69	101	28	1000

The positive direction of x is from left to right and of y vertically downwards. It will suffice to take the τ's to five decimal figures but it will be needful to go further with the τ's if the T's are to be taken correctly to five figures from (viii). The general reduction formula for the T's is:

$$T_t(x) = \frac{T_{t-1}(x)(t-2)\sqrt{t-1}\,x^3 - T_{t-2}(x)(t-3)((t-1)x^2+1)}{\sqrt{t(t-1)}\,(x^2(t-2)+1)} \qquad \ldots\text{(xviii),}$$

or,

$$= \frac{q_t}{x^2(t-2)+1}\left\{\frac{x^3}{p_{t-1}} T_{t-1}(x) - (x^2(t-1)+1)\frac{t-3}{t-2} T_{t-2}(x)\right\} \qquad \text{(xviii) bis.}$$

* See *Biometrika*, Vol. IX. p. 220.

Hence if T_0 and T_1 be found accurately the remaining T's can be determined as accurately as we please without reference to the τ's.

But,
$$T_0 = \tau_1 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(xix)},$$
$$T_1 = \sqrt{2}\,\tau_2 + \tau_0 = \frac{x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx \quad \ldots\ldots\ldots \text{(xx)}.$$

Hence the tables of ordinates and areas of the normal curve readily provide T_0 and T_1 to seven decimal places, and (xviii) provides the higher T's. These were cut down to five figures and an approximate check on their values obtained by (viii).

As a matter of fact if r is of the order ·50 we cannot hope to obtain more than three figure accuracy in r without going to higher τ- and T-functions than the sixth, especially when using (xvii). But three figures in the correlation are usually adequate and the labour of computing is much increased if higher functions are used. Such must, however, be used if the correlation be sensibly higher than ·50.

The following table gives the $\frac{1}{2}(1+\alpha)$'s, h's, H's, $\bar{x}$'s, τ's, $\vartheta\tau$'s, T's, and ϑT's for the x-variate.

TABLE I.

$\frac{1}{2}(1+\alpha)$	0	·036	·358	·622	·802	·871	·972	1·000
h	$-\infty$	$-1·79912$	$-·36381$	$+·31071$	$+·84879$	$+1·13113$	$+1·91104$	$+\infty$
$H = \tau_1$	0	·07908	·37340	·38014	·27827	·21042	·06425	0
τ_0	0	$-·036$	$-·358$	$-·622$	$-·802$	$-·871$	$-·972$	-1
$\tau_1 = T_0$	0	$+·07908$	$+·37340$	$+·38014$	$+·27827$	$+·21042$	$+·06125$	0
τ_2	0	$-·10060$	$-·09606$	$+·08353$	$+·16701$	$+·16830$	$+·08682$	0
τ_3	0	$+·07221$	·13226	$-·14021$	$-·03176$	$+·02401$	$+·06952$	0
τ_4	0	$-·00688$	$+·07952$	$-·07001$	·10990	$-·08359$	$+·01631$	0
τ_5	0	$-·04291$	$+·07579$	$+·08432$	$-·02041$	$-·05839$	$-·03270$	0
τ_6	0	$+·03654$	$-·06933$	$+·06182$	$+·07319$	$+·03408$	$-·03744$	0
T_1	0	$-·17827$	$-·49385$	$-·50388$	$-·56581$	$-·63299$	$-·84922$	-1
T_2	0	$+·23690$	$+·29898$	$+·29475$	$+·33853$	$+·33915$	$+·21135$	0
T_3	0	$-·18799$	$-·00734$	$+·00466$	$+·06947$	$+·12432$	$+·18307$	0
T_4	0	$+·04848$	$-·09506$	$-·09186$	$-·10916$	$-·08255$	$+·06601$	0
T_5	0	$+·07412$	$+·07799$	$-·00511$	$-·06648$	$-·10343$	$-·05518$	0
T_6	0	$-·08325$	$+·05616$	$+·05364$	$+·05379$	$+·01471$	$-·08491$	0

$\bar{x}_s = \dfrac{H_{s-1} - H_s}{\frac{1}{2}(\alpha_s - \alpha_{s-1})}$	$-2·19667$	$-·91401$	$-·02553$	$+·56591$	$+·98333$	$+1·44723$	$+2·29464$
$\vartheta\tau_0$	$-·036$	$-·322$	$-·264$	$-·180$	$-·069$	$-·101$	$-·028$
$\vartheta\tau_1$	$+·07908$	$+·29432$	$+·00674$	$-·10187$	$-·06785$	$-·14617$	$-·06425$
$\vartheta\tau_2$	$-·10060$	$+·00454$	$+·17959$	$+·08348$	$+·00129$	$-·08148$	$-·08682$
$\vartheta\tau_3$	$+·07221$	$-·20417$	$-·00795$	$+·10845$	$+·05577$	$+·04555$	$-·06956$
$\vartheta\tau_4$	$-·00688$	$+·08640$	$-·14953$	$-·03989$	$+·02631$	$+·09993$	$-·01634$
$\vartheta\tau_5$	$-·04291$	$+·11870$	$+·00853$	$-·10473$	$-·03798$	$+·02569$	$+·03270$
$\vartheta\tau_6$	$+·03654$	$-·10587$	$+·13115$	$+·01137$	$-·03911$	$-·07152$	$+·03744$
ϑT_0	$+·07908$	$+·29432$	$+·00674$	$-·10187$	$-·06786$	$-·14617$	$-·06425$
ϑT_1	$-·17827$	$-·31558$	$-·01003$	$-·06193$	$-·06718$	$-·21622$	$-·15078$
ϑT_2	$+·23690$	$+·06208$	$-·00422$	$+·04377$	$+·00063$	$-·12780$	$-·21135$
ϑT_3	$-·18799$	$+·18065$	$+·01200$	$+·06481$	$+·05485$	$+·05874$	$-·18307$
ϑT_4	$+·04848$	$-·14354$	$+·00320$	$-·01731$	$+·02662$	$+·14856$	$-·06601$
ϑT_5	$+·07412$	$-·06613$	$-·01310$	$-·06137$	$-·03695$	$+·04825$	$+·05518$
ϑT_6	$-·08325$	$+·13941$	$-·00253$	$+·00015$	$-·03907$	$-·09962$	$+·08491$

The following table gives the corresponding quantities $\frac{1}{2}(1+\alpha')$'s, k's, K's, y''s, τ''s, $\vartheta\tau_1''$s T''s and $\vartheta T''$s for the y-variate.

TABLE II.

$\frac{1}{2}(1+\alpha')$	0		·034		·335		·619		·756		·861		·959		1·000
k	$-\infty$		−1·82501		−·42615		+·30286		+·69349		+1·08482		+1·73920		$+\infty$
$K=\tau_1'$	0		+·07545		+·36431		+·38106		+·31367		+·22149		+·08792		0
$y_s=\dfrac{K_{s-1}-K_s}{\frac{1}{2}(\alpha_s'-\alpha_{s-1}')}$		−2·21916		·95968		−·05896		+·49188		+·87789		+1·36301		+2·14436	
τ_0'	0		−·034		−·335		−·619		−·756		−·861		·959		−1
$\tau_1'=T_0'$	0		+·07545		+·36431		+·38106		+·31367		+·22149		+·08792		0
τ_2'	0		·09737		−·10978		+·08160		+·15382		+·16990		+·10812		0
τ_3'	0		+·07179		−·12172		−·14130		−·06647		+·01599		+·07268		0
τ_4'	0		−·00929		+·08932		−·06851		−·11185		−·08942		+·00077		0
τ_5'	0		−·04057		+·06463		+·08551		+·00990		−·05411		−·04815		0
τ_6'	0		+·03702		·07647		+·06061		+·08449		+·04134		−·03475		0
$\vartheta\tau_0'$		−·034		−·301		−·284		−·137		−·105		−·098		−·041	
$\vartheta\tau_1'$		+·07545		+·28886		+·01675		−·06739		−·09218		−·13357		−·08792	
$\vartheta\tau_2'$		−·09737		−·01211		+·19138		+·07222		+·01608		−·06178		−·10812	
$\vartheta\tau_3'$		+·07179		−·19351		−·01958		+·07483		+·08246		+·05669		·07268	
$\vartheta\tau_4'$		−·00929		+·09861		−·15783		·04331		+·02243		+·09019		−·00077	
$\vartheta\tau_5'$		−·04057		+·10520		+·02088		−·07561		−·06401		+·00596		+·04815	
$\vartheta\tau_6'$		+·03702		−·11349		+·13708		+·02388		−·04315		−·07609		+·03475	
T_1'	0		−·17170		·19025		−·50360		−·53817		−·62073		−·80610		−1
T_2'	0		+·23105		+·30139		+·29416		+·32847		+·34094		+·25021		0
T_3'	0		−·18723		−·01151		+·00132		+·04271		+·11511		+·18882		0
T_4'	0		+·05286		−·09892		−·09110		−·11081		−·08901		+·03768		0
T_5'	0		+·06989		+·01240		−·00474		−·04316		−·09869		−·08340		0
T_6'	0		−·08112		+·05897		+·05326		+·06263		+·02276		−·08010		0
$\vartheta T_0'$		+·07545		+·28886		+·01675		−·06739		−·09218		−·13358		·08792	
$\vartheta T_1'$		·17170		−·31855		·01334		−·03488		−·08225		−·18537		−·19391	
$\vartheta T_2'$		+·23105		+·07331		−·01023		+·03431		+·01247		−·09072		−·25021	
$\vartheta T_3'$		−·18723		+·17572		+·01583		+·03839		+·07273		+·07338		−·18882	
$\vartheta T_4'$		+·05286		−·15178		+·00752		·01911		+·02179		+·12670		−·03768	
$\vartheta T_5'$		+·06989		−·05749		−·01711		−·03842		−·05553		+·10529		+·08340	
$\vartheta T_6'$		−·08112		+·11309		·00571		+·00937		·03988		−·10286		+·08010	

From Tables I and II we can find from (x) the value of $\bar{n}_{ss'}/N$ for any given value of r, and by equating $\bar{n}_{ss'}/N$ to $n_{ss'}/N$ we should have an equation to determine the correlation r from that cell alone. The weighted mean of these 49 r's would be Ritchie-Scott's polychoric correlation coefficient. But the labour would be immense*.

We are now in a position to give the product of $\vartheta_s\tau_p\vartheta_{s'}\tau_p'$: see Table III, p. 138.

There are certain checks on the accuracy of this table, namely

$$S_{ss'}\,\vartheta_s\,\tau_p\,\vartheta_{s'}\,\tau_p'=0 \text{ except for } p=0, \text{ when it } =1.$$

* We are not underrating the large amount of arithmetic of the present process. It is not likely to be often repeated, and the sole purpose of publishing all these tables for an individual case is to impress the reader with that fact; while at the same time illustrating the actual numerical processes. The amount of arithmetic, great as it is, is relatively small compared with that of solving and weighting the resulting r's in the case of a 49-cell table.

TABLE III.

Values of $\vartheta_s \tau_p \vartheta_s$

s=1			s=4	s=5	s=6	s=7	
+ ·001,224	+ ·010,948	+ ·008,976	+ ·006,120	+ ·002,346	+ ·003,434	+ ·000,952	0
+ ·005,967	+ ·022,206	+ ·000,509	− ·007,686	− ·005,119	− ·011,029	·004,848	1
+ ·009,795	·000,142	·017,487	− ·008,128	− ·000,126	+ ·007,934	+ ·008,154	2
+ ·005,184	− ·014,679	− ·000,571	+ ·007,786	+ ·001,004	+ ·003,270	− ·004,994	3
+ ·000,064	− ·000,803	+ ·001,389	+ ·000,371	− ·000,244	·000,928	+ ·000,152	4
+ ·001,741	− ·004,816	− ·000,346	+ ·004,249	+ ·001,544	− ·001,042	− ·001,327	5
+ ·001,353	·003,919	+ ·004,855	+ ·000,121	− ·001,148	·002,648	+ ·001,386	6
+ ·010,836	+ ·096,922	+ ·079,164	+ ·054,180	+ ·020,769	+ ·030,401	+ ·008,428	0
+ ·022,843	+ ·085,017	+ ·001,947	·029,126	·019,599	− ·042,223	− ·018,559	1
+ ·001,248	− ·000,056	− ·002,229	− ·001,036	− ·000,016	+ ·001,011	+ ·001,07	2
− ·013,973	+ ·039,567	+ ·001,538	− ·020,986	− ·010,792	− ·008,814	+ ·013,461	3
− ·000,678	+ ·008,520	− ·014,745	− ·003,934	+ ·002,594	+ ·009,851	− ·001,611	
− ·004,514	+ ·012,487	+ ·000,897	·011,018	·003,995	+ ·002,703	+ ·003,440	
− ·004,147	+ ·012,015	− ·014,884	− ·001,290	+ ·004,439	+ ·008,117	− ·004,249	
+ ·010,224	+ ·091,448	+ ·074,976	+ ·051,120	+ ·019,596	+ ·028,684	+ ·007,952	0
+ ·001,325	+ ·004,930	+ ·000,113	− ·001,706	− ·001,136	− ·002,448	− ·001,076	
− ·019,253	+ ·000,869	+ ·031,370	+ ·015,976	+ ·000,247	− ·015,594	[illegible]	
·001,414	+ ·004,004	+ ·000,156	·002,123	·001,092	− ·000,892	+ ·001,362	
+ ·001,086	·013,637	+ ·023,600	+ ·006,296	− ·004,153	− ·015,772	+ ·002,579	
·000,896	+ ·002,478	+ ·000,178	− ·002,187	− ·000,793	+ ·000,536	+ ·000,683	
+ ·005,009	− ·011,513	+ ·017,978	+ ·001,559	− ·005,361	·009,804	+ ·005,132	
+ ·004,932	+ ·044,114	+ ·036,168	+ ·024,660	+ ·009,453	+ ·013,837	+ ·003,836	
− ·005,329	− ·019,834	− ·000,154	+ ·006,865	+ ·004,572	+ ·009,850	+ ·004,330	
− ·007,265	+ ·000,328	+ ·012,970	+ ·006,029	+ ·000,093	− ·005,884	− ·006,270	
+ ·005,403	− ·015,300	− ·000,595	+ ·008,115	+ ·004,173	+ ·003,109	− ·005,205	
+ ·000,298	− ·003,715	+ ·006,181	+ ·001,729	− ·001,140	·004,331	+ ·000,708	
+ ·003,214	− ·008,975	− ·000,645	+ ·007,919	+ ·002,872	·001,912	− ·002,172	
+ ·000,873	− ·002,528	+ ·003,132	+ ·000,272	− ·000,934	− ·001,708	+ ·000,894	
+ ·003,780	+ ·033,810	+ ·027,720	+ ·018,900	+ ·007,245	+ ·010,605	+ ·002,940	0
− ·007,290	− ·027,130	·000,621	+ ·009,390	+ ·006,254	+ ·013,474	+ ·005,923	1
− ·001,678	+ ·000,073	+ ·002,888	+ ·001,342	+ ·000,021	− ·001,310	− ·001,396	
·005,954	− ·016,861	− ·000,656	+ ·008,943	+ ·001,599	+ ·003,756	− ·005,736	
− ·000,154	+ ·001,938	− ·003,354	− ·000,895	+ ·000,590	+ ·002,241	− ·000,367	
+ ·002,747	·007,598	− ·000,546	+ ·006,704	+ ·002,431	− ·001,644	− ·002,093	5
·001,577	+ ·004,568	− ·005,659	·000,491	+ ·001,688	+ ·003,086	·001,616	6
+ ·003,528	+ ·031,556	+ ·025,872	+ ·017,640	+ ·006,762	+ ·009,898	+ ·002,744	0
− ·010,563	·039,312	− ·000,900	+ ·013,607	+ ·009,063	+ ·019,524	+ ·008,582	1
+ ·006,215	− ·000,280	− ·011,095	− ·005,157	− ·000,080	+ ·005,034	+ ·005,364	2
+ ·004,094	·011,591	− ·000,151	+ ·006,148	+ ·003,162	+ ·002,582	− ·003,943	3
·000,621	+ ·007,792	− ·013,486	− ·003,598	+ ·002,373	+ ·009,013	− ·001,174	4
− ·000,256	+ ·007,107	+ ·000,051	− ·000,624	− ·000,226	+ ·000,153	+ ·000,195	
− ·002,780	+ ·008,056	·009,979	− ·000,865	+ ·002,976	+ ·005,442	− ·002,849	
− ·001,476	+ ·013,202	+ ·010,824	+ ·007,380	+ ·002,829	+ ·004,141	+ ·001,148	
− ·006,953	− ·025,877	− ·000,593	+ ·008,956	+ ·005,965	+ ·012,851	+ ·005,649	
+ ·010,877	·000,491	− ·019,417	− ·009,026	− ·000,139	+ ·008,810	+ ·009,387	
− ·005,248	+ ·011,861	+ ·000,578	− ·007,882	− ·004,053	− ·003,311	+ ·005,056	
+ ·000,005	− ·000,067	+ ·000,115	+ ·000,031	− ·000,020	− ·000,077	+ ·000,013	
− ·002,066	+ ·005,715	+ ·000,411	− ·005,043	− ·001,829	+ ·001,237	+ ·001,575	
+ ·001,270	− ·003,679	+ ·004,557	+ ·000,395	− ·001,359	− ·002,485	+ ·001,301	

Applying these tests we find:

$$S_{ss'}(\vartheta_s \tau_0 \vartheta_{s'} \tau_0') = \;\;1{\cdot}000{,}000, \quad S_{ss'}(\vartheta_s \tau_1 \vartheta_{s'} \tau_1') = +{\cdot}000{,}001,$$
$$S_{ss'}(\vartheta_s \tau_2 \vartheta_{s'} \tau_2') = +\;{\cdot}000{,}001, \quad S_{ss'}(\vartheta_s \tau_3 \vartheta_{s'} \tau_3') = +{\cdot}000{,}003,$$
$$S_{ss'}(\vartheta_s \tau_4 \vartheta_{s'} \tau_4') = -\;{\cdot}000{,}002, \quad S_{ss'}(\vartheta_s \tau_5 \vartheta_{s'} \tau_5') = +{\cdot}000{,}001,$$

and

$$S_{ss'}(\vartheta_s \tau_6 \vartheta_{s'} \tau_6') = +{\cdot}000{,}002,$$

results as close as we should expect, when we take into account the fact that our $\vartheta\tau$'s were only to five figure accuracy, and our products to six.

The meaning of Table III should be quite intelligible; namely, for example:

$$\begin{aligned}{\cdot}084 = \frac{n_{3,2}}{N} &= {\cdot}079{,}464 + {\cdot}001{,}947\, r - {\cdot}002{,}229\, r^2 \\ &+ {\cdot}001{,}538 r^3 - {\cdot}014{,}745 r^4 + {\cdot}000{,}897 r^5 - {\cdot}014{,}884 r^6 + \ldots \qquad \ldots\text{(xxi)}\end{aligned}$$

is the equation which will give the correlation coefficient r as deduced from the (3, 2) cell. If r be given any other value the right hand of the above expression is equal to the contents of the (3, 2) cell for a normal correlation surface of correlation coefficient r having the observed marginal totals.

Thus far the arithmetic is absolutely comparable with that needed for Ritchie-Scott's "polychoric r." We should have to solve the 49 equations, and then calculate—the stiffest part of the work—the probable errors of the 49 correlation coefficients which are the roots of these equations. Using these probable errors as our weighting data, we should find a mean coefficient. Our purpose is to replace the weighting and the solution of the 49 equations by the solution of a single equation. It will be noticed that both Ritchie-Scott's and our methods have an undesirable limitation, for we both assume the marginal totals to be those of the normal correlation surface. Actually in our case we ought to treat the marginal totals as unknown, or select $h_1, h_2, h_3, \ldots h_q, k_1, k_2, k_3, \ldots k_{q'}$ as well as r to give as closely as possible the observed frequencies. Now the τ's and consequently the T's and $\vartheta\tau$'s and ϑT's all depend upon the h's and k's and the equations obtained by making

$$S_{ss'}\left(\frac{n^2_{ss'}}{\bar{n}_{ss'}}\right) = \text{minimum}$$

do not appear to lend themselves to any reasonably brief system of solutions. We were compelled therefore to introduce the admittedly limited form of solution, i.e. the determination of the best normal correlation surface subject to the restriction of its having the same marginal totals as the observed frequency surface. We consider this a practically necessary but none the less grave restriction.

We next proceeded to determine the value of $\bar{n}_{ss'}/n_{ss'}$ and $(\bar{n}_{ss'}/n_{ss'})^2$ for certain selected values of r in order to build up equation (xvii) and solve it by interpolation. The values chosen were: 0·45, 0·50 and 0·55. These cover the range within which we anticipate the solution of (xvii) for r will lie. We need also the value of the numerator in (xvii). i.e.

$$\nu_{ss'} = \vartheta_s \tau_1 \vartheta_{s'} \tau_1' + 2r\, \vartheta_s \tau_2 \vartheta_{s'} \tau_2' + 2r^2 \vartheta_s \tau_3 \vartheta_{s'} \tau_3' + \ldots,$$

for the same three values of r. These results are given in Table IV.

TABLE IV. *Values of* $(\bar{n}_{ss'}/n_{ss})$, $(\bar{n}_{ss'}/n_{ss'})^2$ *and* $\nu_{ss'}$.

Function	s=1	s=2	s=3	s=4		s=6	s=7
$n_{ss'}/n_{ss'}$	1·602,750	·879,954	·814,714	∞	·388,000	∞	∞
	1·846,000	·902,000	·705,571	∞	·266,000	∞	∞
	2·115,500	·916,409	·587,857	∞	·174,000	∞	∞
	2·568,806	·774,321	·663,759	∞	·150,544	∞	∞
	3·407,716	·813,604	·497,830	∞	·070,756	∞	∞
	4·475,340	·839,805	·345,576	∞	·030,276	∞	∞
	+ ·018,462	+ ·011,177	·014,603	− ·009,218	− ·002,733	− ·002,747	− ·000,336
	+ ·020,480	+ ·008,113	− ·015,910	− ·008,382	− ·002,154	− ·001,929	− ·000,218
	+ ·022,694	+ ·004,477	− ·017,013	− ·007,243	− ·001,519	− ·001,228	− ·000,168
	·867,348	·905,539	·944,250	1·478,500	1·379,000	1·887,333	∞
	·894,565	·944,630	·939,833	1·383,615	1·215,375	1·544,667	∞
	·915,130	·986,935	933,333	1·278,500	1·043,500	1·213,333	∞
	·752,293	·820,001	·891,608	2·185,962	1·901,641	3·562,026	∞
	·800,247	·892,326	·883,286	1·914,390	1·477,136	2·385,996	∞
	·837,463	·974,011	871,110	1·634,562	1·088,892	1·472,177	∞
	+ ·013,846	+ ·116,000	− ·005,963	·046,913	− ·025,552	− ·041,623	− ·009,764
	+ ·011,084	+ ·125,051	− ·009,011	− ·051,854	- ·026,828	− ·040,529	− ·007,913
	+ ·007,767	+ ·135,874	- ·013,006	·057,659	− ·028,171	- ·038,864	·005,940
s/n_{ss}	·857,750	1·075,552	1·108,280	·812,500	·854,818	·984,375	2·194,000
	·751,875	1 076,195	1·138,747	·823,410	·844,773	·930,333	1·846,500
	·635,750	1·075,448	1·175,027	·835,909	·831,636	·866,000	1·486,500
	735,735	1·156,812	1·228,285	·660,156	730,714	·968,994	4·813,636
	·565,316	1 158,196	1 296,745	·678,004	·713,641	·865,519	3 409,562
	·401,178	1·156,588	1·380,688	·698,744	·691,618	·719,956	2·209,682
	·016,095	+ ·002,075	+ ·011,770	+ ·013,402	− ·003,847	− ·023,749	- ·013,555
	− ·017,786	+ ·000,037	+ ·019,827	+ ·015,435	− 005,038	− ·028,268	·011,205
	− ·019,311	·002,805	+ ·059,278	+ ·017,601	·006,601	- ·033,522	- ·014,530
	1·634,000	1·155,897	1 078,222	·808,892	·850,571	1·225,786	·671,833
	1·260,000	1·096,966	1·098,417	·837,135	·877,714	1 239,929	·627,333
	·917,000	1 030,862	1·121,944	·869,568	·907,429	1 250,000	570,000
	2·669,956	1·336,098	1·162,563	654,306	·723,471	1·502,551	·451,360
	1·587,600	1·203,334	1·206,520	·700,795	·770,382	1 537,424	·393,547
	·840,889	1·062,676	1·258,758	·756,149	·823,427	1·562,500	·324,900
	− ·007,715	- ·032,319	+ ·013,131	+ ·019,505	+ ·007,261	+ ·001,459	− ·004,625
	- ·007,215	− ·036,132	+ ·015,696	+ ·022,370	+ ·007,917	+ ·003,430	− ·006,095
	− ·006,471	− ·040,720	+ ·018,237	+ ·025,717	·008,735	+ 002,185	·007,680
$\bar{n}_{ss}/n_{s'}$	∞	1·114,278	1·028,556	·934,423	960,515	·935,111	·946,600
	∞	1·006,167	1·027,185	·969,000	1·008,273	·978,944	·944,400
	∞	·890,389	1 024,148	1·007,692	1·061,727	1·025,111	·927,400
$(n_{ss}/n_{ss}$	∞	1·241,615	1·057,927	·873,146	·922,647	·874,433	·896,052
	∞	1·012,372	1·055,109	·938,961	1·016,614	·958,331	·891,891
	∞	·792,793	1·048,879	1·015,443	1·127,261	1·050,853	·860,071
	- ·004,797	− ·037,653	− ·000,381	+ ·017,025	+ ·009,967	+ ·015,398	+ ·000,440
	− ·003,957	− ·040,252	− ·001,134	+ ·018,995	+ ·011,095	+ ·016,166	− ·000,916
	·002,988	− ·043,158	− ·002,230	+ ·021,305	+ ·012,465	+ 017,113	- ·002,508
	∞	1·461,333	·867,077	·216,474	1·604,286	·701,931	·906,500
	∞	1·224,000	·830,576	·245,526	1·693,714	·754,966	·969,125
	∞	·988,111	786,077	·273,789	1·791,000	·812,828	1 028,375
	∞	2·135,494	·751,823	·479,809	2·573,734	·492,707	·821,742
	∞	1·498,176	·689,856	·551,335	2·868,667	·569,974	·939,203
	∞	·976,363	·617,917	. ·622,538	3·207,681	·660,689	1·057,555
	− ·003,069	- ·042,728	− ·017,170	+ ·011,16[illegible]	+ ·012,060	+ ·029,542	+ ·010,202
	− ·002,189	− ·042,658	− ·020,931	+ ·010,905	+ ·013,028	+ ·032,069	+ ·009,778
	− ·001,381	- ·042,197	·025,479	+ ·010,572	+ ·014,219	+ ·035,116	+ ·009,152
$n_{ss'}/n_s$	∞	·961,333	·747,556	1·462,667	·845,000	1·140,500	·870,286
	∞	·705,000	·618,556	1·411,167	·865,000	1·235,000	1·003,143
	∞	·491,000	·542,000	1·337,333	·877,000	1·331,000	1·150,286
$(n_{ss'}/n_s$	∞	·921,161	·558,840	2·139,395	·714,025	1·300,740	·757,398
	∞	·497,025	·420,625	1·991,392	·748,225	1·525,225	1·006,296
	∞	·241,081	·293,764	1·788,460	·769,129	1·771,561	1·323,158
$\nu_{ss'}$	− ·000,633	− ·016,551	- ·017,086	− ·004,935	+ ·002,845	+ ·018,719	+ ·017,641
	- ·000,417	·014,160	- ·018,536	− ·007,468	+ ·001,950	+ ·019,060	+ ·019,571
	− ·000,309	− ·011,471	- ·019,78	− ·010,293	+ ·000,873	+ ·019,302	+ ·021,685

The values (*a*), (*b*), (*c*) refer respectively to $r=0{\cdot}45$, $0{\cdot}50$, $0{\cdot}55$.

Having obtained $(\bar{n}_{ss'}/n_{ss'})^2$ and $\nu_{ss'}$ for the trial values of r, it is only a matter of adding $\nu_{ss'}/(\bar{n}_{ss'}/n_{ss'})^2$ for all values of s and s' on the machine in order to obtain:

$$u = S_{ss'}\,\{\nu_{ss'}/(\bar{n}_{ss'}/n_{ss'})^2\}.$$

The values obtained were:

$r =$	0·45	0·50	0·55
$u =$	+ ·157,074	+ ·012,276	− ·209,976

Whence by inverse interpolation* we find:

$$u = 0 \text{ for } r_p = {\cdot}5034,$$

which is "polychoric r" as based upon Equation (xvii). We shall compare later the value for r as found by other processes. But the above value is clearly well in accord with the usual result for paternal correlation in man.

Table V gives the working values of $\nu_{ss'}/(\bar{n}_{ss'}/n_{ss'})^2$.

TABLE V. *Values of* $\nu_{ss'}/(\bar{n}_{ss'}/n_{ss'})^2$†.

	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
(a)	+ ·007,186	+ ·014,135	− ·022,001	0	− ·018,159		0
(b)	+ ·006,010	+ ·009,972	·031,958	0	− ·030,424		0
(c)	+ ·005,071	+ ·005,331	·049,232	0	− ·050,132		0
(a)	+ ·018,405	+ ·141,163	− ·006,688	− ·021,475	·013,436	− ·011,685	0
(b)	+ ·013,851	+ ·140,141	− ·010,202	− ·027,086	·018,162	− ·016,986	0
(c)	+ ·009,271	+ ·139,195	− ·011,930	− ·035,275	− ·025,871	− ·026,399	0
(a)	·021,876	+ ·001,791	+ ·034,007	+ ·020,301	− ·005,265	− ·024,509	·002,816
(b)	− ·031,462	+ ·000,032	+ ·038,125	+ ·022,755	− ·007,060	− ·032,660	− ·004,166
(c)	·047,779	− ·002,125	+ ·012,934	+ ·025,189	·009,544	− ·014,832	− ·006,579
	·002,890	− ·024,189	+ ·011,556	+ ·029,810	+ ·010,036	+ ·002,968	·010,247
	·004,543	− ·030,027	+ ·013,009	+ ·031,921	+ ·010,316	+ ·002,231	− ·015,487
	− ·007,695	− ·038,318	+ ·014,188	+ ·034,010	+ ·010,608	+ ·001,398	− ·023,039
(a)	0	− ·030,326	·000,360	+ ·019,198	− ·010,803	+ ·017,610	+ ·000,491
(b)	0	− ·039,760	− ·001,075	+ ·020,230	+ ·010,911	+ ·016,869	− ·001,027
(c)	0	− ·054,439	− ·002,126	+ ·020,981	+ ·014,058	+ ·016,285	− ·002,916
(a)	0	− ·020,008	·022,838	+ ·007,545	+ ·004,686	+ ·059,958	+ ·012,415
(b)	0	− ·028,471	·030,341	+ ·007,029	+ ·004,542	+ ·056,264	+ ·010,411
(c)	0	− ·013,219	− ·041,234	+ ·006,516	+ ·004,133	+ ·053,151	+ ·008,654
(a)	0	·017,910	− ·030,574	·002,307	+ ·003,984	+ ·014,391	+ ·023,291
(b)	0	·028,191	− ·014,067	− ·003,750	+ ·002,606	+ ·012,497	+ ·019,449
(c)	0	− ·017,576	− ·067,356	·005,755	+ ·001,135	+ ·010,895	+ ·016,389

$S(a) = +{\cdot}157,074$, $S(b) = +{\cdot}012,276$, $S(c) = -{\cdot}209,976$.

* The formula used was *Casus* I or $z_\theta = z_0 + \frac{1}{2}\theta(\Delta z_{-1} + \Delta z_0) + \frac{1}{2}\theta^2\delta^2 z_0$, the solution of the quadratic giving θ.

† The table suggests, *a posteriori*, that we should have got quite reasonable results from linear interpolation; we have: from (a) and (b) $r = {\cdot}5012$; from (a) and (c) $r = {\cdot}4928$, and from (b) and (c) $r = {\cdot}5025$, as against our ·5034 It should be noticed that the values in Table V are not always in agreement in the last figure with those obtained by dividing $\nu_{ss'}$ in Table IV by the $(n_{ss'}/n_{ss'})^2$ of that table, because the somewhat more accurate process was adopted of multiplying $\nu_{ss'}$ by $n^2_{ss'}$ and then dividing by $n^2_{ss'}$. Still the physical meanings of $\bar{n}_{ss'}/n_{ss'}$ and $(\bar{n}_{ss'}/n_{ss'})^2$ are so prominent in the work that it seemed desirable to register their values.

Before we consider the graph due to this solution, let us investigate the value of r to be found from (xvi). The values of $\bar{n}_{ss'}/n_{ss'}$ are already provided in Table IV, but we need a table corresponding to Table III giving the product $\vartheta_s T_p \vartheta_{s'} T_p'$ instead of the product $\vartheta_s \tau_p \vartheta_{s'} \tau_p'$. This is provided in Table VI. Further if

$$\kappa_{ss'} = \vartheta_s T_0 \vartheta_s T_0' + r \vartheta_s T_1 \vartheta_{s'} T_1' + r^2 \vartheta_s T_2 \vartheta_{s'} T_2' + \dots,$$

Table VII (p. 143) provides $\kappa_{ss'}$ for the same three values of r, i.e. 0·45, 0·50 and 0·55. Finally Table VIII (p. 143) gives $\kappa_{ss'}/(\bar{n}_{ss'}/n_{ss'})$, whence by summing we obtain

$$v = r - S_{ss'} \{\kappa_{ss'}/(\bar{n}_{ss'}/n_{ss'})\},$$

for the three cases.

Using the same interpolation formula as before in order to discover the value of r for which $v = 0$ we find:

$$r = \cdot 5204.$$

There is thus a difference of ·0170 between the two methods. The probable error found for the product-moment r is ·0160 and the result by the usual product-moment process may be given:

$$r = \cdot 5189 \pm \cdot 0160.$$

Thus either of the values reached by the methods of this paper differ by less than the probable error from the true product-moment value.

(4) If we work out the results by mean square contingency we find:

$$C_2 = \cdot 480{,}690,$$

and the class index correlations are*:

$$\text{For fathers: } r_{c_f} = \cdot 962{,}329.$$
$$\text{For sons: } r_{c_s} = \cdot 964{,}523.$$

Hence correlation from mean square contingency

$$r = C_2/(r_{c_f} r_{c_s}) = \cdot 5179,$$

which is in excellent agreement with the product-moment value.

It would therefore be quite reasonable for such a table as the present to use mean square contingency and class index corrections, and save the heavy labour of Equation (xvi bis) or (xvii). At the same time we cannot assert that this process would always be equally satisfactory for tables with but few broad categories and with much higher correlation.

Our two processes seem to give values slightly in defect and in excess of the true value of r, and we might use their mean, i.e. ·5118, to obtain our graph. We shall, however, first proceed to compare the actual results of solving (xiv) and substituting in (xv) with the result of such approximative processes.

Table IX (p. 145) gives the products of $\vartheta_s T_p \vartheta_{s'} \tau_p'$ and will therefore enable us by aid of Table IV (p. 140) which gives the values of $\bar{n}_{ss'}/n_{ss'}$ to obtain $\bar{h}_s$. for any value of r. Let

$$\lambda_{ss'} = \vartheta_s T_0 \vartheta_s \tau_0' + r \vartheta_s T_1 \vartheta_s \tau_1' + r^2 \vartheta_s T_2 \vartheta_{s'} \tau_2' + \dots \dots\dots\dots \text{(xxii)}.$$

* Using the values of $\bar{x}_s$ and y_s in Tables I and II respectively.

TABLE VI.

Values of $\vartheta_s T_p \vartheta_{s'} T_p'$.

p	$s=1$	$s=2$		$s=4$	$s=5$	$s=6$	$s=7$	
0	+ ·005,966	+ ·022,207	+ ·000,509	− ·007,686	− ·005,120	− ·011,028	− ·004,848	
1	+ ·030,608	+ ·054,185	+ ·001,722	+ ·010,633	+ ·011,536	+ ·037,126	+ ·025,890	
2	+ ·054,736	+ ·014,342	− ·000,976	+ ·010,114	+ ·000,145	− ·029,528	− ·018,833	
3	+ ·035,199	− ·033,825	− ·002,246	− ·012,136	− ·010,270	− ·010,999	+ ·034,276	
4	+ ·002,562	− ·007,587	+ ·000,169	− ·000,915	+ ·001,407	+ ·007,853	− ·003,489	
5	+ ·005,180	− ·004,622	·000,915	− ·004,289	− ·002,583	+ ·003,372	+ ·003,856	
6	+ ·007,003	− ·011,727	+ ·000,213	− ·000,013	+ ·003,287	+ ·008,380	− ·007,142	
0	+ ·022,842	+ ·085,018	+ ·001,918	− ·029,426	− ·019,601	− ·042,222	− ·018,559	0
1	+ ·056,787	+ ·100,528	+ ·003,195	+ ·019,728	+ ·021,402	+ ·068,878	+ ·018,033	1
2	+ ·017,375	+ ·001,553	− ·000,310	− ·003,210	+ ·000,046	− ·009,373	− ·015,501	2
3	− ·033,035	+ ·031,745	+ ·002,108	+ ·011,389	+ ·009,639	+ ·010,323	− ·032,169	3
4	·007,358	+ ·021,786	·000,486	+ ·002,627	− ·004,040	− ·022,548	+ ·010,019	4
5	− ·001,261	+ ·003,802	+ ·000,753	+ ·003,528	+ ·002,124	− ·002,774	− ·003,172	
6	·011,913	+ ·019,949	·000,362	+ ·000,022	− ·005,591	− ·014,255	+ ·012,150	
0	+ ·001,321	+ ·004,929	+ ·000,113	− ·001,706	− ·001,136	− ·002,448	− ·001,076	
1	+ ·002,378	+ ·001,211	+ ·000,134	+ ·000,826	+ ·000,896	+ ·002,885	+ ·002,012	
2	·002,423	·000,635	+ ·000,043	− ·000,448	− ·000,006	+ ·001,307	+ ·002,162	
3	− ·002,976	+ ·002,860	+ ·000,190	+ ·001,026	+ ·000,868	+ ·000,930	− ·002,898	
	+ ·000,365	·001,080	+ ·000,024	− ·000,130	+ ·000,200	+ ·001,118	− ·000,497	
	·001,271	+ ·001,131	+ ·000,225	+ ·001,052	+ ·000,631	− ·000,827	− ·000,946	
	+ ·000,475	− ·000,796	+ ·000,014	− ·000,001	+ ·000,223	+ ·000,569	− ·000,485	
	− ·005,329	− ·019,833	·000,455	+ ·006,865	+ ·004,573	+ ·009,850	+ ·004,330	0
	+ ·006,217	+ ·011,006	+ ·000,350	+ ·002,160	+ ·002,343	+ ·007,541	+ ·005,259	1
	+ ·008,127	+ ·002,130	·000,145	+ ·001,502	+ ·000,022	− ·001,385	− ·007,251	2
	·007,217	+ ·006,935	+ ·000,461	+ ·002,188	+ ·002,106	+ ·002,255	− ·007,028	3
	·000,911	+ ·002,786	− ·000,062	+ ·000,336	− ·000,517	− ·002,883	+ ·001,281	4
	− ·002,847	+ ·002,540	+ ·000,503	+ ·002,358	+ ·001,419	− ·001,854	− ·002,120	5
	− ·000,780	+ ·001,306	·000,021	+ ·000,001	− ·000,366	− ·000,934	+ ·000,796	6
0	·007,289	·027,130	·000,622	+ ·009,390	+ ·006,255	+ ·013,173	+ ·005,923	0
1	+ ·011,662	+ ·025,956	+ ·000,825	+ ·005,094	+ ·005,526	+ ·017,784	+ ·012,402	1
2	+ ·002,953	+ ·000,771	− ·000,053	+ ·000,546	+ ·000,008	− ·001,593	− ·002,635	2
3	− ·013,673	+ ·013,139	+ ·000,873	+ ·001,714	+ ·003,990	+ ·004,273	− ·013,315	3
4	+ ·001,057	− ·003,128	+ ·000,070	·000,377	+ ·000,580	+ ·003,238	− ·001,439	4
5	·001,116	+ ·003,672	+ ·000,727	+ ·003,408	+ ·002,052	− ·002,679	− ·003,064	5
	+ ·003,320	− ·005,559	+ ·000,101	− ·000,006	+ ·001,558	+ ·003,973	− ·003,386	6
0	·010,563	− ·039,314	− ·000,901	+ ·013,607	+ ·009,064	+ ·019,524	+ ·008,582	0
1	+ ·033,046	+ ·058,500	+ ·001,860	+ ·011,180	+ ·012,454	+ ·040,082	+ ·027,952	1
2	·021,493	·005,632	+ ·000,383	− ·003,971	·000,057	+ ·011,595	+ ·019,175	2
3	− ·013,795	+ ·013,256	+ ·000,880	+ ·004,756	+ ·004,025	+ ·004,311	− ·013,433	3
4	+ ·006,142	·018,186	+ ·000,106	− ·002,193	+ ·003,372	+ ·018,822	− ·008,364	4
	[illegible]	− ·001,011	− ·000,200	·000,939	·000,565	+ ·000,738	+ ·000,841	
	+ ·008,563	− ·011,340	+ ·000,260	− ·000,016	+ ·004,019	+ ·010,217	− ·008,733	
		− ·025,876	− ·000,593	+ ·008,956	+ ·005,966	+ ·012,851	+ ·005,649	0
	+ ·034,867	+ ·061,193	+ ·001,915	+ ·012,009	+ ·013,028	+ ·011,927	+ ·029,238	
	− ·059,276	·015,532	+ ·001,057	− ·010,953	− ·000,157	+ ·031,978	+ ·052,883	
	+ ·035,498	·034,111	− ·002,265		− ·010,357	− ·011,092	+ ·034,567	
	·001,827	+ ·005,409	− ·000,121	+ ·000,652	·001,003	− ·005,599	+ ·002,488	
	+ ·006,181	− ·005,515	− ·001,092	− ·005,118	[illegible]	+ ·004,024	+ ·004,602	
	− ·006,668	+ ·011,167	− ·000,202	+ ·000,202	− ·003,130	− ·007,980	+ ·006,801	

TABLE VII. *Values of* κ_{ss}

r	s=1	s=2	s=3	s=4	s=5	s=6	s=7
(a)	+ ·034,290	+ ·045,918	+ ·000,873	− ·002,076	− ·000,798	− ·000,849	− ·000,094
(b)	+ ·039,786	+ ·047,855	+ ·000,831	− ·001,549	− ·000,541	− ·000,496	− ·000,036
(c)	+ ·045,903	+ ·049,468	+ ·000,762	− ·001,097	− ·000,350	− ·000,251	− ·000,001
(a)	+ ·048,425	+ ·135,200	+ ·003,506	− ·018,688	− ·009,255	− ·013,278	− ·002,561
(b)	+ ·050,671	+ ·142,180	+ ·003,719	− ·017,061	− ·007,957	− ·010,554	− ·001,722
(c)	+ ·052,617	+ ·149,704	+ ·003,946	− ·015,291	− ·006,630	− ·008,054	− ·001,089
(a)	+ ·001,628	+ ·006,926	+ ·000,205	− ·001,317	− ·000,633	− ·000,765	− ·000,039
(b)	+ ·001,526	+ ·007,188	+ ·000,223	− ·001,252	− ·000,545	− ·000,509	+ ·000,040
(c)	+ ·001,386	+ ·007,465	+ ·000,245	− ·001,175	− ·000,444	− ·000,235	+ ·000,096
(a)	− ·001,641	− ·013,645	− ·000,278	+ ·008,425	+ ·005,826	+ ·012,401	+ ·004,608
(b)	− ·001,250	− ·012,657	·000,247	+ ·008,726	+ ·006,019	+ ·012,553	+ ·004,294
(c)	− ·000,903	− ·011,563	− ·000,211	+ ·009,071	+ ·006,233	+ ·012,663	+ ·003,892
(a)	− ·001,344	− ·014,202	− ·000,165	+ ·012,270	+ ·009,181	+ ·021,659	+ ·009,613
(b)	− ·000,940	− ·012,484	− ·000,085	+ ·012,745	+ · 09,643	+ ·022,682	+ ·009,562
(c)	− ·000,625	− ·010,689	+ ·000,007	+ ·013,278	+ ·010,160	+ ·023,755	+ ·009,352
(a)	− ·000,958	− ·013,805	+ ·000,109	+ ·018,295	+ ·015,185	+ ·041,172	+ ·023,419
(b)	− ·000,584	− ·011,207	+ ·000,258	+ ·018,782	+ ·016,036	+ ·044,362	+ ·025,040
(c)	− ·000,328	− ·008,749	+ ·000,419	+ ·019,263	+ ·016,957	+ ·047,837	+ ·026,557
(a)	− ·000,047	− ·004,380	+ ·000,263	+ ·010,961	+ ·010,729	+ ·036,961	+ ·032,908
(b)	− ·000,076	− ·003,086	+ ·000,316	+ ·010,574	+ ·010,938	+ ·040,073	+ ·038,215
(c)	+ ·000,159	− ·002,067	+ ·000,348	+ ·010,019	+ ·011,027	+ ·043,208	+ ·044,126

TABLE VIII. *Values of* $\kappa_{ss}'(\bar{n}_{ss}/n_{ss})$

		s=1	s=2	s=3	s=4	s=5	s=6	s=7
	(a)	+ ·021,394	+ ·052,182	+ ·001,072	0	− ·002,057	0	0
	(b)	+ ·021,552	+ ·053,054	+ ·001,178	0	− ·002,033	0	0
	(c)	+ ·021,698	+ ·053,980	+ ·001,296	0	− 022,011	0	0
	(a)	+ 055,831	+ 149,303	+ ·003 713	− ·012,640	− ·006,711	− ·007,035	0
	(b)	+ ·056,643	+ 150,514	+ ·003 57	− 012,331	·006,547	− 006,833	0
	(c)	+ ·057,197	+ ·151,686	+ ·001,228	011,960	− ·006,354	− 006,638	0
	(a)	+ ·001,898	+ ·006,439	+ ·000,185	− 001,621	− ·000,741	− ·000,777	− 000,018
3	(b)	+ ·002,029	+ ·006,679	+ 000,196	− ·001,521	− ·000,645	− ·000,547	+ ·000,022
		+ ·002,180	+ ·006,941	+ ·000,209	− 001,406	− ·000,534	− ·000,271	+ ·000,065
		− ·001,004	− 011,805	·000,258	+ 010,415	+ ·006,850	+ ·010,117	+ ·006,859
		− 000,992	011,538	·000,225	+ 010,124	+ ·006,858	+ 010,124	+ ·006,845
		− ·000,985	− ·011,217	000,188	+ 010,432	+ ·006,869	+ ·010,130	+ ·006,828
	(a)	0	− ·012,745	·000 160	+ ·013,131	+ ·009,558	+ 023,162	+ ·010,155
	(b)	0	− ·012,407	− ·000,083	+ ·013,153	+ ·009,564	+ ·023,170	+ ·010,125
	(c)	0	− ·012,005	− ·000,007	+ ·013,177	+ ·009,569	+ ·023,173	+ ·010,084
	(a)	0	− ·009,447	+ 000,126	+ ·015,039	+ ·009,465	+ ·058,655	+ ·025,835
	(b)	0	− ·009,156	+ ·000,311	+ ·015,079	+ ·009,468	+ ·058,760	+ ·025,838
	(c)	0	− ·008,854	+ ·000,533	+ ·015,123	+ ·009,468	+ ·058,853	+ ·025,824
	(a)	0	− ·004,556	+ ·000,352	+ ·007,494	+ ·012,697	+ ·032,408	+ ·037,813
	(b)	0	− ·004,377	+ ·000,487	+ ·007,493	+ ·012,645	+ ·032,448	+ ·038,095
	(c)	0	− ·004,210	+ ·000,642	+ ·007,492	+ ·012,574	+ ·032,463	+ ·038,361

$S(a) = +·510,573$, $S(b) = +·517,476$, $S(c) = +·524,735$,

$v_a =$ 060,573, $v_b = -·017,476$, $v_c = +·025,265$.

respectively. The numerical values of $\bar{h}_s$ and $\bar{k}_{s'}$ can be easily ascertained from the table published recently of ordinates of normal curve to permilles of area*. Care must be taken in every case to give the correct sign to $\bar{h}_s$ and $\bar{k}_{s'}$.

Now if there were no correlation, $\bar{h}_s$ and $\bar{k}_{s'}$ combined would give the mean of the group $n_{ss'}$, and they give a fair approximation to the result if there are numerous categories, that is if the range of the categories be small.

The correlation found from these marginal centroids would then be

$$r_c = S\,(n_{ss'}\bar{h}_s\bar{k}_{s'})/N \text{ ....(ii)},$$

but as Ritchie-Scott has shown† this r_c diverges much more than r_ϕ the mean square contingency value from the true correlation, and considerably more than the tetrachoric or polychoric coefficients do. The reason for this is clear and was pointed out by one of us in 1913‡. Namely $\bar{h}_s$ and $\bar{k}_{s'}$ do not give the coordinates of the mean of $n_{ss'}$. In fact $n_{ss'}\bar{h}_s\bar{k}_{s'}$ is not the contribution of $n_{ss'}$ to the product-moment.

We propose in the present paper to give first the actual contributions of $n_{ss'}$ to the means and product-moments of the two variates and then to apply these results in order to obtain (a) a polychoric coefficient, and (b) a graph of the relation of the two variates.

The essential assumptions that will be made are the following:

(i) The marginal totals having been reduced to a normal scale, and the correlation being supposed to be r, we shall calculate what the contents of the sth-s'th cell would be on the assumption that the frequency surface is the normal surface represented by the given correlation and the marginal totals reduced to normal scales. We shall further calculate the x-moment, the y-moment and the xy product-moment of the sth-s'th cell on the same hypothesis.

(ii) From these data we shall determine the most suitable value to give to r, so that the actually observed frequencies differ least from those that would be given by such a correlation surface. We shall also obtain a formula for calculating the mean value of y for the array of B-variates, $n_{s.}$ in number, which corresponds to the sth category of A. We shall thus be in a position to plot the regression line of B on A and test at the same time the closeness with which it fits the thus calculated array means, both variates being represented on a normal scale.

We shall write the real coefficient of correlation of the population r, the coefficient as found from a single sth-s'th cell, as $r_{ss'}$, and those found from the $n_{s.}$ and $n_{.s'}$ arrays as $r_{s.}$ and $r_{.s'}$ respectively.

$\bar{h}_{ss'}$, $\bar{k}_{ss'}$ will be the A- and B-variate means of the sth-s'th cell and $\pi_{ss'}$ the product-moment, per unit of the population, of the frequency in the sth-s'th cell about the mean axes as determined from the marginal totals on the normal scale.

* See *Biometrika*, Vol. XIII. pp. 426–8.

† *Biometrika*, Vol. XII. p. 122.

‡ *Biometrika*, Vol. IX. p. 138.

(2) The developments we require involve the use of the tetrachoric functions. The tetrachoric function of the order t is given by*

$$\tau_t = \frac{1}{\sqrt{t!}}\left(-\frac{d}{dx}\right)^{t-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad \text{.......................(iii).}$$

The tetrachoric functions τ_1 to τ_6 are tabled for *positive* values of x in *Tables for Statisticians and Biometricians*† to five decimal places. For *negative* values of x tetrachoric functions of an odd order remain unchanged, but those of an even order must have their sign as given in the tables reversed.

It will frequently be needful to take the difference of the tetrachoric functions at the boundaries of a marginal category. Thus if $\tau_t(h)$ denotes the value of the tetrachoric function for $x = h$, we shall need for the sth marginal total

$$\tau_t(h_s) - \tau_t(h_{s-1})$$

This difference we shall write, for brevity,

$$\vartheta_s \tau_t,$$

and in obtaining its numerical value from tables of the tetrachoric functions it is essential to remember that s (or s') is supposed to increase in the positive direction of the axis of x (or y), and that when h (or k) is negative attention must be paid to changing the sign of the tabled value of τ_t, if t be even.

The formula for determining the successive tetrachoric functions for a given value of x is

$$\tau_t = xp_t\tau_{t-1} - q_t\tau_{t-2} \quad \text{..............................(iv),}$$

where p_t and q_t are given by the following table:

t	p_t	q_t	t	p_t	q_t
2	·707,1068	·000,0000	14	·267,2612	·889,4990
3	·577,3503	·408,2483	15	·258,1989	·897,0851
4	·500,0000	·577,3503	16	·250,0000	·903,6962
5	·447,2136	·670,8204	17	·242,5356	·909,5085
6	·408,2483	·730,2968	18	·235,7023	·914,6592
7	·377,9645	·771,5168	19	·229,4157	·919,2547
8	·353,5534	801,7838	20	·223,6068	·923,3804
9	·333,3333	·824,9578	21	·218,2179	·927,1051
10	316,2278	843,2740	22	·213,2007	·930,4842
11	·301,5113	858,1163	23	·208,5144	·933,5637
12	·288,6751	·870,3880	24	·204,1241	·936,3819
13	·277,3501	·880,7047	25	·200,0000	·938,9709

Since $\tau_1 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, it can be found at once from the tables for the ordinates of the normal curve, and will indeed have been computed at each division in order

* The reasons why the tetrachoric functions are tabled with the factor $1/\sqrt{t!}$ are: (*a*) because this factor greatly simplifies our formulae and (*b*) because a factor of some such order is essential, if we are to have manageable tabulated values. As a matter of fact the factor chosen reduces all tetrachoric functions to numerical values lying between 0 and 1.

† Cambridge University Press, pp. 42—51.

to determine $\bar{h}_{s\cdot}$ and $\bar{h}_{\cdot s'}$. It is then often simpler to work directly with (iv) rather than interpolate into the tabled values of the functions.

In an earlier paper* dealing with the tetrachoric functions one of us has shown that if

$$z = \frac{N}{2\pi\sqrt{(1-r^2)}}\,e^{-\frac{1}{2}\frac{x^2-2rxy+y^2}{1-r^2}}$$

be the equation to a normal correlation surface the variates being measured in the standard deviations as units, then

$$z/N = \tau_1\tau_1' + 2r\tau_2\tau_2' + 3r^2\tau_3\tau_3' + \ldots + (t+1)\,r^t\tau_{t+1}\tau'_{t+1} + \ldots \qquad \ldots\ldots(\text{v}),$$

where $\tau_t = \tau_t(x)$ and $\tau_t' = \tau_t(y)$.

Now in order to proceed further it is needful to determine the following integrals:

$$\int_{h_{s-1}}^{h_s} \tau_t\,dx, \qquad \int_{h_{s-1}}^{h_s} x\tau_t\,dx.$$

We can determine these by using (iii) after in the second case integrating by parts. We have:

$$\begin{aligned}\int_{h_{s-1}}^{h_s} \tau_t\,dx &= \frac{1}{\sqrt{t!}}\int_{h_{s-1}}^{h_s}\left(-\frac{d}{dx}\right)^{t-1}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}\,dx\\ &= -\frac{1}{\sqrt{t!}}\left[\left(-\frac{d}{dx}\right)^{t-2}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}\right]_{h_{s-1}}^{h_s}\\ &= -\frac{1}{\sqrt{t}}\left[\tau_{t-1}\right]_{h_{s-1}}^{h_s}.\end{aligned}$$

Again

$$\begin{aligned}\int_{h_{s-1}}^{h_s} x\tau_t\,dx &= \int_{h_{s-1}}^{h_s}\frac{1}{\sqrt{t!}}\,x\left(-\frac{d}{dx}\right)^{t-1}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}\,dx\\ &= \frac{1}{\sqrt{t!}}\left[(-x)\left(-\frac{d}{dx}\right)^{t-2}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}\right]_{h_{s-1}}^{h_s}\\ &\quad+\frac{1}{\sqrt{t!}}\int_{h_{s-1}}^{h_s}\left(-\frac{d}{dx}\right)^{t-2}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}\,dx\\ &= -\frac{1}{\sqrt{t}}\,\tau_{t-1}x\Big]_{h_{s-1}}^{h_s}+\frac{1}{\sqrt{t}}\int_{h_{s-1}}^{h_s}\tau_{t-1}\,dx\\ &= -\frac{1}{\sqrt{t}}\,\tau_{t-1}x-\frac{1}{\sqrt{t\,(t-1)}}\,\tau_{t-2}\Big]_{h_{s-1}}^{h_s}\\ &= -\frac{1}{\sqrt{t}}\left[\tau_{t-1}x+\frac{1}{\sqrt{t-1}}\,\tau_{t-2}\right]_{h_{s-1}}^{h_s}\qquad\ldots\ldots\ \ldots\ldots(\text{vi})\ \text{bis}.\end{aligned}$$

But by (iv):

$$\tau_{t-1}x = \frac{\tau_t + q_t\tau_{t-2}}{p_t},$$

where

$$p_t = 1/\sqrt{t}, \quad q_t = (t-2)/\sqrt{t\,(t-1)}.$$

* *Phil. Trans.* Vol. 195 A, p. 4, Equation (xiv), with a slight change of notation. In that paper, $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\frac{v_n}{\sqrt{(n+1)!}}$ is written for τ_{n+1} and $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\frac{w_n}{\sqrt{(n+1)!}}$ for τ'_{n+1}.

Thus: $$\tau_{t-1}x + \frac{1}{\sqrt{t-1}}\tau_{t-2} = \sqrt{t}\,\tau_t + \sqrt{t-1}\,\tau_{t-2}.$$

Accordingly
$$\int_{h_{s-1}}^{h_s} x\tau_t dx = -\frac{1}{\sqrt{t}}\left[\sqrt{t}\,\tau_t + \sqrt{t-1}\,\tau_{t-2}\right]_{h_{s-1}}^{h_s}$$
$$= -\frac{1}{\sqrt{t}}(\sqrt{t}\,\vartheta_s\tau_t + \sqrt{t-1}\,\vartheta_s\tau_{t-2}) \quad \text{.............. (vii).}$$

The latter form throws us back on $\vartheta_s\tau_t$ which will have to be calculated to determine the integral in (vi) for the successive values of t and s.

On the other hand a table of
$$T_{t-1} = \sqrt{t}\,\tau_t + \sqrt{t-1}\,\tau_{t-2} \quad \text{.......................... (viii)}$$
would be a convenient method of determining the integral and tables of T might be easily formed, say up to T_6.

In this case we may write (vii):
$$\int_{h_{s-1}}^{h_s} x\tau_t dx = -\frac{1}{\sqrt{t}}\vartheta_s T_{t-1} \quad \text{.......................... (ix).}$$

We are now in the position to compute all the requisite integrals we need; if we write $\bar{n}_{ss'}$ for the contents of the sth-s'th cell, then on the supposition that the surface is normal, has correlation r and follows the actual marginal frequencies, we have:
$$\frac{\bar{n}_{ss'}}{N} = \int_{h_{s-1}}^{h_s}\int_{k_{s'-1}}^{k_{s'}} \frac{z}{N}dxdy$$
$$= \vartheta_s\tau_0\vartheta_{s'}\tau_0' + r\vartheta_s\tau_1\vartheta_{s'}\tau_1' + r^2\vartheta_s\tau_2\vartheta_{s'}\tau_2' + \ldots + r^p\vartheta_s\tau_p\vartheta_{s'}\tau_p' + \ldots \quad \text{(x),}$$
$$\frac{\bar{n}_{ss'}}{N}\bar{h}_{ss'} = \int_{h_{s-1}}^{h_s}\int_{k_{s'-1}}^{k_{s'}} \frac{xzdxdy}{N} = -\vartheta_s T_0\vartheta_{s'}\tau_0' + r\vartheta_s T_1\vartheta_{s'}\tau_1' + r^2\vartheta_s T_2\vartheta_{s'}\tau_2'$$
$$+ \ldots + r^p\vartheta_s T_p\vartheta_{s'}\tau_p' + \ldots \quad \text{.......... (xi),}$$
$$\frac{\bar{n}_{ss'}}{N}\bar{k}_{ss'} = \int_{h_{s-1}}^{h_s}\int_{k_{s'-1}}^{k_{s'}} \frac{yzdxdy}{N} = \vartheta_s\tau_0\vartheta_{s'}T_0' + r\vartheta_s\tau_1\vartheta_{s'}T_1' + r^2\vartheta_s\tau_2\vartheta_{s'}T_2'$$
$$+ \ldots + r^p\vartheta_s\tau_p\vartheta_{s'}T_p' + \ldots \quad \text{.......... (xii),}$$
$$\frac{\bar{n}_{ss'}}{N}\pi_{ss'} = \int_{h_{s-1}}^{h_s}\int_{k_{s'-1}}^{k_{s'}} \frac{xyzdxdy}{N} = \vartheta_s T_0\vartheta_{s'}T_0' + r\vartheta_s T_1\vartheta_{s'}T_1' + r^2\vartheta_s T_2\vartheta_{s'}T_2'$$
$$+ \ldots + r^p\vartheta_s T_p\vartheta_{s'}T_p' + \ldots \quad \text{......... (xiii).}$$

It is desirable to say a few words about the functions τ_0 and T_0 which may at first present difficulties to the reader. $-\tau_0$ clearly stands for the integral
$$\int_{h_{s-1}}^{h_s}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx, \text{ i.e. } \int_{h_{s-1}}^{h_s}\tau_1 dx,$$
and is therefore simply $n_{s\cdot}/N$.

Similarly $-\tau_0' = n_{\cdot s'}/N$.

Next clearly $-\vartheta_s T_0$ stands for

$$\int_{h_{s-1}}^{h_s} \tau_1 x\,dx = \int_{h_{s-1}}^{h_s} \frac{1}{\sqrt{2\pi}} x e^{-\frac{1}{2}x^2} dx$$

$$= -\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\right]_{h_{s-1}}^{h_s}$$

$$= -\vartheta_s \tau_1,$$

or

$$\vartheta_s T_0 = \vartheta_s \tau_1,$$

which is precisely the value given by (viii).

Thus (viii) is shown to be correct even for this special case although a form like (vi)bis through which it is reached shows difficulties.

Similarly $\vartheta_s T_0' = \vartheta_s \tau_1'$ *.

The remainder of the τ's knowing τ_0 and τ_1 come directly from (iv) and the T's are always given by (viii).

Now it is clear that (x) to (xiii) provide a large number of ways of determining r. We might find r, i.e. $r_{ss'}$, from the single cell by writing in (x) $n_{ss'}$ for $\bar{n}_{ss'}$. Or we may find

$$\bar{h}_{s\cdot} = \frac{1}{n_{s\cdot}} \underset{s'}{S}(n_{ss'}\bar{h}_{ss'})$$

$$= \frac{N}{n_{s\cdot}} \underset{s'}{S}\left(\frac{n_{ss'}}{\bar{n}_{ss'}}\{\vartheta_s T_0' \vartheta_{s'} \tau_0' + r\vartheta_{s'} T_1' \vartheta_{s'} \tau_1' + \ldots + r^p \vartheta_s T_p \vartheta_{s'} \tau_p' + \ldots\}\right) \quad \text{(xiv)},$$

where $\bar{n}_{ss'}$ is given by (x). But $\bar{h}_{s\cdot}$ is the known centroid of the $n_{s\cdot}$ marginal total, and accordingly the above is an equation to find r, i.e. $r_{s'}$, from a given column of the table.

If we use this value of $r_{s\cdot}$ in (x) and (xii) to find $\bar{n}_{ss'}$, and $\bar{k}_{ss'}$, we obtain the theoretical cell frequency and y-mean of the cell as found from a column.

Now sum $\bar{k}_{ss'}$ for every value of s' and we find $\bar{k}_{s\cdot}$ the y mean of a column depending on the data as found from the column, i.e.

$$\bar{k}_{s\cdot} = \frac{N}{n_{s\cdot}} \underset{s'}{S}\left(\frac{n_{ss'}}{\bar{n}_{ss'}}\{\vartheta_s \tau_0 \vartheta_s T_0' + r\vartheta_s \tau_1 \vartheta_{s'} T_1' + \ldots + r^p \vartheta_s \tau_p \vartheta_{s'} T_p' + \ldots\}\right) \quad \text{(xv)},$$

where $n_{ss'}$ is the observed cell frequency and $\bar{n}_{ss'}$ the frequency found by (x) when we insert the value of r as found from (xiv). We are thus in a position theoretically to determine on a normal scale the mean of a column from the correlation actually determined from that column. This would be the ideal method of determining the mean of a row or column; but it would involve a great deal of hard work, as with the two regression curves we should need to find r for every row and column by an equation of a high order. Hence in most cases we are likely to content ourselves by finding r for the whole table and then use this value in (x) to determine $\bar{n}_{ss'}$ and in (xv) to find the mean of the array. $\bar{k}_{s\cdot}$ plotted to the known $\bar{h}_{s\cdot}$ on the normal scale will give the regression curve.

* We can thus take $T_0 = \tau_1$ and $T_0' = \tau_1'$.

The question now arises as to the manner in which we can find r for the whole table most effectively.

Clearly we might assume the product-moment components from (xiii) and sum for all cells. We should have

$$\sum_{s,s'}\left(\frac{n_{ss'}\pi_{ss'}}{N}\right)=r,$$

since the coordinates are measured from the means in terms of the standard deviations as units.

Hence substituting from (xiii) we have:

$$r=\underset{s,s'}{S}\left(\frac{n_{ss'}}{\bar{n}_{ss'}}\{\vartheta_s T_0\vartheta_{s'}T_0'+r\vartheta_s T_1\vartheta_{s'}T_1'+\ldots+r^p\vartheta_s T_p\vartheta_{s'}T_p'+\ldots\}\right)\quad\text{(xvi)}.$$

Here $\bar{n}_{ss'}$ must be substituted from (x) and we have finally

$$r=\underset{s,s'}{S}\left(\frac{n_{ss'}}{N}\left\{\frac{\vartheta_s T_0\vartheta_{s'}T_0'+r\vartheta_s T_1\vartheta_{s'}T_1'+\ldots+r^p\vartheta_s T_p\vartheta_{s'}T_p'+\ldots}{\vartheta_s\tau_0\vartheta_{s'}\tau_0'+r\vartheta_s\tau_1\vartheta_{s'}\tau_1'+\ldots+r^p\vartheta_s\tau_p\vartheta_{s'}\tau_p'+\ldots}\right\}\right)\quad\text{(xvi) bis.}$$

This equation based upon the product-moment method of finding r is clearly likely to be very complicated, and although it can be proved that the product-moment method is the "best" method of finding r when we are dealing with a series of quantitatively measured individuals, we have no certainty that it is the best method in the present case of broad categories. It may indeed be questioned whether another method now to be considered cannot be shown to be better or at least equally efficacious.

Let us consider for a moment what we have in view. We observe $n_{ss'}$ as the frequency of the sth-s'th cell; we find that with a given correlation r the frequency of this cell would be $\bar{n}_{ss'}$ on the assumption that the frequency surface is the normal frequency surface corresponding to the observed marginal totals. Accordingly the most probable value to give to r would be that which made

$$\chi^2=\underset{s,s'}{S}\frac{(\bar{n}_{ss'}-n_{ss'})^2}{\bar{n}_{ss'}}=\text{minimum},$$

or, what is the same thing,

$$\underset{s,s'}{S}\left(\frac{n^2_{ss'}}{\bar{n}_{ss'}}\right)=\text{minimum}.$$

This leads us, differentiating with regard to r, to

$$\underset{s,s'}{S}\left\{\left(\frac{n_{ss'}}{\bar{n}_{ss'}}\right)^2\frac{d\bar{n}_{ss'}}{dr}\right\}=0,$$

or, writing at length, our equation for r is:

$$\underset{s,s'}{S}\left\{\left(\frac{n_{ss'}}{N}\right)^2\frac{\vartheta_s\tau_1\vartheta_{s'}\tau_1'+2r\vartheta_s\tau_2\vartheta_{s'}\tau_2'+\ldots+pr^{p-1}\vartheta_s\tau_p\vartheta_{s'}\tau_p'+\ldots}{(\vartheta_s\tau_0\vartheta_{s'}\tau_0'+r\vartheta_s\tau_1\vartheta_{s'}\tau_1'+r^2\vartheta_s\tau_2\vartheta_{s'}\tau_2'+\ldots+r^p\vartheta_s\tau_p\vartheta_{s'}\tau_p'+\ldots)^2}\right\}=0\quad\text{(xvii)}.$$

Neither (xvi) nor (xvii) are very readily solved. Probably the easiest way will be to obtain an approximate value of r by existing methods either from a good fourfold table, or from contingency, and then evaluate (xvi) or (xvii) for values of r, one well above and one well below this result, so that the real value of r lies

between the two. A linear interpolation will probably suffice in most cases to determine r with sufficient accuracy.

It will be observed that what we are trying to do is to fit a normal correlation surface to a series of cell frequencies. We may do this by equating product-moments, or actual cell frequencies properly weighted. The factors $\frac{n_{ss'}}{\bar{n}_{ss'}}$ and $\left(\frac{n_{ss'}}{\bar{n}_{ss'}}\right)^2$ come into our equations as a form of weights. When $n_{ss'}$ is small as compared with $\bar{n}_{ss'}$ that cell will contribute less to the general equations for r, and when $n_{ss'}$ is large as compared with $\bar{n}_{ss'}$, the contribution will be considerable. If the observed results were closely normal then $n_{ss'}$ would be nearly $\bar{n}_{ss'}$. If we might assume the differences of $n_{ss'}$ and $\bar{n}_{ss'}$ so small as to be negligible we should have:

$$r = \underset{s,\,s'}{S}\,(\vartheta_s T_0 \vartheta_{s'} T_0' + r\vartheta_s T_1 \vartheta_{s'} T_1' + \ldots + r^p \vartheta_s T_p \vartheta_{s'} T_p' + \ldots \qquad \text{(xvi) ter,}$$

and

$$0 = \underset{s,\,s'}{S}\,(\vartheta_s \tau_1 \vartheta_{s'} \tau_1' + 2r\vartheta_s \tau_2 \vartheta_{s'} \tau_2' + \ldots + pr^{p-1} \vartheta_s \tau_p \vartheta_{s'} \tau_p' + \ldots \text{(xvii) bis,}$$

instead of (xvi) bis and (xvii). These equations it will be found are identically satisfied. Hence our values for r from (xvi) and (xvii) depend on $\bar{n}_{ss'}$ differing from $n_{ss'}$.

(3) We now proceed to illustrate the application of these results.

Stature of Father and Son.

The following table gives a correlation table for the inheritance of stature in Father and Son made up in broad categories corresponding to eye-colour groups*. Upon this material we shall be able to test our correlations and our graph against those found by definite numerical groupings

Stature of Father (Broad Categories).

Stature of Son (Broad Categories).	1	2	3	4	5	6	7	Totals
1′	4	22	7	—	1	—	—	34
2′	23	154	84	26	8	6	-	301
3′	8	87	75	66	22	24	2	284
4′	1	29	36	37	14	14	6	137
5′		18	27	26	11	18	5	105
6′	.	9	26	19	7	29	8	98
7′		3	9	6	6	10	7	41
Totals	36	322	264	180	69	101	28	1000

The positive direction of x is from left to right and of y vertically downwards. It will suffice to take the τ's to five decimal figures but it will be needful to go further with the τ's if the T's are to be taken correctly to five figures from (viii). The general reduction formula for the T's is:

$$T_t(x) = \frac{T_{t-1}(x)(t-2)\sqrt{t-1}\,x^3 - T_{t-2}(x)(t-3)((t-1)x^2+1)}{\sqrt{t}\,(t-1)(x^2(t-2)+1)} \qquad \ldots\text{(xviii),}$$

or,

$$= \frac{q_t}{x^2(t-2)+1}\left\{\frac{x^3}{p_{t-1}}T_{t-1}(x) - (x^2(t-1)+1)\frac{t-3}{t-2}T_{t-2}(x)\right\} \quad \text{(xviii) bis.}$$

* See *Biometrika*, Vol. IX. p. 220.

Hence if T_0 and T_1 be found accurately the remaining T's can be determined as accurately as we please without reference to the τ's.

But,
$$T_0 = \tau_1 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xix)},$$
$$T_1 = \sqrt{2}\,\tau_2 + \tau_0 = \frac{x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \quad \ldots\ldots\ldots\ldots\text{(xx)}.$$

Hence the tables of ordinates and areas of the normal curve readily provide T_0 and T_1 to seven decimal places, and (xviii) provides the higher T's. These were cut down to five figures and an approximate check on their values obtained by (viii).

As a matter of fact if r is of the order ·50 we cannot hope to obtain more than three figure accuracy in r without going to higher τ- and T-functions than the sixth, especially when using (xvii). But three figures in the correlation are usually adequate and the labour of computing is much increased if higher functions are used. Such must, however, be used if the correlation be sensibly higher than ·50.

The following table gives the $\frac{1}{2}(1+\alpha)$'s, h's, H's, x's, τ's, $\vartheta\tau$'s, T's, and ϑT's for the x-variate.

TABLE I.

$\frac{1}{2}(1+\alpha)$		·036	·358	·622	·802	·871	·972	1·000
h	$-\infty$	−1·79912	−·36381	+·31074	+·84879	+1·13113	+1·91101	$+\infty$
$H=\tau_1$	0	·07908	·37340	·38014	·27827	·21042	·06425	0
$x_s = \dfrac{H_{s-1}-H_s}{\frac{1}{2}(\alpha_s-\alpha_{s-1})}$		−2·19667	−·91401	−·02553	+·56594	+·98333	+1·44723	+2·29464
τ_0	0	−·036	−·358	·622	−·802	−·871	·972	1
$\tau_1 = T_0$	0	+·07908	+·37340	+·38014	+·27827	+·21042	+·06425	0
τ_2	0	−·10060	−·09606	+·08353	+·16701	+·16830	+·08682	0
τ_3	0	+·07221	−·13226	−·14021	·03176	+·02401	+·06952	0
τ_4	0	·00688	+·07952	−·07001	−·10990	−·08359	+·01634	0
τ_5	0	−·04291	+·07579	+·08432	−·02041	−·05839	−·03270	0
τ_6	0	+·03654	−·06933	+·06182	+·07319	+·03408	−·03744	0
$\vartheta\tau_0$		·036	−·322	−·264	·180	−·069	−·101	−·028
$\vartheta\tau_1$		·07908	+·29432	+·00674	−·10187	−·06785	·14617	−·06125
$\vartheta\tau_2$		−·10060	+·00154	+·17959	+·08348	+·00129	−·08148	−·08682
$\vartheta\tau_3$		+·07221	·20447	−·00795	+·10845	+·0557	+·04555	−·06956
$\vartheta\tau_4$		−·00688	+·08640	−·14953	−·03989	+·02631	+·09993	−·01634
$\vartheta\tau_5$		−·04291	+·11870	+·00853	−·10473	−·03798	+·02569	+·03270
$\vartheta\tau_6$		+·03654	−·10587	+·13115	+·01137	−·03911	−·07152	+·03744
T_1	0	−·17827	−·19385	−·50388	−·56581	−·63299	·84922	1
T_2	0	+·23690	+·29898	+·29475	+·33853	+·33915	+·21135	0
T_3	0	−·18799	−·00734	+·00466	+·06947	+·12432	+·18307	0
T_4	0	+·04848	−·09506	−·09186	−·10916	−·08255	+·06601	0
T_5	0	+·07412	+·07799	−·00511	−·06648	−·10343	·05518	0
T_6	0	−·08325	+·05616	+·05364	+·05379	+·01471	−·08491	0
ϑT_0		+·07908	+·29432	+·00674	−·10187	−·06786	−·14617	−·06425
ϑT_1		−·17827	−·31558	−·01003	−·06193	−·06718	−·21622	−·15078
ϑT_2		+·23690	+·06208	−·00422	+·04377	+·00063	−·12780	−·21135
ϑT_3		·18799	+·18065	+·01200	+·06481	+·05485	+·05874	−·18307
ϑT_4		+·04848	−·14354	+·00320	−·01731	+·02662	+·14856	−·06601
ϑT_5		−·07412	−·06613	−·01310	−·06137	−·03695	+·04825	+·05518
ϑT_6		−·08325	+·13941	−·00253	+·00015	−·03907	−·09962	+·08491

The following table gives the corresponding quantities $\frac{1}{2}(1+\alpha')$'s, k's, K's, y''s, τ''s, $\vartheta\tau_1'$'s T''s and $\vartheta T'$'s for the y-variate.

TABLE II.

$\frac{1}{2}(1+\alpha')$		·034	335	·619	·756	·861	·959	1·000
k		− 1·82501	− ·42615	+ ·30286	+ ·69349	+ 1·08482	+ 1·73920	+ ∞
$K - \tau_1'$		+ ·07545	+ ·36431	+ ·38106	+ ·31367	+ ·22149	+ ·08792	0
$y_s = \dfrac{K_{s-1} - K_s}{\frac{1}{2}(a_s' - a'_{s-1})}$		− 2·21916	·95968	− ·05896	+ ·49188	+ ·87789	+ 1·36301	+ 2·14436
	0	− ·031	− ·335	− ·619	− ·756	− ·861	− ·959	− 1
T_0'	0	+ ·07545	+ ·36431	+ ·38106	+ ·31367	+ ·22149	+ ·08792	0
	0	·09737	− ·10978	+ ·08160	+ ·15382	+ ·16990	+ ·10812	0
	0	+ ·07179	− ·12172	− ·14130	− ·06647	+ ·01599	+ ·07268	0
	0	·00929	+ ·08932	− ·06851	− ·11185	− ·08942	+ ·00077	0
	0	·04057	+ ·06463	+ ·08551	+ ·00990	− ·05411	− ·04815	0
τ_6'	0	+ ·03702	·07647	+ ·06061	+ ·08419	+ ·04134	− ·03475	0
$\vartheta\tau_0'$		·034	− ·301	·284	− ·137	− ·105	− ·098	− ·041
$\vartheta\tau_1'$		+ ·07545	+ ·28886	+ ·01675	− ·06739	− ·09218	− ·13357	− ·08792
$\vartheta\tau_2'$		− ·09737	− ·01241	+ ·19138	+ ·07222	+ ·01608	− ·06178	− ·10812
$\vartheta\tau_3'$		+ ·07179	− ·19351	·01958	+ ·07483	+ ·08246	+ ·05669	·07268
$\vartheta\tau_4'$		− ·00929	+ ·09861	− ·15783	− ·04331	+ ·02213	+ ·09019	− ·00077
$\vartheta\tau_5'$		− ·04057	+ ·10520	+ ·02088	− ·07561	− ·06401	+ ·00596	+ ·04815
$\vartheta\tau_6'$		+ ·03702	− ·11349	+ ·13708	+ ·02388	− ·04315	− ·07609	+ ·03475
T_1'	0	− ·17170	− ·19025	− ·50360	− ·53847	− ·62073	− ·80610	− 1
T_2'	0	+ ·23105	+ ·30439	+ ·29416	+ ·32847	+ ·34094	+ ·25021	0
T_3'	0	− ·18723	− ·01151	+ ·00132	+ ·04271	+ ·11544	+ ·18882	0
T_4'	0	+ ·05286	− ·09892	− ·09140	·11081	− ·08901	+ ·03768	0
T_5'	0	+ ·06989	+ ·01210	− ·00471	− ·04316	− ·09869	− ·08340	0
T_6'	0	·08412	+ ·05897	+ ·05326	+ ·06263	+ ·02276	− ·08010	0
		+ ·07515	+ ·28886	+ ·01675	− ·06739	− ·09218	·13358	·08792
		·17170	− ·31855	·01331	− ·03188	− ·08225	·18537	− ·19391
$\vartheta T_2'$		+ ·23105	+ ·07334	− ·01023	+ ·03431	+ ·01247	− ·09072	− ·25021
$\vartheta T_3'$		− ·18723	+ ·17572	+ ·01583	+ ·03839	+ ·07273	+ ·07338	− ·18882
$\vartheta T_4'$		+ ·05286	− ·15178	+ ·00752	·01941	+ ·02179	+ ·12670	− ·03768
$\vartheta T_5'$		+ ·06989	·05749	− ·01714	− ·03842	− ·05553	+ ·10529	+ ·08340
$\vartheta T_6'$		− ·08112	+ ·11309	·00571	+ ·00937	− ·03988	·10286	+ ·08010

From Tables I and II we can find from (x) the value of $\bar{n}_{ss'}/N$ for any given value of r, and by equating $\bar{n}_{ss'}/N$ to $n_{ss'}/N$ we should have an equation to determine the correlation r from that cell alone. The weighted mean of these 49 r's would be Ritchie-Scott's polychoric correlation coefficient. But the labour would be immense*.

We are now in a position to give the product of $\vartheta_s\tau_p\vartheta_{s'}\tau_p'$: see Table III, p. 138.

There are certain checks on the accuracy of this table, namely

$$S_{ss'}\,\vartheta_s\,\tau_p\,\vartheta_{s'}\,\tau_p' = 0 \text{ except for } p=0, \text{ when it} = 1.$$

* We are not underrating the large amount of arithmetic of the present process. It is not likely to be often repeated, and the sole purpose of publishing all these tables for an individual case is to impress the reader with that fact; while at the same time illustrating the actual numerical processes. The amount of arithmetic, great as it is, is relatively small compared with that of solving and weighting the resulting r's in the case of a 49-cell table.

TABLE III.

Values of $\vartheta_s\tau_p\vartheta_{s'}\tau_p{}'$.

		s=1		s=3	s=4		s=6	s=7		
	0	+ ·001,224	+ ·010,948	+ ·008,976	+ ·006,120	+ ·002,346	+ ·003,434	+ ·000,952	0	
	1	+ ·005,967	+ ·022,206	+ ·000,509	− ·007,686	− ·005,119	− ·011,029	− ·001,848	1	
	2	+ ·009,795	− ·000,112	·017,487	− ·008,128	− ·000,126	+ ·007,934	+ ·008,451		
	3	+ ·005,184	− ·014,679	− ·000,571	+ ·007,786	+ ·004,004	+ ·003,270	− ·004,994		
	4	+ ·000,064	·000,803	+ 001,389	+ ·000,371	− ·000,244	− ·000,928	+ ·000,152		
		+ ·001,741	− ·004,816	− ·000,316	+ ·004,219	+ ·001,541	− ·001,042	− ·001,327		
	6	+ ·001,353	− ·003,919	+ ·004,855	+ ·000,421	− ·001,148	− ·002,648	+ ·001,386		
	0	+ ·010,836	+ ·096,922	+ ·079,464	+ ·054,180	+ ·020,769	+ ·030,401	+ ·008,428	0	
	1	+ ·022,843	+ ·085,017	+ 001,947	·029,426	− ·019,599	·042,223	− ·018,559	1	
	2	+ ·001,248	− ·000,056	− ·002,229	− ·001,036	− ·000,016	+ ·001,011	+ ·001,077	2	
	3	− 013,973	+ ·039,567	+ ·001,538	−− ·020,986	− ·010,792	− ·008,814	+ ·013,461	3	
	4	− ·000,678	+ ·008,520	− ·014,745	− ·003,934	+ ·002,594	+ ·009,854	− ·001,611	4	
		− ·004,514	+ ·012,487	+ ·000,897	− ·011,018	− ·003,995	+ ·002,703	+ ·003,410	5	
		·004,147	+ ·012,015	− ·014,884	·001,290	+ ·001,439	+ ·008,117	− ·004,249	6	
	0	+ ·010,224	+ ·091,448	+ ·074,976	+ ·051,120	+ ·019,596	+ ·028,684	+ ·007,952	0	
		+ ·001,325	+ ·004,930	+ ·000,113	− ·001,706	− ·001,136	002,418	·001,076	1	
		·019,253	+ ·000,869	+ ·031,370	+ ·015,976	+ ·000,247	− ·015,594	− 016,616	2	
		·001,414	+ ·004,004	+ ·000,156	·002,123	− ·001,092	− ·000,892	+ ·001,362	3	
		+ ·001,086	− ·013,637	+ ·023,600	+ ·006,296	− ·004,153	− ·015,772	+ ·002,579	4	
		·000,896	+ ·002,478	+ ·000,178	− ·002,187	− ·000,793	+ ·000,536	+ ·000,683	5	
		+ ·005,009	− ·014,513	+ ·017,978	+ ·001,559	− ·005,361	·009,801	+ ·005,132	6	
	0	+ ·004,932	+ ·011,114	+ ·036,168	+ ·024,660	+ ·009,153	+ 013,837	+ ·003,836	0	
	1	− ·005,329	− ·019,834	− ·000,451	+ ·006,865	+ ·004,572	+ ·009,850	+ ·004,330	1	
	2	− ·007,265	+ ·000,328	+ ·012,970	+ ·006,029	+ ·000,093	005,884	− ·006,270	2	
1	3	+ ·005,403	− 015,300	− ·000,595	+ 008,115	+ ·004,173	+ ·003,409	·005,205	3	
	4	+ ·000,298	− ·003,745	+ ·006,481	+ ·001,729	− ·001,140	·004,331	+ 000,708	4	
	5	+ ·003,244	− ·008,975	− ·000,615	+ ·007,919	+ ·002,872	·001,942	− ·002,472	5	
	6	+ ·000,873	− ·002,528	+ ·003,132	+ ·000,272	− ·000,934	− ·001,708	+ ·000,894	6	
		+ ·003,780	+ ·033,810	+ ·027,720	+ ·018,900	+ ·007,245	+ ·010,605	+ 002,940	0	
		− ·007,290	− ·027,130	− ·000,621	+ ·009,390	+ ·006,254	+ ·013,474	+ ·005,923	1	
		− 001,678	+ ·000,073	+ ·002,888	+ ·001,342	+ ·000,021	− ·001,310	− ·001,396	2	
		+ ·005,954	·016,861	·000,656	+ ·008,913	+ ·004,599	+ ·003,756	− ·005,736	3	
		− ·000,154	+ ·001,938	·003,354	− ·000,895	+ ·000,590	+ ·002,241	− ·000,367	4	
		+ ·002,747	− ·007,598	− ·000,546	+ ·006,704	+ ·002,431	− ·001,644	·002,093	5	
		− ·001,577	+ ·004,568	− ·005,659	·000,491	+ ·001,688	+ ·003,086	− ·001,616	6	
	0	+ ·003,528	+ ·031,556	+ ·025,872	+ ·017,640	+ ·006,762	+ ·009,898	+ ·002,714	0	
	1	− ·010,563	− ·039,312	− ·000,900	+ 013,607	+ ·009,063	+ ·019,524	+ ·008,582	1	
	2	+ ·006,215	− ·000,280	− ·011,095	− 005,157	− ·000,080	+ ·005,031	+ ·005,364	2	
	3	+ ·004,094	− ·011,591	− ·000,451	+ ·006,148	+ ·003,162	+ ·002,582	− ·003,943	3	6
	4	− ·000,621	+ ·007,792	− ·013,486	− ·003,598	+ ·002,373	+ ·009,013	− ·001,474	4	
	5	− ·000,256	+ ·007,107	+ ·000,051	− ·000,624	− ·000,226	+ ·000,153	+ ·000,195	5	
	6	− ·002,780	+ ·008,056	− ·009,979	− ·000,865	+ ·002,976	+ ·005,442	− ·002,849	6	
	0	− ·001,476	+ ·013,202	+ ·010,824	+ ·007,380	+ ·002,829	+ ·004,141	+ ·001,148	0	
	1	− ·006,953	− ·025,877	− ·000,593	+ ·008,956	+ ·005,965	+ ·012,851	+ ·005,649	1	
	2	+ ·010,877	− ·000,491	− ·019,417	− ·009,026	−− ·000,139	+ ·008,810	+ ·009,387	2	
	3	− ·005,248	+ ·014,861	+ ·000,578	− ·007,882	− ·004,053	− ·003,311	+ ·005,056	3	
	4	+ ·000,005	− ·000,067	+ ·000,115	+ ·000,031	− ·000,020	− ·000,077	+ ·000,013	4	
	5	− ·002,066	+ ·005,715	+ ·000,411	− ·005,043	− ·001,829	+ ·001,237	+ ·001,575	5	
	6	+ ·001,270	− ·003,679	+ ·004,557	+ ·000,395	− ·001,359	− ·002,485	+ ·001,301	6	

Applying these tests we find:

$$S_{ss'}(\vartheta_s \tau_0 \vartheta_{s'} \tau_0') = \ \ 1{\cdot}000{,}000, \quad S_{ss'}(\vartheta_s \tau_1 \vartheta_{s'} \tau_1') = +\cdot000{,}001,$$
$$S_{ss'}(\vartheta_s \tau_2 \vartheta_{s'} \tau_2') = +\ \cdot000{,}001, \quad S_{ss'}(\vartheta_s \tau_3 \vartheta_{s'} \tau_3') = +\cdot000{,}003,$$
$$S_{ss'}(\vartheta_s \tau_4 \vartheta_{s'} \tau_4') = -\ \cdot000{,}002, \quad S_{ss'}(\vartheta_s \tau_5 \vartheta_{s'} \tau_5') = +\cdot000{,}001,$$

and

$$S_{ss'}(\vartheta_s \tau_6 \vartheta_s \tau_6') = +\cdot000{,}002,$$

results as close as we should expect, when we take into account the fact that our $\vartheta\tau$'s were only to five figure accuracy, and our products to six.

The meaning of Table III should be quite intelligible; namely, for example:

$$\cdot084 = \frac{n_{3,2}}{N} = \cdot079{,}464 + \cdot001{,}947\,r - \cdot002{,}229\,r^2$$
$$+ \cdot001{,}538\,r^3 - \cdot014{,}745\,r^4 + \cdot000{,}897\,r^5 - \cdot014{,}884\,r^6 + \ldots \quad \ldots(\text{xxi})$$

is the equation which will give the correlation coefficient r as deduced from the (3, 2) cell. If r be given any other value the right hand of the above expression is equal to the contents of the (3, 2) cell for a normal correlation surface of correlation coefficient r having the observed marginal totals.

Thus far the arithmetic is absolutely comparable with that needed for Ritchie-Scott's "polychoric r." We should have to solve the 49 equations, and then calculate—the stiffest part of the work—the probable errors of the 49 correlation coefficients which are the roots of these equations. Using these probable errors as our weighting data, we should find a mean coefficient. Our purpose is to replace the weighting and the solution of the 49 equations by the solution of a single equation. It will be noticed that both Ritchie-Scott's and our methods have an undesirable limitation, for we both assume the marginal totals to be those of the normal correlation surface. Actually in our case we ought to treat the marginal totals as unknown, or select $h_1, h_2, h_3, \ldots h_q, k_1, k_2, k_3, \ldots k_{q'}$ as well as r to give as closely as possible the observed frequencies. Now the τ's and consequently the T's and $\vartheta\tau$'s and ϑT's all depend upon the h's and k's and the equations obtained by making

$$S_{ss'}\left(\frac{n^2_{ss'}}{\bar{n}_{ss'}}\right) = \text{minimum}$$

do not appear to lend themselves to any reasonably brief system of solutions. We were compelled therefore to introduce the admittedly limited form of solution, i.e. the determination of the best normal correlation surface subject to the restriction of its having the same marginal totals as the observed frequency surface. We consider this a practically necessary but none the less grave restriction.

We next proceeded to determine the value of $\bar{n}_{ss'}/n_{ss'}$ and $(\bar{n}_{ss'}/n_{ss'})^2$ for certain selected values of r in order to build up equation (xvii) and solve it by interpolation. The values chosen were: 0·45, 0·50 and 0·55. These cover the range within which we anticipate the solution of (xvii) for r will lie. We need also the value of the numerator in (xvii), i.e.

$$\nu_{ss'} = \vartheta_s \tau_1 \vartheta_{s'} \tau_1' + 2r\,\vartheta_s \tau_2 \vartheta_{s'} \tau_2' + 2r^2\,\vartheta_s \tau_3 \vartheta_{s'} \tau_3' + \ldots,$$

for the same three values of r. These results are given in Table IV.

TABLE IV. *Values of* $(\bar{n}_{ss}/n_{ss})$, $(\bar{n}_{ss}/n_{ss})^2$ *and* $\nu_{ss'}$.

Function		$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
$n_{ss'}/n_{ss'}$	(a)	1·602,750	·879,951	·814,714	∞	·388,000	∞	∞
	(b)	1·846,000	·902,000	·705,571	∞	·266,000	∞	∞
	(c)	2·115,500	·916,409	·587,857	∞	·174,000	∞	∞
$(\bar{n}_{ss}/n_{ss})^2$	(a)	2·568,806	·774,321	·663,759	∞	·150,544	∞	∞
	(b)	3·407,716	·813,604	·497,830	∞	·070,756	∞	∞
	(c)	4·475,340	·839,805	·315,576	∞	·030,276	∞	∞
ν_{ss}	(a)	+ ·018,462	+ ·011,177	− ·014,603	− ·009,218	·002,733	− ·002,747	·000,336
	(b)	+ ·020,480	+ ·008,113	− ·015,910	− ·008,382	· ·002,154	− ·001,929	− ·000,218
	(c)	+ ·022,694	+ ·001,177	− ·017,013	− ·007,243	· ·001,519	− ·001,228	− ·000,168
n_{ss}/n_{ss}	(a)	·867,348	·905,539	·914,250	1·478,500	1·379,000	1·887,333	∞
	(b)	·894,565	·944,630	·939,833	1·383,615	1·215,375	1·544,66'	∞
	(c)	·915,130	·986,935	·933,333	1·278,500	1·043,500	1·213,333	∞
$(n_{ss}/n_{ss})^2$	(a)	752,293	·820,001	·891,608	2·185,962	1·901,641	3·562,026	∞
	(b)	800,247	·892,326	·883,286	1·914,390	1·477,136	2·385,996	∞
	(c)	·837,463	·974,041	·871,110	1·634,562	1·088,892	1·472,177	∞
	(a)	+ ·013,846	+ ·116,000	− ·005,963	·046,943	− ·025,552	− ·041,623	− ·009,764
	(b)	+ ·011,084	+ ·125,051	009,011	- ·051,854	− ·026,828	− ·040,529	− ·007,913
	(c)	+ ·007,767	+ ·135,874	− ·013,006	·057,659	− ·028,171	·038,864	·005,940
s/n_{ss}	(a)	·857,750	1·075,552	1·108,280	·812,500	·851,818	·981,375	2·194,000
	(b)	·751,875	1·076,195	1·138,747	·823,410	·844,773	·930,333	1·846,500
	(c)	·635,750	1·075,448	1·175,027	·835,909	831,636	·866,000	1·486,500
$(n_{ss}/n_{ss})^2$	(a)	·735,735	1·156,812	1·228,285	·660,156	·730,714	·968,994	1·813,636
	(b)	·565,316	1·158,196	1·296,745	·678,001	·713,641	·865,519	3·409,562
	(c)	·404,178	1·156,588	1·380,688	·698,741	·691,618	·719,956	2 209,682
ν_{ss}	(a)	·016,095	+ ·002,075	+ ·041,770	+ ·013,402	·003,817	− 023,749	·013,555
	(b)	·017,786	+ ·000,037	+ ·049,827	+ ·015,435	− 005,038	− 028,268	·011,205
	(c)	·019,311	− ·002,805	+ ·059,278	+ ·017,601	·006,601	033,522	− ·014,539
$\bar{n}_{ss}/n_{ss}$	(a)	1·634,000	1·155,897	1·078,222	·808,892	·850 571	1·225,786	·671,833
	(b)	1·260,000	1·096,966	1·098,117	·837,135	877,714	1 239,929	627,333
	(c)	·917,000	1·030,862	1·121,944	·869,568	·907,129	1·250,000	·570,000
$)^2$	(a)	2·669,956	1·336,098	1 162,563	·654,306	·723,471	1 502,551	·451,360
	(b)	1·587,600	1·203,334	1·206,520	·700,795	·770,382	1·537,121	·393,547
	(c)	·840,889	1·062,676	1·258,758	·756,119	·823,127	1·562,500	321,900
	(a)	− ·007,715	− ·032,319	+ ·013,434	+ ·019,505	+ 007,261	+ ·004,459	004,625
	(b)	007,215	·036,132	+ ·015,696	+ 022,370	+ 007,947	+ ·003,430	− ·006,095
	(c)	− ·006,471	·040,720	+ ·018,237	+ ·025,717	+ 008,735	+ ·002,185	007,680
	(a)	∞	1 114,278	1·028,556	·934,423	·960,545	·935,111	·946,600
	(b)	∞	1·006,167	1·027,185	·969,000	1·008,273	978,941	914,400
	(c)	∞	·890,389	1·024,148	1·007,692	1·061,727	1 025,111	·927,100
	(a)	∞	1 241,615	1·057,927	·873,146	·922,647	·874,433	·896,052
	(b)	∞	1·012,372	1·055,109	·938,961	1·016,614	·958,331	·891,891
	(c)	∞	792,793	1·048,879	1·015,443	1·127,261	1·050,853	·860,071
	(a)	·004,797	− ·037,653	− ·000,381	+ ·017,025	+ ·009,967	+ ·015,398	+ ·000,440
	(b)	·003,957	− 040,252	− ·001,134	+ 018,995	+ ·011,095	+ ·016,166	- ·000,916
	(c)	·002,988	− ·043,158	− ·002,230	+ ·021,305	+ ·012,165	+ 017,113	- ·002,508
n_{ss} n_{ss}	(a)		1 461,333	·867,077	1·216,471	1·604,286	701,931	·906,500
	(b)	∞	1 224,000	·830,576	1·245,526	1·693,714	·754,966	·969,125
	(c)	∞	·988,111	·786,077				
6 $(n_{ss}/n_{ss})^2$	(a)	∞	2·135,494	·751,823	1·179,809	2·573,734	·492,707	·821,742
	(b)	∞	1·498,176	·689,856	1·551,335	2·868,667	·569,974	·939,203
	(c)	∞	·976,363	·617,917	1·622,538	3·207,681	·660,689	1·057,555
ν_{ss}	(a)	− ·003,069	- ·042,728	− 017,170	+ ·011,165	+ ·012,060	+ 029,542	+ ·010,202
	(b)	− ·002,189	− ·042,658	- ·020,931	+ ·010,905	+ ·013,028	+ ·032,069	+ ·009,778
	(c)	− ·001,381	− ·042,197	− ·025,479	+ ·010,572	+ ·014,219	+ ·035,116	+ ·009,152
n_{ss}/n_{ss}	(a)	∞	·961,333	·747,556	1·462,667	·845,000	1·140,500	·870,286
	(b)	∞	705,000	·648,556	1·411,167	·865,000	1·235,000	1·003,143
	(c)	∞	·491,000	·542,000	1·337,333	·877,000	1·331,000	1·150,286
7 $(n_{ss'}/n_{ss})^2$	(a)	∞	·924,161	·558,840	2·139,395	·714,025	1·300,740	·757,398
	(b)	∞	·497,025	·420,625	1·991,392	·748,225	1·525,225	1·006,296
	(c)	∞	·241,081	·293,764	1·788,460	·769,129	1·771,561	1·323,158
$\nu_{ss'}$	(a)	− ·000,633	− ·016,551	− ·017,086	- ·004,935	+ ·002,845	+ ·018,719	+ ·017,641
	(b)	− ·000,417	- ·014,160	- ·018,536	− ·007,468	+ ·001,950	+ ·019,060	+ ·019,571
	(c)	− ·000,309	− ·011,471	- ·019,787	− ·010,293	+ ·000,873	+ ·019,302	+ ·021,685

The values (a), (b), (c) refer respectively to $r=0·45$, $0·50$, $0·55$.

Having obtained $(\bar{n}_{ss'}/n_{ss'})^2$ and $\nu_{ss'}$ for the trial values of r, it is only a matter of adding $\nu_{ss'}/(\bar{n}_{ss'}/n_{ss'})^2$ for all values of s and s' on the machine in order to obtain:

$$u = S_{ss'}\,\{\nu_{ss'}/(\bar{n}_{ss'}/n_{ss'})^2\}.$$

The values obtained were:

$r =$	0·45	0·50	0·55
$u =$	+ ·157,074	+ ·012,276	− ·209,976

Whence by inverse interpolation* we find:

$$u = 0 \text{ for } r_p = ·5034,$$

which is "polychoric r" as based upon Equation (xvii). We shall compare later the value for r as found by other processes. But the above value is clearly well in accord with the usual result for paternal correlation in man.

Table V gives the working values of $\nu_{ss'}/(\bar{n}_{ss'}/n_{ss'})^2$

TABLE V. *Values of* $\nu_{ss'}/(\bar{n}_{ss'}/n_{ss'})^2$†.

	$s=1$	$s=2$	$s=3$	$s=4$		$s=6$	$s=7$
	+ ·007,186	+ ·014,435	− ·022,001	0	·018,159	0	0
	+ ·006,010	+ ·009,972	·031,958	0	− ·030,421	0	0
	+ ·005,071	+ ·005,331	·049,232	0	− ·050,132	0	0
	+ ·018,405	+ ·141,163	·006,688	− ·021,475	− ·013,436	− ·011,685	0
	+ ·013,851	+ ·140,141	·010,202	− ·027,086	− ·018,162	− ·016,986	0
	+ ·009,271	+ ·139,495	− ·014,930	− ·035,275	− ·025,871	− ·026,399	0
	·021,876	+ ·001,791	+ ·034,007	+ ·020,301	− ·005,265	− ·024,509	− ·002,816
	·031,462	+ ·000,032	+ ·038,425	+ ·022,755	− ·007,060	− ·032,660	·004,166
	·047,779	− ·002,125	+ ·042,934	+ ·025,189	·009,541	− ·044,832	− ·006,579
	·002,890	− ·024,189	+ ·011,556	+ ·029,810	+ ·010,036	+ ·002,968	·010,247
	·004,543	− ·030,027	+ ·013,003	+ ·031,921	+ ·010,316	+ ·002,231	− ·015,487
	·007,695	− ·038,318	+ ·014,488	+ ·034,010	+ ·010,608	+ ·001,398	− ·023,039
(a)	0	·030,326	·000,360	+ ·019,198	− ·010,803	+ ·017,610	+ ·000,491
(b)	0	− ·039,760	− ·001,075	+ ·020,230	+ ·010,914	+ ·016,869	·001,027
(c)	0	− ·054,139	·002,126	+ ·020,981	+ ·011,058	+ ·016,285	− ·002,916
	0	− ·020,008	− ·022,838	+ ·007,545	+ ·004,686	+ ·059,958	+ ·012,415
	0	·028,471	·030,341	+ ·007,029	+ ·004,542	+ ·056,261	+ ·010,411
	0	− ·043,219	− ·041,231	+ ·006,516	+ ·004,433	+ ·053,151	+ ·008,654
	0	− ·017,910	− ·030,571	·002,307	+ ·003,984	+ ·014,391	+ ·023,291
	0	·028,191	− ·044,067	− ·003,750	+ ·002,606	+ ·012,497	+ ·019,449
	0	− ·047,576	·067,356	− ·005,755	+ ·001,135	+ ·010,895	+ ·016,389

$S(a) = +·157,074$, $S(b) = +·012,276$, $S(c) = -·209,976$.

* The formula used was *Casus* I or $z_\theta = z_0 + \frac{1}{2}\theta(\Delta z_{-1} + \Delta z_0) + \frac{1}{2}\theta^2\delta^2 z_0$, the solution of the quadratic giving θ.

† The table suggests, *a posteriori*, that we should have got quite reasonable results from linear interpolation; we have: from (a) and (b) $r = ·5042$; from (a) and (c) $r = ·4928$, and from (b) and (c) $r = ·5025$, as against our ·5034. It should be noticed that the values in Table V are not always in agreement in the last figure with those obtained by dividing $\nu_{ss'}$ in Table IV by the $(\bar{n}_{ss'}/n_{ss'})^2$ of that table, because the somewhat more accurate process was adopted of multiplying $\nu_{ss'}$ by $n^2_{ss'}$ and then dividing by $\bar{n}^2_{ss'}$. Still the physical meanings of $\bar{n}_{ss'}/n_{ss'}$ and $(\bar{n}_{ss'}/n_{ss'})^2$ are so prominent in the work that it seemed desirable to register their values.

Before we consider the graph due to this solution, let us investigate the value of r to be found from (xvi). The values of $\bar{n}_{ss'}/n_{ss'}$ are already provided in Table IV, but we need a table corresponding to Table III giving the product $\vartheta_s T_p \vartheta_{s'} T_p'$ instead of the product $\vartheta_s \tau_p \vartheta_s \tau_p'$. This is provided in Table VI. Further if

$$\kappa_{ss'} = \vartheta_s T_0 \vartheta_s T_0' + r \vartheta_s T_1 \vartheta_{s'} T_1' + r^2 \vartheta_s T_2 \vartheta_{s'} T_2' + \ldots,$$

Table VII (p. 143) provides κ_{ss} for the same three values of r, i.e. 0·45, 0·50 and 0·55. Finally Table VIII (p. 143) gives $\kappa_{ss'}/(\bar{n}_{ss'}/n_{ss'})$, whence by summing we obtain

$$v = r - S_{ss'} \{\kappa_{ss'}/(\bar{n}_{ss}/n_{ss'})\},$$

for the three cases.

Using the same interpolation formula as before in order to discover the value of r for which $v = 0$ we find:

$$r = \cdot 5204.$$

There is thus a difference of ·0170 between the two methods. The probable error found for the product-moment r is ·0160 and the result by the usual product-moment process may be given:

$$r = \cdot 5189 \pm \cdot 0160.$$

Thus either of the values reached by the methods of this paper differ by less than the probable error from the true product-moment value.

(4) If we work out the results by mean square contingency we find:

$$C_2 = \cdot 480{,}690,$$

and the class index correlations are*:

$$\text{For fathers: } r_{c_f} = \cdot 962{,}329.$$
$$\text{For sons: } r_{c_s} = \cdot 964{,}523.$$

Hence correlation from mean square contingency

$$r = C_2/(r_{c_f} r_{c_s}) = \cdot 5179,$$

which is in excellent agreement with the product-moment value.

It would therefore be quite reasonable for such a table as the present to use mean square contingency and class index corrections, and save the heavy labour of Equation (xvi bis) or (xvii). At the same time we cannot assert that this process would always be equally satisfactory for tables with but few broad-categories and with much higher correlation.

Our two processes seem to give values slightly in defect and in excess of the true value of r, and we might use their mean, i.e. ·5118, to obtain our graph. We shall, however, first proceed to compare the actual results of solving (xiv) and substituting in (xv) with the result of such approximative processes.

Table IX (p. 145) gives the products of $\vartheta_s T_p \vartheta_{s'} \tau_p'$ and will therefore enable us by aid of Table IV (p. 140) which gives the values of $\bar{n}_{ss'}/n_{ss}$ to obtain $\bar{h}_s$ for any value of r. Let

$$\lambda_{ss'} = \vartheta_s T_0 \vartheta_s \tau_0' + r \vartheta_s T_1 \vartheta_s \tau_1' + r^2 \vartheta_s T_2 \vartheta_{s'} \tau_2' + \ldots \ldots \ldots \text{(xxii)}.$$

* Using the values of x_s and y_s in Tables I and II respectively.

TABLE VI.

Values of $\vartheta_s T_p \vartheta_{s'} T_p'$.

s'	p	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$	
	0	+ ·005,966	+ ·022,207	+ ·000,509	- ·007,686	− ·005,120	− ·011,028	− ·004,848	0
	1	+ ·030,608	+ ·054,185	+ ·001,722	+ ·010,633	+ ·011,536	+ ·037,126	+ ·025,890	1
	2	+ ·054,736	+ ·014,342	− ·000,976	+ ·010,114	+ ·000,145	− ·029,528	− ·048,833	2
	3	+ ·035,199	− ·033,825	- ·002,246	- ·012,136	− ·010,270	− ·010,999	+ ·034,276	3
	4	+ ·002,562	− ·007,587	+ ·000,169	- ·000,915	+ ·001,407	+ ·007,853	− ·003,489	4
	5	+ ·005,180	− ·004,622	·000,915	·004,289	− ·002,583	+ ·003,372	+ ·003,856	5
	6	+ ·007,003	− ·011,727	·000,213	− ·000,013	+ ·003,287	+ ·008,380	- ·007,142	6
	0	+ ·022,812	+ ·085,018	+ ·001,948	− ·029,426	− ·019,601	− ·042,222	− ·018,559	0
	1	+ ·056,787	+ ·100,528	+ ·003,195	+ ·019,728	+ ·021,402	+ ·068,878	+ ·048,033	1
	2	+ ·017,375	+ ·004,553	·000,310	+ ·003,210	+ ·000,046	− ·009,373	− ·015,501	2
	3	− ·033,035	+ ·031,745	+ ·002,108	+ ·011,389	+ ·009,639	+ ·010,323	- ·032,169	3
	4	·007,358	+ ·021,786	·000,486	+ ·002,627	− ·004,040	− ·022,548	+ ·010,019	4
		·004,261	+ 003,802	+ ·000,753	+ ·003,528	+ ·002,124	− ·002,771	− ·003,172	5
		·011,913	+ ·019,949	·000,362	+ ·000,022	− ·005,591	− ·014,255	+ ·012,150	6
		+ ·001,324	+ ·004,929	+ ·000,113	- ·001,706	− ·001,136	− ·002,448	- ·001,076	
		+ 002,378	+ ·004,211	+ ·000,131	+ 000,826	+ ·000,896	+ ·002,885	+ ·002,012	
	2	·002,423	000,635	+ 000,043	− ·000,148	− ·000,006	+ ·001,307	+ ·002,162	
	3	·002,976	+ ·002,860	+ ·000,190	+ 001,026	+ ·000,868	+ ·000,930	− ·002,898	
	4	+ 000,365	·001,080	+ ·000,024	− ·000,130	+ ·000,200	+ ·001,118	·000,497	
		·001,271	+ ·001,134	+ ·000,225	+ ·001,052	+ ·000,634	− ·000,827	- ·000,946	
		+ ·000,475	796	+ ·000,014	− ·000,001	+ ·000,223	+ ·000,569	− ·000,485	
		− ·005,329	− ·019,833	− ·000,155	+ ·006,865	+ ·004,573	+ ·009,850	+ ·001,330	0
		+ ·006,217	+ ·011,006	+ ·000,350	+ ·002,160	+ ·002,343	+ ·007,541	+ ·005,259	1
	2	+ 008,127	+ ·002,130	·000,145	+ ·001,502	+ 000,022	− ·004,385	− ·007,251	2
	3	·007,217	+ ·006,935	+ ·000,461	+ ·002,488	+ ·002,106	+ ·002,255	− ·007,028	3
	4	·000,941	+ ·002,786	000,062	+ ·000,336	− ·000,517	− ·002,883	+ ·001,281	4
		·002,817	+ ·002,540	+ ·000,503	+ ·002,358	+ ·001,119	− ·001,854	− ·002,120	
		− ·000,780	+ ·001,306	·000,024	+ ·000,001	− ·000,366	− ·000,934	+ ·000,796	
	0	·007,289	·027,130	- ·000,622	+ ·009,390	+ ·006,255	+ ·013,473	+ ·005,923	
	1	+ 01 ,662	+ ·025,956	+ ·000,825	+ ·005,094	+ 005,526	+ ·017,784	+ ·012,102	
	2	+ ·002,953	+ ·000,774	− ·000,053	+ ·000,546	+ 000,008	− ·001,593	− 002,635	
	3	·013,673	+ ·013,139	+ ·000,873	+ ·004,714	+ ·003,990	+ ·004,273	− ·013,315	3
	4	+ ·001,057	− 003,128	+ ·000,070	− ·000,377	+ ·000,580	+ ·003,238	− 001,439	4
	5	− 004,116	+ 003,672	+ ·000,727	+ ·003,408	+ ·002,052	− ·002,679	− ·003,064	
	6	+ 003,320	− ·005,559	+ ·000,101	− ·000,006	+ ·001,558	+ ·003,973	− ·003,386	
	0	·010,563	− ·039,314	− ·000,901	+ ·013,607	+ ·009,064	+ ·019,524	+ ·008,582	
	1	+ ·033,046	+ ·058,500	+ ·001,860	+ ·011,480	+ ·012,454	+ ·040,082	+ ·027,952	
	2	− ·021,493	005,632	+ ·000,383	− ·003,971	- ·000,057	+ ·011,595	+ ·019,175	2
	3	− ·013,795	+ ·013,256	+ 000,880	+ ·004,756	+ ·004,025	+ ·004,311	− ·013,433	3
	4	+ ·006,142	- ·018,186	+ ·000,406	− ·002,193	+ 003,372	+ ·018,822	− ·008,364	4
	5	+ ·001,134	− ·001,011	− ·000,200	− ·000,939	- ·000,565	+ ·000,738	+ ·000,844	
	6	+ ·008,563	− ·014,340	+ ·000,260	·000,016	+ ·004,019	+ ·010,247	− ·008,733	
	0	− ·006,952	− ·025,876	− ·000,593	+ ·008,956	+ ·005,966	+ ·012,851	+ ·005,649	0
	1	+ ·034,867	+ ·061,193	+ ·001,945	+ ·012,009	+ ·013,028	+ ·041,92.	+ ·029,238	1
	2	− ·059,276	− ·015,532	+ ·001,057	- ·010,953	− ·000.157	+ ·031,978	+ ·052,883	2
	3	+ ·035,498	- ·034,111	− ·002,265	- ·012,238	− ·010,357	− ·011,092	+ ·034,567	3
	4	- ·001,827	+ ·005,409	− ·000,121	+ ·000,652	- ·001,003	− ·005,599	+ ·002,488	4
	5	+ ·006,181	− ·005,515	− ·001,092	− ·005,118	− ·003,082	+ ·004,024	+ ·004,602	
	6	− ·006,668	+ ·011,167	− ·000,202	+ ·000,202	− ·003,130	− ·007,980	+ ·006,801	

TABLE VII. *Values of* κ_{ss}.

s'	r	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
	(a)	+ ·034,290	+ ·045,918	+ ·000,873	− ·002,076	− ·000,798	− ·000,849	− ·000,094
1	(b)	+ ·039,786	+ ·047,855	+ ·000,831	− ·001,549	− ·000,541	− ·000,496	− ·000,036
	(c)	+ ·045,903	+ ·049,468	+ ·000,762	− ·001,097	− ·000,350	− ·000,251	− ·000,001
	(a)	+ ·048,425	+ ·135,200	+ ·003,506	− ·018,688	− ·009,255	− ·013,278	− ·002,561
2	(b)	+ ·050,671	+ ·142,180	+ ·003,719	− ·017,061	− ·007,957	− ·010,554	− ·001,722
	(c)	+ ·052,617	+ ·149,704	+ ·003,946	− ·015,291	− ·006,630	− ·008,054	− ·001,089
	(a)	+ ·001,628	+ ·006,926	+ ·000,205	− ·001,317	− ·000,633	− ·000,765	− ·000,039
3	(b)	+ ·001,526	+ ·007,188	+ ·000,223	·001,252	− ·000,545	− ·000,509	+ ·000,040
	(c)	+ ·001,386	+ ·007,465	+ ·000,245	− ·001,175	− ·000,444	− ·000,235	+ ·000,096
	(a)	− ·001,641	− ·013,645	− ·000,278	+ ·008,425	+ ·005,826	+ ·012,401	+ ·004,608
4	(b)	− ·001,250	− ·012,657	·000,247	+ ·008,726	+ ·006,019	+ ·012,553	+ ·004,294
	(c)	·000,903	− ·011,563	− ·000,211	+ ·009,071	+ ·006,233	+ ·012,663	+ ·003,892
	(a)	− ·001,344	− ·014,202	− ·000,165	+ ·012,270	+ ·009,181	+ ·021,659	+ ·009,613
	(b)	− ·000,940	− ·012,484	− ·000,085	+ ·012,745	+ ·009,643	+ ·022,682	+ ·009,562
	(c)	− ·000,625	− ·010,689	+ ·000,007	+ ·013,278	+ ·010,160	+ ·023,755	+ ·009,352
	(a)	− ·000,958	− ·013,805	+ ·000,109	+ ·018,295	+ ·015,185	+ ·041,172	+ ·023,419
	(b)	− ·000,584	− ·011,207	+ ·000,258	+ ·018,782	+ ·016,036	+ ·044,362	+ ·025,040
	(c)	− ·000,328	− ·008,749	+ ·000,419	+ ·019,263	+ ·016,957	+ ·047,837	+ ·026,557
	(a)	− ·000,047	− ·004,380	+ ·000,263	+ ·010,961	+ ·010,729	+ ·036,961	+ ·032,908
	(b)	− ·000,076	− ·003,086	+ ·000,316	+ ·010,571	+ ·010,938	+ ·040,073	+ ·038,215
	(c)	+ ·000,159	− ·002,067	+ ·000,348	+ ·010,019	+ ·011,027	+ ·043,208	+ ·044,126

TABLE VIII. *Values of* $\kappa_{ss}/(\bar{n}_{ss}/n_{ss})$.

		$s=1$	$s=2$		$s=4$			$=7$
	(a)	+ ·021,391	+ ·052,182	+ ·001,072	0	− ·002,057	0	0
	(b)	+ ·021,552	+ ·053,054	+ ·001,178	0	·002,033	0	0
	(c)	+ ·021,698	+ ·053,980	+ ·001,296	0	− ·022,011	0	0
	(a)	+ ·055,831	+ ·149,303	+ ·003,713	− ·012,640	− ·006,711	·007,035	0
2	(b)	+ ·056,643	+ ·150,514	+ ·003,957	− ·012,331	·006,547	·006,833	0
	(c)	+ ·057,497	+ ·151,686	+ ·004,228	− ·011,960	− ·006,351	− ·006,638	0
	(a)	+ ·001,898	+ ·006,139	+ ·000,185	− ·001,621	− ·000,741	·000,777	− ·000,018
3	(b)	+ ·002,029	+ ·006,679	+ ·000,196	·001,521	− ·000,645	− ·000,547	+ ·000,022
	(c)	+ ·002,180	+ ·006,941	+ ·000,209	− ·001,106	− ·000,534	·000,271	+ ·000,065
		·001,004	− ·011,805	·000,258	+ ·010,115	+ ·006,850	+ ·010,117	+ ·006,859
		·000,992	·011,538	·000,225	+ ·010,424	+ ·006,858	+ ·010,124	+ ·006,845
		·000,985	·011,217	·000,188	+ ·010,132	+ ·006,869	+ ·010,130	+ ·006,828
		0	− ·012,745	− ·000,160	+ ·013,131	+ ·009,558	+ ·023,162	+ ·010,155
		0	− ·012,407	− ·000,083	+ ·013,153	+ ·009,564	+ ·023,170	+ ·010,125
		0	− ·012,005	− ·000,007	+ ·013,177	+ ·009,569	+ ·023,173	+ ·010,084
		0	− ·009,147	+ ·000,126	+ ·015,039	+ ·009,465	+ ·058,655	+ ·025,835
			− ·009,156	+ ·000,311	+ ·015,079	+ ·009,468	+ ·058,760	+ ·025,838
		0	− ·008,851	+ ·000,533	+ ·015,123	+ ·009,468	+ ·058,853	+ ·025,824
	(a)	0	− ·004,556	+ ·000,352	+ ·007,491	+ ·012,697	+ ·032,408	+ ·037,813
	(b)	0	− ·004,377	+ ·000,487	+ ·007,493	+ ·012,645	+ ·032,448	+ ·038,095
	(c)	0	− ·004,210	+ ·000,642	+ ·007,492	+ ·012,574	+ ·032,463	+ ·038,361

$S(a) = + \cdot 510,573$, $S(b) = + \cdot 517,476$, $S(c) = + \cdot 524,735$,

$r_a = - \cdot 060,573$, $r_b = - \cdot 017,476$, $r_c = + \cdot 025,265$

TABLE IX.

Values of $\vartheta_s T_p \vartheta_{s'} \tau_p'$.

$s=7$

0	− ·002,689	− ·010,007	− ·000,229	+ ·003,464	+ ·002,307	+ ·004,970	+ ·002,185	0	
1	− ·013,450	− ·023,810	− ·000,757	− ·004,673	·005,069	− ·016,314	− ·011,377	1	
2	− ·023,067	− ·006,044	+ ·000,411	− ·004,262	− ·000,061	+ ·012,444	+ ·020,57	2	
3	− ·013,496	+ ·012,969	+ ·000,861	+ ·004,653	+ ·003,938	+ ·004,217	− ·013,14	3	
4	− ·000,450	+ ·001,333	− ·000,030	+ ·000,161	− ·000,247	− ·001,380	+ ·000,613	4	
5	− ·003,007	+ ·002,683	+ ·000,531	+ ·002,490	+ ·001,499	− ·001,958	− ·002,239	5	
6	− ·003,082	+ ·005,161	− ·000,091	+ ·000,006	− ·001,446	·003,688	+ ·003,143	6	
0	− ·023,802	− ·088,590	− ·002,030	+ ·030,662	+ ·020,425	+ ·043,996	+ ·019,339	0	
1	·051,494	− ·091,158	·002,898	− ·017,889	− ·019,407	− ·062,458	− ·043,556	1	
2	− ·002,940	− ·000,770	+ ·000,052	·000,543	·000,008	+ ·001,586	+ ·002,623	2	
3	+ ·036,379	·034,958	− ·002,322	− ·012,542	− ·010,614	− ·011,368	+ ·035,125	3	
4	+ ·004,781	− ·014,154	+ ·000,316	·001,707	+ ·002,625	+ ·014,650	− ·006,510	4	
	+ ·007,797	·006,957	·001,378	− ·006,156	·003,887	+ ·005,076	+ ·005,805		
	+ ·009,148	− ·015,822	+ ·000,287	− ·000,017	+ ·004,434	+ ·011,306	− ·009,636		
0	− ·022,458	·083,587	− ·001,915	+ ·028,931	+ ·019,271	+ ·041,511	+ ·018,247		
1	·002,986	·005,286	− ·000,168	− ·001,037	− ·001,125	·003,622	− ·002,526		
	+ ·015,338	+ ·011,880	− ·000,808	+ ·008,377	+ ·000,120	− ·024,459	− ·040,449		
	+ ·003,681	− ·003,537	·000,235	− ·001,269	− ·001,074	− ·001,150	+ ·003,584		
	·007,651	+ ·022,655	·000,506	+ ·002,732	·004,201	·023,147	+ ·010,419		
	+ ·001,548	·001,381	·000,273	− ·001,281	− ·000,772	+ ·001,007	+ ·001,152		
	− ·011,412	+ ·019,111	·000,346	+ ·000,021	− ·005,356	− ·013,656	+ ·011,639	6	
	·010,833	− ·010,322	− ·000,924	+ ·013,956	+ ·009,296	+ ·020,025	+ ·008,802	0	
	+ ·012,013	+ ·021,267	+ ·000,676	+ ·004,173	+ ·001,528	+ ·011,571	+ ·010,161	1	
	+ ·017,109	+ ·004,483	− ·000,305	+ ·003,161	+ ·000,045	− ·009,230	·015,264	2	
	− ·014,068	+ ·013,518	+ ·000,898	+ ·004,850	+ ·004,105	+ ·004,396	− ·013,699	3	
	− ·002,101	+ ·006,221	·000,139	+ ·000,750	− ·001,154	− ·006,439	+ ·002,861	4	
	·005,604	+ ·005,000	+ ·000,990	+ ·004,640	+ ·002,794	− ·003,648	− ·004,172	5	
	− ·001,988	+ ·003,329	·000,060	+ ·000,004	− ·000,933	− ·002,379	+ ·002,028	6	
0	− ·008,303	− ·030,904	− ·000,708	+ ·010,696	+ ·007,125	+ ·015,347	+ ·006,746	0	
1	+ ·016,433	+ ·029,090	+ ·000,925	+ ·005,709	+ ·006,[illegible]	+ ·019,931	+ ·013,899	1	
2	+ ·003,809	+ ·000,998	·000,068	+ ·000,704	+ ·000,010	− ·002,055	− ·003,399	2	
3	− ·015,502	+ ·014,897	+ ·000,989	+ ·005,314	+ ·004,523	+ ·004,844	− ·015,096	3	
4	+ ·001,087	− ·003,220	+ ·000,072	·000,388	+ ·000,597	+ ·003,332	− ·001,481	4	
5	− ·001,711	+ ·004,233	+ ·000,838	+ ·003,928	+ ·002,365	·003,088	− ·003,532	5	
6	+ ·003,592	·006,016	+ ·000,109	− ·000,00[illegible]	+ ·001,686	+ ·004,299	− ·003,664	6	
0	− ·007,749	− ·028,843	− ·000,661	+ ·009,983	+ ·006,650	+ ·014,324	+ ·006,297	0	
1	+ ·023,811	+ ·042,152	+ ·001,340	+ ·008,272	+ ·008,974	+ ·028,881	+ ·020,140	1	
2	− ·014,636	·003,835	+ ·000,261	− ·002,701	− ·000,039	+ ·007,896	+ ·013,057	2	
3	− ·010,657	+ ·010,241	+ ·000,680	+ ·003,674	+ ·003,110	+ ·003,330	− ·010,378	3	
4	+ ·004,372	− ·012,946	+ ·000,289	− ·001,561	+ ·002,401	+ ·013,399	− ·005,954	4	
5	+ ·000,142	− ·000,394	− ·000,078	− ·000,366	− ·000,220	+ ·000,288	+ ·000,329	5	
6	+ ·006,335	− ·010,608	+ ·000,192	− ·000,012	+ ·002,973	+ ·007,580	− ·006,461		
0	·003,212	·012,067	− ·000,276	+ ·004,177	+ ·002,782	+ ·005,993	+ ·002,634		
1	+ ·015,673	+ ·027,746	+ ·000,882	+ ·005,445	+ ·005,907	+ ·019,010	+ ·013,257		
2	− ·025,614	− ·006,712	+ ·000,457	− ·004,733	·000,068	+ ·013,818	+ ·022,851		
3	+ ·013,663	− ·013,130	− ·000,872	− ·004,711	− ·003,987	− ·004,270	+ ·013,305	3	7
4	− ·000,037	+ ·000,111	− ·000,002	+ ·000,013	·000,020	− ·000,114	+ ·000,051	4	
	+ ·003,569	− ·003,181	− ·000,631	− ·002,955	− ·001,779	+ ·002,323	+ ·002,657		
	− ·002,893	+ ·004,845	− ·000,088	+ ·000,005	− ·001,358	·003,462	+ ·002,951		

TABLE X.

Values of $\vartheta_s \tau_p \vartheta_s T_p'$.

	s=1	s=2				s=6	s=7	
0	− ·002,716	− ·024,296	− ·019,919	− ·013,581	− ·005,206	− ·007,621	− ·002,113	0
1	− ·013,578	− ·050,535	− ·001,157	+ ·017,491	+ ·011,650	+ ·025,097	+ ·011,032	1
2	− ·023,244	+ ·001,049	+ ·011,194	+ ·019,288	+ ·000,298	− ·018,826	− ·020,060	2
3	− ·013,520	+ ·038,284	+ ·001,489	− ·020,306	− ·010,442	− ·008,529	+ ·013,024	3
4	− ·000,364	+ ·004,567	− ·007,904	− ·002,108	+ ·001,391	+ ·005,282	− ·000,864	4
	− ·002,999	+ ·008,296	+ ·000,596	− ·007,320	− ·002,654	+ ·001,795	+ ·002,285	5
	− ·003,074	+ ·008,906	− ·011,032	000,956	+ ·003,290	+ ·006,016	− ·003,149	6
0	− ·010,399	− ·093,014	− ·076·260	− ·051,995	− ·019,931	− ·029,175	− ·008,088	0
1	− ·025,191	·093,756	·002,147	+ ·032,451	+ ·021,614	+ ·046,563	+ ·020,467	1
2	− ·007,378	+ ·000.333	+ 013,171	+ ·006,123	+ 000.095	− ·005,976	− ·006,367	2
3	+ ·012,689	− ·035,930	− ·001,397	+ ·019,057	+ ·009,800	+ ·008,004	− ·012,223	3
4	+ ·001,044	·013,114	+ ·022,696	+ ·006,054	− ·003,993	− ·015,167	+ ·002,480	4
5	+ ·002,467	− ·006,824	− ·000,490	+ 006,021	+ ·002,183	− ·001,477	− ·001,880	5
6	+ ·005,229	− ·015,149	+ ·018,767	+ 001,627	− ·005,596	− ·010,234	+ ·005,357	6
0	− ·000,603	·005,392	− ·004,421	− ·003,014	− ·001,155	− ·001,691	− ·000,469	
1	− ·001,055	− ·003,927	− ·000,090	+ 001,359	+ ·000,905	+ ·001,950	+ ·000,857	
2	+ ·001,029	− ·000,046	001,837	− ·000,854	− ·000,013	+ ·000,833	+ ·000,888	
	+ ·001,143	− ·003,237	− ·000,126	+ ·001,717	+ ·000,883	+ ·000,721	− ·001,101	
	− ·000,052	+ ·000,650	− ·001,125	− ·000,300	+ ·000,198	+ ·000,752	− ·000,123	
	+ ·000,736	− ·002,035	− ·000,146	+ 001,795	+ ·000,651	− ·000,440	− 000,561	
	− ·000,209	+ ·000,605	− ·000,749	·000,065	+ ·000,223	+ ·000,108	− 000,214	
	+ 002,426	+ ·021,699	+ ·017,790	+ ·012,130	+ ·004,650	+ ·006,806	+ ·001,887	
	·002,758	− ·010,265	− ·000,235	+ ·003,553	+ ·002,366	+ ·005,098	+ ·002,241	
2	− ·003,451	+ ·000,156	+ ·006,161	+ ·002,864	+ ·000,014	− ·002,795	− ·002,979	
3	+ ·002,772	− ·007,849	− ·000,305	+ ·004,163	+ ·002,141	+ ·001,749	− ·002,670	
1	+ 000,134	− ·001,677	+ ·002,902	+ 000,771	− ·000,511	001,939	+ ·000,317	
5	+ ·001,648	·004,560	− ·000,328	+ ·004,023	+ ·001,459	− ·000,987	− ·001,256	
6	+ ·000,342	− ·000,992	+ ·001,229	+ ·000,107	− ·000,366	− ·000,670	+ ·000,351	
	+ ·003,318	+ ·029,682	+ ·024,335	+ ·016,592	+ 006,360	+ ·009,310	+ ·002,581	
	− ·006,504	− ·021,907	− ·000,554	+ 008,379	+ ·005,581	+ ·012,022	+ ·005,284	
2	− ·001,254	+ ·000,057	+ ·002,239	+ ·001,041	+ 000,016	− ·001,016	·001,082	
3	+ ·005,252	− ·014,872	·000,578	+ ·007,888	+ ·004,056	+ ·003,313	− ·005,059	3
1	·000,150	+ ·001,883	− ·003,259	− ·000,869	+ ·000,573	+ ·002,178	− ·000,356	1
5	+ ·002,383	− ·006,592	·000,474	+ ·005,816	+ ·002,109	− ·001,427	− ·001,816	
	− 001,457	+ ·004,222	− ·005,230	− 000,453	+ 001,560	+ ·002,852	− ·001,193	
0	+ ·004,809	+ ·043,011	+ ·035,261	+ ·024,044	+ ·009,217	+ ·013,491	+ ·003,740	0
1	− ·014,659	− ·054,559	− ·001,249	+ ·018,884	+ ·012,578	+ ·027,096	+ ·011,910	1
2	+ ·009,127.	− ·000,412	− ·016,293	− ·007,574	− ·000,117	+ ·007,392	+ ·007,876	2
3	+ ·005,299	− ·015,004	− ·000,583	+ 007,958	+ ·001,092	+ ·003,342	− ·005,104	3
4	− ·000,872	+ ·010,946	− ·018,945	− ·005,054	+ ·003,333	+ ·012,661	− ·002,070	4
5	− ·000,656	+ ·001,815	+ ·000,130	− ·001,602	− ·000,581	+ ·000,393	+ ·000,500	
6	− 003,758	+ ·010,889	− 013,490	− ·001,169	+ ·004,023	+ ·007,356	·003,851	
0	+ ·003,165	+ ·028,310	+ ·023,211	+ ·015,825	+ ·006,066	+ ·008,880	+ ·002,462	
1	− 015,334	− ·057,071	·001,307	+ ·019,753	+ ·013,157	+ ·028,344	+ ·012,459	
2	+ ·025,172	·001,136	− ·044,936	− ·020,888	− ·000,323	+ ·020,387	+ ·021,724	
3	− ·013,635	+ ·038,608	+ ·001,501	− ·020,478	− ·010,531	− ·008,601	+ ·013,134	
4	+ ·000,259	− ·003,256	+ ·005,635	+ ·001,503	− ·000,991	− ·003,766	+ ·000,616	
	− ·003,579	+ ·009,899	+ ·000,711	− ·008,734	− ·003,167	+ 002,142	+ ·002,727	
	+ ·002,927	·008,480	+ ·010,505	+ ·000,911	− ·003,133	− ·005,729	+ ·002,999	

We shall proceed to calculate $\lambda_{ss'}$ for three values of r which lie near the probable value of r as found from each column. We will take these as ·45, ·50 and ·55; from these values we shall obtain $\bar{h}_{s}.$ for each column from (xiv) and interpolating the real $\bar{h}_{s}.$ between them find the corresponding columnar r, which will be then substituted in (xv) by aid of Table X to obtain the columnar mean $\bar{k}_{s}.$. Table XI gives the values of λ_{ss} for $r = ·45$, ·50 and ·55, and Table XII the resulting values of $\bar{h}_{s}$.

TABLE XI. *Values of $\lambda_{ss'}$ for $r = ·45$, ·50 and ·55.*

		$s=1$	$s=2$	$s=3$	$s=4$		$s=6$	$s=7$
	(a)	·014,742,01	− ·020,616,58	− ·000,400,18	+ 000,974,70	+ ·000,377,97	+ ·000,409,54	+ ·000,044,95
	(b)	· ·017,038,00	− 021,554,08	− ·000,383,88	+ ·000,731,59	+ ·000,258,31	+ ·000,246,06	+ ·000,015,95
	(c)	· ·019,587,49	− ·022,373,22	− ·000,356,40	+ ·000,518,95	+ ·000,168,60	+ ·000,136,31	− ·000,003,29
	(a)	− ·043,836,23	− 133,792,74	− ·003,545,25	+ ·021,169,83	+ ·010,795,76	+ ·015,963,45	+ ·003,258,21
	(b)	− ·045,046,53	− 140,080,50	− ·003,775,08	+ ·019,705,30	+ ·009,504,62	+ ·012,993,41	+ ·002,268,84
	(c)	− 045,869,07	− ·146,859,24	− 004,026,98	+ ·018,090,53	+ ·008,150,14	+ ·010,141,50	+ ·001,500,21
	(a)	− ·014,665,26	− ·082,820,10	− 002,204,29	+ 030,133,27	+ ·018,460,19	+ ·033,767,07	+ 009,791,12
	(b)	− 012,764,50	− 082,030,73	− ·002,275,91	+ 030,479,17	+ ·018,233,88	+ ·031,794,16	+ ·008,188,80
	(c)	− ·010,711,23	− 080,956,49	− ·002,360,54	+ 030,869,67	+ ·017,938,38	+ ·029,455,85	+ 006,551,72
		003,450,60	− ·028,237,21	− ·000,587,66	+ ·017,032,32	+ ·011,713,27	+ ·024,762,35	+ ·009,092,34
		− ·002,645,25	− 026,280,92	− ·000,528,69	+ ·017,630,94	+ 012,084,98	+ ·024,998,89	+ ·008,434,25
		− 001,920,26	− ·024,106,94	− ·000,459,56	+ ·018,316,53	+ ·012,492,17	+ 025,139,70	+ ·007,601,99
	(a)	− ·001,562,59	− ·016,357,80	− ·000,196,08	+ ·013,951,10	+ ·010,408,16	+ ·024,456,57	+ 010,780,30
	(b)	− ·001,096,19	− ·011,410,34	− ·000,106,48	+ ·014,492,89	+ 010,926,94	+ ·025,583,17	+ ·010,698,56
	(c)	− 000,731,63	− ·012,372,26	− ·000,003,19	+ ·015,100,01	+ 011,507,01	+ ·026,761,82	+ ·010,435,95
		− ·000,728,92	− ·010,344,20	+ ·000,068,82	+ ·013,421,77	+ ·011,082,88	+ ·029,840,53	+ ·016,766,62
		·000,448,58	− ·008,432,81	+ 000,177,88	+ 013,793,06	+ 011,705,64	+ ·032,119,62	+ ·017,871,20
		− ·000,255,73	− 006,613,75	+ ·000,295,92	+ ·014,164,31	+ 012,382,26	+ ·034,601,52	+ ·018,889,99
	(a)	− ·000,090,63	− ·002,150,92	+ ·000,121,52	+ ·005,185,57	+ ·005,018,14	+ 016,965,98	+ ·014,515,02
7	(b)	− 000,037,11	− ·001,530,11	+ ·000,119,03	+ ·005,035,92	+ ·005,142,06	+ ·018,430,12	+ 016,770,70
	(c)	− ·000,000,75	− ·001,037,56	+ ·000,167,89	+ ·004,808,83	+ ·005,217,99	+ ·019,928,67	+ ·019,271,47

TABLE XII. *Values of $\bar{h}_{s}.$ for Columns.*

r	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
·45	− 2·19299	− ·92114	− ·02549	+ ·56650	+ ·98336	+ 1·45054	+ 2·30569
·50	− 2·18505	·91848	− ·02538	+ ·56616	+ ·98315	+ 1·44900	+ 2·29880
·55	− 2·17567	− ·91485	− ·02521	+ ·56580	+ ·98286	+ 1·44685	+ 2·28999
Actual h_s	− 2·19667	− ·91404	·02553	+ ·56594	+ ·98333	+ 1·44723	+ 2·29464
Extra- or Interpolated r	·4229	·5599	·4167	·5309	·4585	·5426	·5249

We have thus the values of r found from each column*.

We now turn to Table X and calculate in exactly the same way the values of

$$\lambda'_{ss} = \vartheta_s \tau_0 \vartheta_{s'} T_0' + r \vartheta_s \tau_1 \vartheta_{s'} T_1' + \ldots + r^p \vartheta_s \tau_p \vartheta_{s'} T_p' + \ldots,$$

for the r peculiar to each column for that column. We thus obtain Table XIII.

TABLE XIII.

Values of λ'_{ss} for r of each Vertical Column.

	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
1	− ·013,707,54	− ·044,362,34	− ·013,376,98	− ·002,394,74	− ·000,770,04	− ·000,212,73	− ·000,006,10
2	− ·021,315,40	− ·153,841,07	− ·074,192,36	− ·029,418,02	− ·009,240,63	− ·006,036,02	− ·000,641,41
3	− ·000,771,58	− ·008,202,77	− ·004,826,27	− ·002,225,87	− ·000,633,67	− ·000,217,60	+ ·000,030,11
4	+ ·000,880,64	+ ·011,176,58	+ ·018,829,61	+ ·015,680,02	+ ·005,954,01	+ ·008,797,09	+ ·001,837,83
5	+ ·000,759,52	+ ·013,488,41	+ ·024,318,82	+ ·022,680,28	+ ·009,395,75	+ ·016,257,72	+ ·004,194.22
6	+ ·000,584,54	+ ·011,211,78	+ ·031,232,08	+ ·032,630,32	+ ·015,526,73	+ ·032,207,15	+ ·011,205,66
7	+ ·000,127,47	+ ·002,739,83	+ ·015,206,16	+ ·017.131,65	+ ·010,878,46	+ ·028,515,80	+ ·017,104,71
$h_{s\cdot}$	− ·963,72	− ·507,66		+ ·303,04	+ ·425,95	+ ·789,11	+ 1·238,75

The values in Table XIII divided by $\bar{n}_{ss}/n_{ss'}$ from Table XIV and summed for each column give, on multiplication by $N/n_{s\cdot}$, the $\bar{h}_{s\cdot}$ of the last row of the table.

To obtain Table XIV we must return to Equation (λ), use the appropriate r for the column and the values in Table III of $\vartheta_s \tau_p \vartheta_s \tau_p'$. Taking σ_x and σ_y as units of the horizontal and vertical variates we can plot $k_{s\cdot}$ in Table XIII to $\bar{h}_s$ from Table XII and so obtain the regression line as formed by the means of each column, and set against it the regression lines as found from polychoric $r_p = {}$·5034, or ·5204.

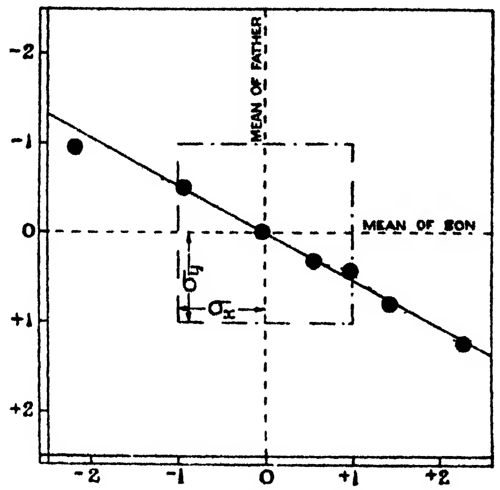
TABLE XIV.

Values of $\frac{\bar{n}_{ss'}}{n_{ss}}$ for columnar Values of r.

s'	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
1	1·481,160	·918,247	·881,901	∞	·366,258	∞	∞
2	·850,270	·995,746	·946,290	1·319,975	1·351,776	1·261,565	∞
3	·910,673	1·075,087	1·090,803	·830,945	·853,291	·876,250	1·668,232
4	1·846,116	1·016,776	1·066,432	·856,640	·855,012	1·248,823	·600,417
5	∞	·866,469	1·028,801	·992,395	·968,288	1·018,112	·937,952
6	∞	·980,885	·887,677	1·263,099	1·619,040	·803,917	·999,089
7	∞	·454,171	·808,851	1·368,315	·848,918	1·316,691	1·074,517

* The mean value of r weighted with the column totals is ·5022 which is in reasonable accord with (i.e. within the probable error of) the results on p. 142.

This is done in Diagram I. But what we actually desire is to compare the observations and the regression lines as given by the present polychoric method with those obtained by product-moment methods.



Stature of Father in Inches.

Diagram I.

Our actual data from which the table on p. 135 was obtained are given in Table XV. The following are the values of the constants in inches:

Mean Stature of Father: $x = 67''\cdot878$.

Mean Stature of Son: $y = 68''\cdot845$.

Standard Deviation of Father: $\sigma_x = 2''\cdot6576$.

Standard Deviation of Son: $\sigma_y = 2''\cdot6885$.

Correlation of Father and Son: $r = \cdot5189 \pm \cdot0160$.

In Diagram II the regression line (slope, ·5245) with means of the arrays as dark circles is given. Against this we have put as hollow circles the values of $\bar{h}_s$. and $\bar{k}_s$. multiplied by their respective S.D.'s to indicate the result as worked out in the present paper. The closeness of the polychoric coefficient ·5204 and the product-moment coefficient does not permit of two regression lines being drawn. It will be seen that the fit to the observations by use of broad categories and the polychoric method is really quite as satisfactory as the fit by the product-moment method. But the amount of arithmetical work is incomparably greater by the former, even if it be less than Ritchie-Scott's process with 49 cells would be.

Accordingly we now proceeded to investigate the extent to which approximations shortening the arithmetic would introduce serious error. The first question to be answered is: To what extent in finding the means $\bar{k}_s$. of the arrays is it needful to use the actual value of the correlation coefficient as found for each column? In order to test this we proceeded to find the $\bar{k}_s$. for each columnar

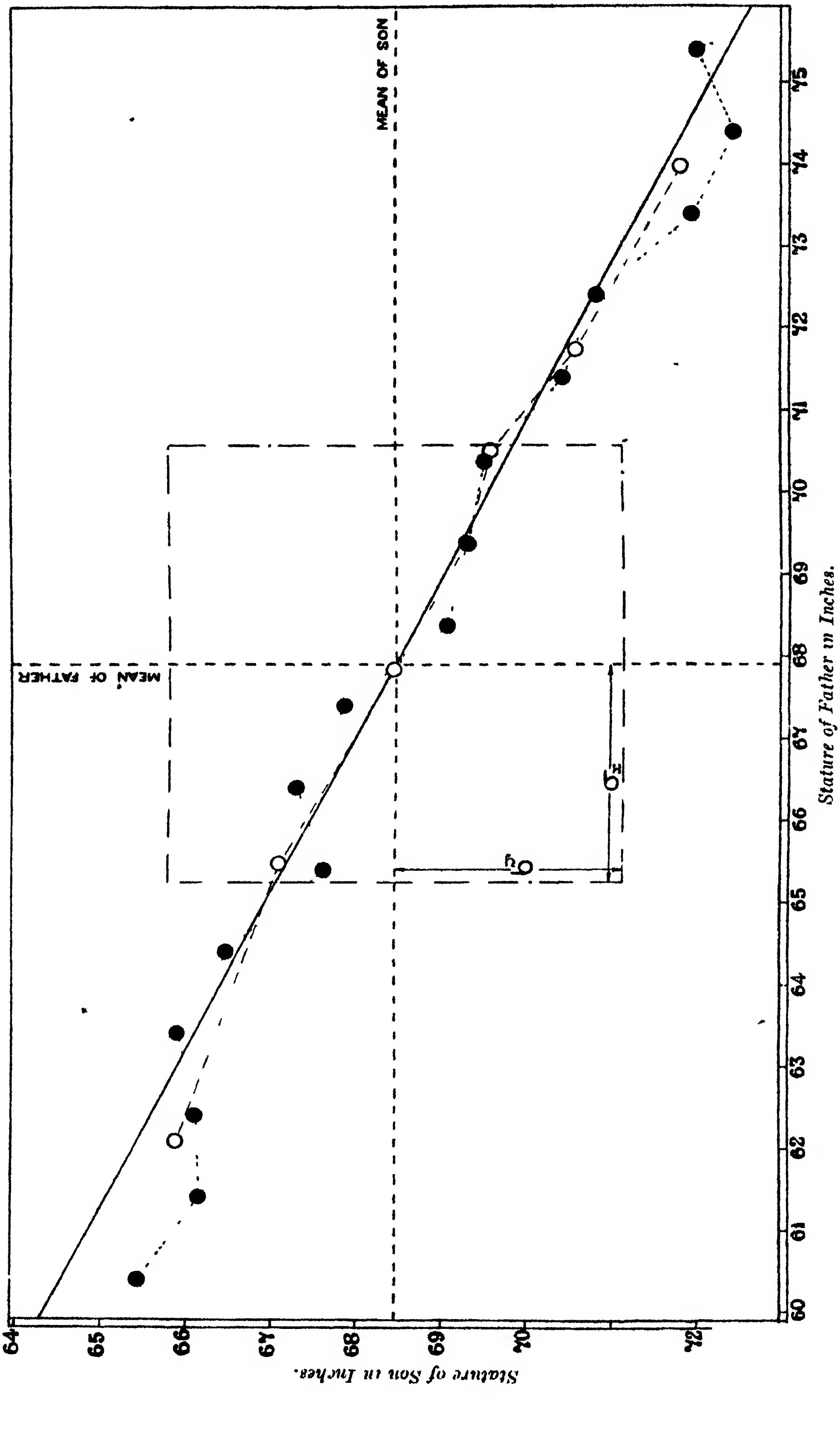

MEAN OF SON
MEAN OF FATHER
Stature of Father in Inches.
60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75
Stature of Son in Inches.
64 65 66 67 68 69 70 72

Diagram II.

TABLE XV.

Correlation of Stature in 1000 pairs, Father and Son.

Stature of Father.

Stature of Son	59"·875—	60"·875—	61"·875—	62"·875—	63"·875—	64"·875—	65"·875—	66"·875—	67"·875—	68"·875—	69"·875—	70"·875—	71"·875—	72"·875—	73"·875—	74"·875—	75"·875—	Totals
59"·875—	—	—	—	1	—	1		—	—	—	—	—	—	—	—	—	—	2
60"·875—	-	—		—	—	—	—	—	1	—	—	—	—	—	—	—	—	1
61"·875—	1		—		3	—	1	1	—	-	—	—	—	-	—	—	-	6
62"·875—		—	2	4	4	3	1	3	2	-	1	—		-	—		—	20
63"·875—	1		1	5	6	5	9	2	2	1	—	—	—		—	—	—	32
64"·875—	2	3	3	5	11	11	10	17	4	1	2	—	1	—	—	—	—	70
65"·875—	1	1	2	6	9	10	20	17	15	7	6	—	—		—	-	—	94
66"·875—	1	—	6	4	11	24	21	28	10	12	7	4	1	—	—		—	129
67"·875—	—	2	2	7	9	20	16	33	27	26	20	13	6	—	—		-	181
68"·875—	1	-	1	4	1	12	13	10	22	26	21	6	2	2	1	—		125
69"·875—	-		—	—	5	11	15	18	18	23	18	13	4	4	1	1	-	131
70"·875—			-	—	2	5	4	13	12	12	13	8	7	3	1			80
71"·875—	—	—	—		-	4	1	7	7	9	9	9	7	3	1	—	-	57
72"·875—	-	-	—	-	1	2		—	13	4	2	9	1	1	1	2	—	36
73"·875—	—	—		-	1	1	—	—	4	1	5	4	3	2	—	—		21
74"·875—	—		-		—	-	—	-	9	2	—	1	—	2	1	—	—	8
75"·875—	—		—	—	-		—	—	—	-	—		1	—	—		—	1
76"·875—	—	—	—		—	—			—	-	—	—	—	1	1	—	—	2
77"·875—	—	—	—	—	—		—	—		1	1	—	1	—	—		—	3
78"·875—	—	—	—	—		—	—		—	—	1	—		—	—	—	—	1
Totals	7	6	17	36	63	109	111	149	139	125	109	67	34	18	7	3	0	1000

array for the *same* correlation coefficient, and we took for the value of that coefficient ·5000, somewhat under the value found by either polychoric coefficient.

Table XVI gives our results. It involved finding a new series of values for λ'_{ss}, but those for $\bar{n}_{ss}/n_{ss}$ have already been computed under (*b*) in Table IV. The results are given in terms of inches.

TABLE XVI.

Columnar Means by Different Processes.

s	$h_{s.} \times \sigma_x$	$\bar{h}_{s.} \times \sigma_y$ Each column its own r	$\bar{h}_{s.} \times \sigma_y$ Each column for $r = \cdot 50$	$\bar{h}_{s.} \times \sigma_y$ Each column assumed Normal	Common base
1	−5·8379	−2·5881	−2·6498	−2·4809	2
2	−2·4292	−1·3633	1·3531	−1·4357	3
3	− ·0678	− ·0276	− ·0176	− ·0701	3
4	+1·5040	+ ·8138	+ ·8087	+ ·7632	3
5	+2·6133	+1·1439	+1·1511	+1·0866	3′+4′
6	+3·8462	+2·1192	+2·1122	+2·1744	4′+5′
7	+6·0982	+3·3267	+3·3194	+3·1109	5′

An examination of the fourth column of Table XVI shows us that we have not for practical purposes seriously modified the columnar means by using $r = \cdot 50$ instead of the individual value for each column. This is illustrated in Diagram III, where except in the case of the first array there is hardly daylight between the two series of points.

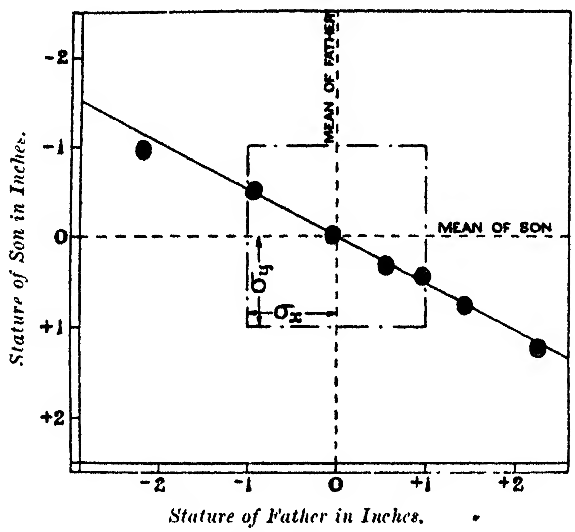


Diagram III.

In Diagram III the hollow circles give the means with r obtained for each column, the nearly superposed dark circles the means with $r = \cdot 5000$.

The solution of the problem therefore falls back on Equations (x), (xvi) and (xv). We should still have to calculate $\vartheta_s \tau_p$, $\vartheta_s \tau_p'$, $\vartheta_s T_p$ and $\vartheta_s T_p'$, but we should only need the three series of products $\vartheta_s \tau_p \vartheta_{s'} \tau_p'$, $\vartheta_s T_p \vartheta_s T_p'$ and $\vartheta_s \tau_p \vartheta_s T_p'$, and to obtain $\bar{k}_s$ it would be adequate to use a value of r for which $\bar{n}_{ss'}/n_{ss'}$ had been found for the final interpolation. Still this involves very lengthy arithmetic, and we naturally crave for a still easier process. The present full working out of a numerical example enables us for the first time really to test the adequacy of an easier method of dealing with such polychoric tables which has been long in use as an approximate method in the Biometric Laboratory.

(6) It is clear that if we could find the means of the columnar arrays, we could readily obtain the correlation and the regression line by aid of the correlation ratio corrected for class index. The whole problem accordingly turns on a ready means of reaching—at any rate—an approximate value of the mean of a columnar array. This array is the slice between two parallel planes of a normal correlation surface.

In the case of a surface of zero correlation

$$Z = Z_0 e^{-\frac{1}{2}(X^2 + Y^2)/a^2},$$

the slice between X_1 and X_2 has for its volume on dY

$$\int_{X_1}^{X_2} e^{-\frac{1}{2}X^2/a^2}\,dX\,e^{-\frac{1}{2}Y^2/a^2}\,dY;$$

the slice is therefore given by the normal curve:

$$\text{Ordinate} = \text{const.} \times e^{-\frac{1}{2}Y^2/a^2}.$$

It seems therefore not unreasonable after the surface of revolution is stretched and slid into a correlation surface to assume the slice to be still approximately a normal curve. Unfortunately the determination of the best mean and standard deviation for normal material given in broad categories does not admit of very easy solution. What we need is the difference between the means of a columnar array and of a marginal frequency as a multiple of the standard deviation of the latter. We shall obtain results differing more or less from each other according to the individual broad category we take as the basis of comparison between σ_s the standard deviation of the sth slice and σ_y the standard deviation of the marginal frequency. In fact the range of any broad category or of any combination of broad categories, except the tail categories, can be made a means of linking up σ_s and σ_y. A little experience, however, shows (a) that it is undesirable to find the σ_s of any array from a category of small frequency, and (b) that for arrays of small total frequency symmetrical tripartite divisions as far as feasible are the best*. The last column in Table XVI shows the system selected for each of our columnar arrays.

Take, for example, $s = 5$, the columnar array may be taken on the base of 3′ and 4′ categories as

$$\begin{array}{rr}
1'+2' & 9 \\
3'+4' & 36 \\
5'+6'+7' & 24 \\
\hline
\text{Totals} & 69
\end{array}
\Bigg\}\ \text{and compared with}\ \Bigg\{
\begin{array}{r}
335 \\
421 \\
244 \\
\hline
1000
\end{array}$$

as the corresponding marginal distribution. The corresponding proportional frequencies up to the dichotomic planes are: $\begin{matrix}\cdot1304\\ \cdot6521\end{matrix}$ and $\begin{matrix}\cdot3350\\ \cdot7560\end{matrix}$. The distances of the mean† from the two dichotomic planes in the first case are

$$-1{\cdot}1245\,\sigma_5 \text{ and } +{\cdot}3910\,\sigma_5,$$

and in the second case

$$-{\cdot}4261\,\sigma_y \text{ and } +{\cdot}6935\,\sigma_y,$$

where σ_5 is the standard deviation of the normal curve assumed to represent the columnar array 5. Accordingly the range of 3′ + 4′ categories

$$= 1{\cdot}5155\,\sigma_5 = 1{\cdot}1196\,\sigma_y,$$

which gives σ_5 in terms of σ_y.

* The probable error of a standard deviation found in this way is discussed in *Biometrika*, Vol. XIII. p. 129.

† Found from the Probability Integral Table.

Hence the distance between the means is

$$\begin{aligned} & \cdot 6935\,\sigma_y - \cdot 3910\,\sigma_s \\ &= \{\cdot 6935 - \cdot 3910 \times 1\cdot 1196/1\cdot 5155\}\,\sigma_y \\ &= \cdot 4046\,\sigma_y \\ &= 1\cdot 0866, \text{ if we introduce the value of } \sigma_y. \end{aligned}$$

This and the corresponding values are recorded in the fifth column of Table XVI. It will be seen that these values approximate to those in the third column, the greatest differences being in the small first and last arrays.

Of course in actually working with material solely given in broad categories we use the value ·4046, treating σ_y as our unit of measurement. The means of the columnar arrays can be found with great ease and with considerable approximation by this method.

If we now proceed to take the mean of our means duly weighted with their frequencies, we find it to be − ·0510,—not a very serious divergence from zero. However, we subtract it* from the means in the fifth column of Table XVI, multiply the squares of the remainders by the corresponding frequencies, sum and divide by the square of σ_y. Thus we obtain

$$\eta^2 = \frac{1\cdot 818,8034}{7\cdot 211,9103} = \cdot 25210144,$$

or:

$$\eta = \cdot 502148.$$

If we divide by the class index correlation of the x-variate, i.e. ·962,329†, we obtain

$$\eta = \cdot 5218,$$

which correlation ratio we may take to be the correlation coefficient and compare with our polychoric coefficient ·5204 (p. 142) Clearly although our means as found by the hypothesis of normal distribution of the columnar arrays agree only approximately with the polychoric means of the third column of Table XVI, they lie practically on the same regression line, as Diagram IV indicates. We conclude, therefore, that in this case as probably in many like cases, it is quite adequate to obtain the means of the columnar arrays by treating them as normal distributions, then determining their correlation ratio and correcting it for the class index. The corresponding regression line with the means of the columnar arrays indicated will be for many purposes an adequate graph showing the general nature of the correlation.

The general purpose of this paper has now been fulfilled; it has been shown how a general polychoric coefficient covering all the data provided in a given contingency table may be found, and how a graph may be drawn representing such a table effectively. At the same time such a process is very laborious and probably will not be lightly undertaken or only in cases of grave uncertainty. The method

* Correlation ratio without subtraction = ·5222.

† See p. 142.

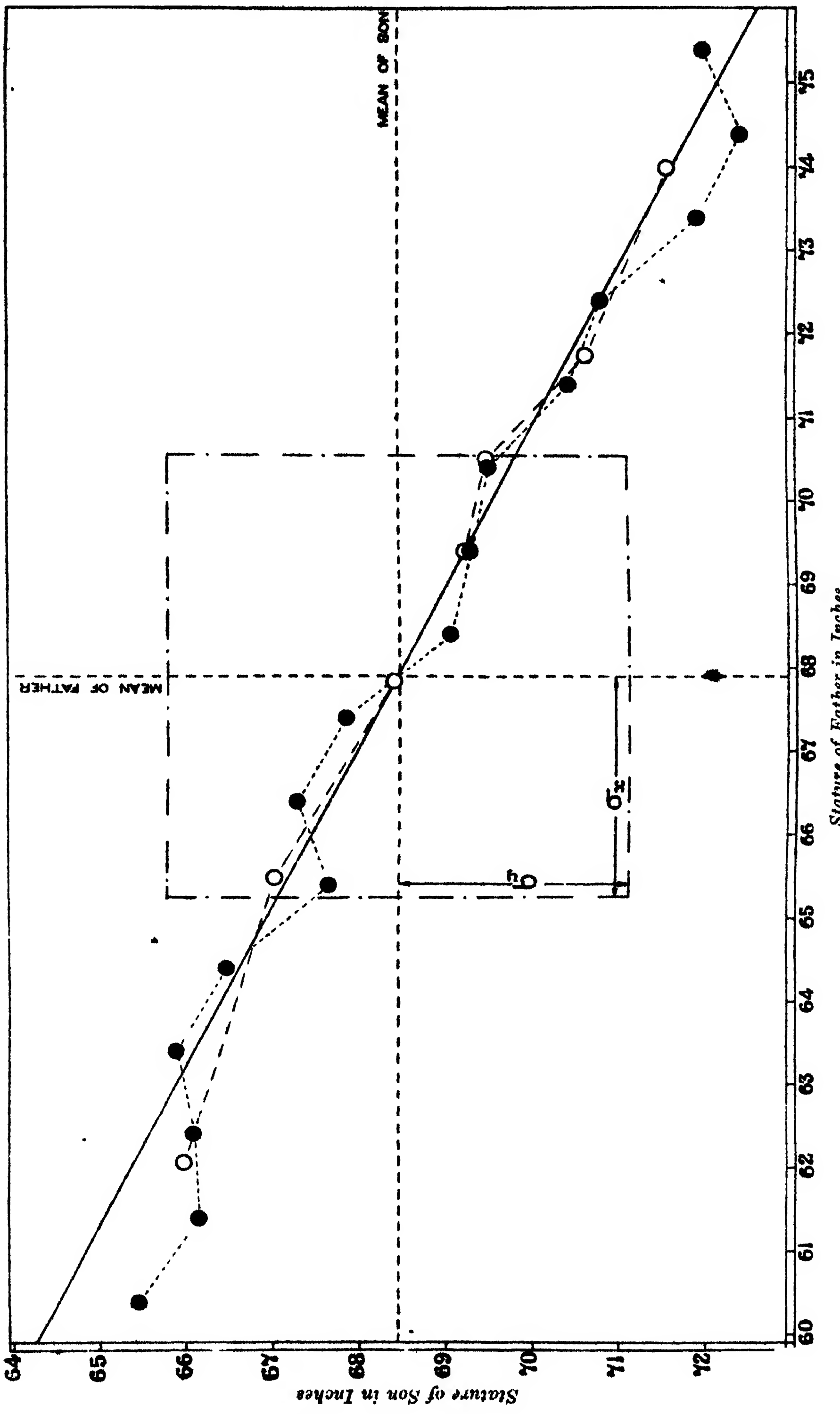

MEAN OF SON
MEAN OF FATHER
σ_x
σ_y
60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75
64 65 66 67 68 69 70 71 72
Stature of Father in Inches.
Stature of Son in Inches

Diagram IV.

is one of fitting the "best" normal surface to the data subject to the limitation that the marginal totals are exactly reproduced, and this limits the generality.

An example has been given of the process, but it is seen from this example that the heavy arithmetic does not lead us to any more accurate value for the correlation than far simpler methods. Thus:

Correlation from product-moment	$= \cdot 5189 \pm \cdot 0160$.
Polychoric Correlation Coefficient "Best Fit"	$= \cdot 5034$.
Polychoric Correlation Coefficient "Product Moment"	$= \cdot 5204$.
Mean Square Contingency, Corrected for Class Indices	$= \cdot 5179$.
Correlation Ratio from means of arrays	$= \cdot 5218$.

The latter method, which has been long in use in the Biometric Laboratory, is thus, when used with due precaution, seen to be justified by the theoretically preferable polychoric method. If a method could be discovered of finding uniquely the mean of a columnar array, *using all its cells at the same time*, this method would still more effectively replace the polychoric correlation coefficient.

ON EXPANSIONS IN TETRACHORIC FUNCTIONS.

By JAMES HENDERSON, M.A., B.Sc.

(1) We define the *tetrachoric function* of order s to be $\tau_s(x)$, where

$$\tau_s(x) = \frac{1}{\sqrt{s!}} \left(-\frac{d}{dx}\right)^{s-1} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \quad \text{.........................(i).}$$

Other writers have adopted various other values for the external numerical factor but this is immaterial. The factor $\frac{1}{\sqrt{s!}}$ was chosen because it gives an extremely simple expression for the volume of a quadrant of the normal bivariate frequency surface, and because for tabulating the numerical values of the functions it is necessary to have some reduction factor of this kind to keep them of manageable size. We can usually drop the argument x and speak of τ_s. The values of τ_s for $s = 1$ up to $s = 6$ are tabled to five decimal places in the book, *Tables for Statisticians and Biometricians**, for values of $\frac{1}{2}(1-\alpha)$ (which is really τ_0, when the argument is negative) from ·000 to ·500 at intervals of ·001. With a different multiplier they have been tabled by Charlier† to four decimal places only for $s = 1$, 4 and 5 (x = ·00 to 3).

The general form of the tetrachoric function of order s is

$$\tau_s(x) = \frac{1}{\sqrt{s!}} \left\{x^{s-1} - \frac{(s-1)(s-2)}{2.1!} x^{s-3} + \frac{(s-1)(s-2)(s-3)(s-4)}{2^2.2!} x^{s-5} - \text{etc.}\right\}$$
$$\times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad \text{..............................(ii),}$$

that is, the ordinate of the normal curve of errors multiplied by a polynomial of degree $(s-1)$. τ_1 is simply the ordinate of the normal curve, while τ_0 is the area of the tail of the normal curve up to a given abscissa x, with the addition of an arbitrary constant. This constant may be so selected that $\tau_0 = \int_{-\infty}^{x} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx$, and will be found from the tables of the probability integral. It will be equal to $\frac{1}{2}(1+\alpha)$, if x is positive and $\frac{1}{2}(1-\alpha)$, if x be negative in the usual notation. Accordingly the expansion of a function of x, $f(x)$ in a series of tetrachoric functions, is really the expansion of the difference of the function and a multiple of the probability integral in terms of

$$\left(c_0 + c_1\frac{x}{\sigma} + c_2\frac{x^2}{\sigma^2} + \dots\right) e^{-\frac{1}{2}x^2/\sigma^2},$$

where σ and $c_0, c_1, c_2 \dots$ are at our choice.

* Cambridge University Press, p. 1, and pp. 42—51.

† *Vorlesungen über die Grundzüge der mathematischen Statistik*, 1920.

The real reason for adopting

$$c_0'\tau_0 + c_1'\tau_1 + c_2'\tau_2 + c_3'\tau_3 + \ldots,$$

instead of the above expression, is that the calculation of the constants $c_0', c_1', c_2' \ldots$ is more direct than that of $c_0, c_1, c_2 \ldots$ because the tetrachoric functions are semi-orthogonal functions*. It will be seen that the problem of expansion in tetrachoric functions is closely related to a theorem of Laplace. If U be a unimodal function of x within the range under discussion and the integral $I = \int U dx$ be required, Laplace transfers to the mode m as origin so that $x = m + \xi$ and writes U in the following form:

$$U = U_m e^{-\frac{1}{2}a_0\xi^2}(1 + a_3\xi^3 + a_4\xi^4 + \ldots).$$

He extends the limits to ∞ in both directions by supposing $U = 0$ outside the given range and in the integration applies the well-known values of $\int_{-\infty}^{\infty} \xi^s e^{-\frac{1}{2}a_0\xi^2} d\xi$, i.e. zero if s be odd, and again if s be even $(= 2r)$,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\xi^2/\sigma^2}\xi^{2r}\, d\xi = (2r-1)(2r-3)\ldots 3 \,.\, 1 \,.\, \sqrt{2\pi}\,\sigma^{2r+1}.$$

It will be seen that Laplace is really proceeding by expansion in tetrachoric functions as the process is precisely the same whatever be the limits of the integral of U. Following Laplace we develop our function in "incomplete normal moment functions," i.e. $\int_{-\infty}^{x} \frac{x^s e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx$†; it is better to use tetrachoric functions. The series in tetrachoric functions seems to converge slightly better than that in incomplete normal moment functions.

If we have

$$F(x) = a_0\tau_1 + a_1\tau_2 + a_2\tau_3 + \ldots + a_{s-1}\tau_s + \ldots,$$

then, assuming we may integrate the right-hand side of this equation term by term (i.e. assuming uniform convergence) between x and ∞,

$$\int F(x)\, dx = a_0\tau_0 + \frac{a_1\tau_1}{\sqrt{2}} + \frac{a_2\tau_2}{\sqrt{3}} + \ldots,$$

since

$$\int_x^{\infty} \tau_s dx = \frac{\tau_{s-1}}{\sqrt{s}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

* A series of functions $f_1(x), f_2(x) \ldots f_s(x) \ldots f_{s'}(x)$ is orthogonal if $\int f_s(x) f_{s'}(x)\, dx = 0$ when s and s' are not equal, the integration being throughout the range. They are semi-orthogonal if

$$\int f_s(x) f_{s'}(x)\, \phi(x)\, dx = 0,$$

$\phi(x)$ being a function of x peculiar to the series. In other words a system is orthogonal if the sums of the products of different order functions vanish *without weighting* for x. A system is semi-orthogonal if we require to weight the values of x to obtain the vanishing of the product sum. This weighting is the great disadvantage of semi-orthogonal functions. In our case of the tetrachoric functions the weighting factor is $e^{\frac{1}{2}x^2}$ or the tails of series are excessively weighted.

† Discussed *Biometrika*, Vol. VI. p. 59. Tables of these functions up to $s = 10$ are given in *Tables for Statisticians*, pp. 22—3.

Let $\tau_s = \frac{1}{\sqrt{s!}} p_{s-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, where p_{s-1} is the polynomial in x of degree $(s-1)$ in (ii).

Let $\tau_{s'}$ be another tetrachoric function and suppose s' is greater than s. Then

$$\int_{-\infty}^{\infty} \tau_s \tau_{s'} e^{\frac{1}{2}x^2} dx = \frac{1}{\sqrt{s!}} \frac{1}{\sqrt{s'!}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p_{s-1} \left(-\frac{d}{dx}\right)^{s'-1} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx.$$

Now since $\tau_s(\infty)$ and $\tau_s(-\infty)$ will always be zero owing to the exponential factor $(s > 0)$ we can integrate by parts transferring the $\frac{d}{dx}$ from the exponential to the polynomial, therefore

$$\int_{-\infty}^{x} \tau_s \tau_{s'} e^{\frac{1}{2}x^2} dx = \frac{1}{\sqrt{s!}} \frac{1}{\sqrt{s'!}} \frac{1}{\sqrt{2\pi}} \left[\left\{-p_{s-1}\left(-\frac{d}{dx}\right)^{s'-2} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\right\}_{-\infty}^{x} + \int_{-\infty}^{x} \left(-\frac{d}{dx}\right)^{s'-2} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \frac{d}{dx} p_{s-1}\, dx\right].$$

The integrated part at every step vanishes at the limits and ultimately

$$\int_{-\infty}^{x} \tau_s \tau_{s'} e^{\frac{1}{2}x^2} dx = \frac{1}{\sqrt{s!}\sqrt{s'!}\sqrt{2\pi}} \int_{-x}^{x} \frac{d^{s'-1}}{dx^{s'-1}} p_{s-1} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx.$$

Since p_{s-1} is a polynomial of degree $(s-1)$ and s' is $> s$ the differential of the polynomial vanishes, i.e.

$$\int_{-\infty}^{x} \tau_s \tau_{s'} e^{\frac{1}{2}x^2} dx = 0, \quad s \neq s' \quad \text{.......... ....(iv).}$$

If $s' = s$ then the differential of p_{s-1} reduces to $(s-1)!$ so that

$$\int_{-\infty}^{\infty} \tau_s^2 e^{\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} \frac{(s-1)!}{s!} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{s}, \quad s > 0 \quad \text{.............. ....(v).}$$

These equations (iv) and (v), which give the fundamental properties of the tetrachoric functions, enable us to expand any function $F(x)$ in terms of tetrachoric functions if we can find the value of the integral

$$\int_{-\infty}^{\infty} F(x)\, \tau_s e^{\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{s!}} p_{s-1} F(x)\, dx \quad \text{...... ....(vi).}$$

Since p_{s-1} is an integral function of x, this amounts to saying that we can expand any function of which we are able to determine the successive moment-coefficients.

The *practical value* of the functional expansion when obtained is, however, a very different matter. That depends on the convergency of the series and our experience has shown us that in the most common cases the convergency is so slight or non-existent as to render the expansion idle.

The matter is a very important one for Thiele*, Edgeworth† and Charlier‡ have proposed to treat skew frequency distributions by a process, which amounts to the same thing as the expansion by tetrachoric functions.

An attempt made many years ago§ to expand Incomplete Γ- and B-functions by Laplace's method in Incomplete Moment Functions convinced Professor Pearson that little was to be gained by a series expansion in the form of a polynomial multiplied by the ordinate of a normal curve. A variant of this method, that of expressing Incomplete Γ- and B-functions in a series of tetrachoric functions, was tried a year ago and it was found that except for a small distance round the mode this method of expressing a frequency distribution was quite ineffectual. The matter is of considerable importance because quite recently a Scandinavian actuary in America‖ has been analysing mortality curves by tetrachoric functions and asserts not only that they give a good fit but apparently believes that each function of the series has some natural physiological meaning! It is quite possible to represent the survivors of 100,000 persons born in the same year of life by a Fourier's series from 0 to 100 years but one would hardly claim any special physiological significance for the individual periodic terms¶. Such a series however is far easier to deal with in later treatment, such as differencing, than a series in tetrachoric functions.

For the numerical calculation of the tetrachoric functions the difference equation of these functions is invaluable, i.e.

$$\tau_s = x\beta_s\tau_{s-1} - \gamma_s\tau_{s-2},$$

where x is the argument of the functions and

$$\beta_s = \frac{1}{\sqrt{s}}, \quad \gamma_s = \frac{s-2}{\sqrt{s(s-1)}}.$$

Tables of β_s and γ_s are given in *Tables for Statisticians* (p. 1 of introduction) to five decimal places for $s=7$ to $s=24$ (the first six tetrachoric functions being given on pp. 42—51) and in *Biometrika*, Vol. XIV. p. 130 to 7 decimal places.

For our work β_s and γ_s were required to 7 places (sometimes to 8) to obtain the requisite accuracy. The procedure consists in calculating τ_1, which is equal to $\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$, directly to the required degree of accuracy and then by means of the tables referred to above the higher tetrachoric functions are obtained in rapid succession on the machine for a given value of the argument. In the testing of our tetrachoric series seven-place accuracy was aimed at so that it was necessary to calculate τ_1 to eight places, which was done with the help of Vega's ten-figure logarithms.

* *Forlaesninger over Almindelig Iagttagelseslaere*, Kjobenhaven, 1889.

† *Royal Soc. Proc.* Vol. LVI. p. 271, and in many papers, *Journal of R. Statistical Society*.

‡ *Vorlesungen über die Grundzuge der mathematischen Statistik* (Hamburg, 1920), p. 67.

§ *Biometrika*, Vol. VI. p. 68, 1908.

‖ Arne Fisher, *Casualty, Actuarial and Statistical Society of America. Proceedings*, Vol. IV. Part I. No. 9.

¶ A normal curve, for example, is quite adequately represented by two or three periodic terms: see *Phil. Trans.* Vol. CLXXXVI A, p. 355, 1895.

(2) It is well known that a wide range of frequency distributions can be adequately represented by one or other of the curves

$$\left.\begin{aligned} y &= y_0 e^{-\gamma x/a}\left(1+\frac{x}{a}\right)^{p-1} \ldots\ldots\ldots\ldots\ \ldots\ldots\ldots\ldots(a) \\ \text{and}\qquad y &= y_0\left(1+\frac{x}{a_1}\right)^{m_1-1}\left(1-\frac{x}{a_2}\right)^{m_2-1}\ \ldots\ldots\ldots\ \ldots(b) \end{aligned}\right\} \quad \text{(vii)}.$$

By a change of origin and the appropriate stretch or squeeze these may be reduced to

$$\left.\begin{aligned} y &= y_0 x^{p-1} e^{-x} \ \ldots\ldots\ldots\ldots\ldots\ \ldots\ldots\ldots(a) \\ \text{and}\qquad y &= y_0 x^{m_1-1}(1-x)^{m_2-1} \ldots\ldots\ldots\ldots\ldots\ .(b) \end{aligned}\right\} \text{(vii) bis.}$$

Now, generally, it is not the ordinates of these curves which are required but the areas of certain portions, or in other words the probability integrals of these skew curves. The total range for (vii) bis (a) is 0 to ∞ and for (b) is 0 to 1; since

$$\int_0^\infty x^{p-1} e^{-x}\, dx = \Gamma(p)$$

and

$$\int_0^1 x^{m_1-1}(1-x)^{m_2-1}\, dx = \mathrm{B}(m_1, m_2)$$

we may take these probability integrals to be

$$I(p, v) \qquad = \frac{1}{\Gamma(p)}\int_0^v v^{p-1} e^{-v}\, dv$$

and

$$\mathrm{B}(v, m_1, m_2) = \frac{1}{\mathrm{B}(m_1, m_2)}\int_0^v v^{m_1-1}(1-v)^{m_2-1}\, dv,$$

which are the ratios of the incomplete to the complete Γ- and B-functions

The equations on p. 158 show us that if either of the frequency functions (vii) is expressible in a series of tetrachoric functions their probability integrals (assuming convergence) will also be. Now there is no doubt that a large mass of material does not differ practically from the forms in (vii) and accordingly if the above probability integrals cannot be adequately expressed in a series of tetrachoric functions, we may be certain that tetrachoric functions do not furnish a suitable method of representing skew frequency. Accordingly our problem reduces itself to the following one: Can $I(p, v)$ and $\mathrm{B}(v, m_1, m_2)$, or the Incomplete Γ- and B-functions, be represented with adequate convergency by a series of tetrachoric functions? After examination of the numerical and graphical results obtained, we are obliged to conclude that the answer to this question is in the negative.

(3) Let us first consider the expansion in tetrachoric functions of the function

$$y = x^{p-1} e^{-x}/\Gamma(p) \ \ldots\ldots\ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\text{(viii)}.$$

In expanding this expression there are at least two methods, which we ought to consider, and one may have advantages over the other as far as convergency is

concerned. It may be expanded with regard: (i) to the mean and the standard deviation, or (ii) to the mode in the manner of Laplace*.

(i) The mean of the function (viii) is easily found to be at $x = p$, the mode is at $x = p - 1$ and the standard deviation is $\sqrt{p}$.

Referring to the mean as origin the function becomes

$$y = \frac{(\xi + p)^{p-1} e^{-(\xi+p)}}{\Gamma(p)} \quad \text{.............................(ix).}$$

Let $$y = \phi(-D) \frac{e^{-\xi^2/2p}}{\sqrt{2\pi}}, \text{ where } D = \frac{d}{dz} \text{ and } z = \frac{\xi}{\sqrt{p}} \quad \text{..............(x).}$$

Except for a numerical factor the right-hand side is a series of tetrachoric functions.

Let $$\phi(-D) = c_0 - c_1 D + c_2 D^2 \ldots (-1)^s c_s D^s + \ldots.$$

The function $\phi(-D)$ has to be determined, i.e. we require to find the successive c's:

$$\phi(-D)\left\{\frac{e^{-\xi^2/2p}}{\sqrt{2\pi}}\right\} = \phi(-D)\left\{\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\right\}$$

$$= c_0 \tau_1 + c_1 \sqrt{2!}\, \tau_2 + c_2 \sqrt{3!}\, \tau_3 + \ldots + c_{s-1} \sqrt{s!}\, \tau_s + \ldots \quad \text{(xi).}$$

To determine the c's. With the origin at the mean the function y must be taken as zero from $-\infty$ to $-p$, while from $-p$ to $+\infty$ it is given by (ix). The c's will be obtained most easily by multiplying both sides of (x) by $e^{\theta\xi}$ and equating the coefficients of powers of θ on both sides of the equation, i.e. we make all the moments of the two expressions for the curve the same, for the coefficient of θ^s on either side is the sth moment†. Thus

$$\int_{-\infty}^{\infty} y e^{\theta\xi}\, d\xi = \int_{-\infty}^{\infty} e^{\theta\xi} \phi(-D) \frac{e^{-\xi^2/2p}}{\sqrt{2\pi}}\, d\xi;$$

but $y = 0$ from $x = -\infty$ to $-p$.

Accordingly $$\int_{-p}^{\infty} \frac{e^{\theta\xi}(\xi + p)^{p-1} e^{-(\xi+p)}}{\Gamma(p)}\, d\xi = \int_{-\infty}^{\infty} e^{\theta\xi} \phi(-D) \frac{e^{-\xi^2/2p}}{\sqrt{2\pi}}\, d\xi.$$

Now $x = p + \xi$ and $z = \xi/\sqrt{p}$.

Thus $$\int_0^{\infty} \frac{e^{\theta(x-p)} x^{p-1} e^{-x}}{\Gamma(p)}\, dx = \sqrt{p} \int_{-\infty}^{\infty} e^{(\theta\sqrt{p}\, z)} \phi(-D) \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\, dz.$$

The left-hand side is equal to

$$e^{-p\theta} \int_0^{\infty} \frac{x^{p-1} e^{-x(1-\theta)}}{\Gamma(p)}\, dx$$

$$= e^{-p\theta} \int_0^{\infty} \frac{u^{p-1}}{(1-\theta)^{p-1}} \frac{e^{-u}}{\Gamma(p)} \frac{du}{(1-\theta)} \qquad \text{Let } x(1-\theta) = u.$$

$$= e^{-p\theta}(1-\theta)^{-p}.$$

* Laplace's method is really an expansion in incomplete normal moment functions but as we have seen (p. 158) these may be replaced by tetrachoric functions.

† We owe this elegant method of determining the c's to Mr H. E. Soper. Originally the c's were determined by use of the fundamental property of the tetrachoric functions but that method, while leading to the same result, is more laborious.

To find the value of the integral on the right-hand side, consider the term $c_s(-D)^s$ in the function $\phi(-D)$. Its contribution to the integral is

$$c_s \int_{-\infty}^{\infty} e^{\theta' z}\left(-\frac{d}{dz}\right)^s \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,dz,$$

where $\theta' = \theta\sqrt{p}$.

On integrating by parts the term between limits vanishes owing to the factor $e^{-\frac{1}{2}z^2}$. Hence the integral

$$= c_s\theta' \int_{-\infty}^{\infty} e^{\theta' z}\left(-\frac{d}{dz}\right)^{s-1} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,dz,$$

and ultimately

$$= c_s\theta'^s \int_{-\infty}^{\infty} \frac{e^{\theta' z-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,dz$$

$$= c_s\theta'^s \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(z-\theta')^2+\frac{1}{2}\theta'^2}}{\sqrt{2\pi}}\,dz$$

$$= c_s\theta'^s e^{\frac{1}{2}\theta'^2}.$$

Therefore the whole integral on the right is

$$\phi(\theta')\,e^{\frac{1}{2}\theta'^2},$$

i.e.
$$e^{-p\theta}(1-\theta)^{-p} = \sqrt{p}\,\phi(\sqrt{p}\theta)\,e^{\frac{1}{2}p\theta^2},$$

or
$$\sqrt{p}\,\phi(\sqrt{p}\theta) = e^{-p\theta-\frac{1}{2}p\theta^2}(1-\theta)^{-p},$$

and
$$\phi(\sqrt{p}\theta) = c_0 + c_1(\sqrt{p}\theta) + c_2(\sqrt{p}\theta)^2 + \ldots + c_s(\sqrt{p}\theta)^s + \ldots$$
$$= c_0 + c_1'\theta + c_2'\theta^2 + \ldots\ldots\ldots\ldots + c_s'\theta^s + \ldots.$$

where
$$c_s' = c_s(\sqrt{p})^s \quad\text{or}\quad c_s = c_s'(\sqrt{p})^{-s}.$$

Now
$$e^{-p\theta-\frac{1}{2}p\theta^2}(1-\theta)^{-p}$$
$$= e^{-p\theta-\frac{1}{2}p\theta^2-p\log(1-\theta)}$$
$$= e^{-p\theta-\frac{1}{2}p\theta^2+p\theta+\frac{1}{2}p\theta^2+\frac{1}{3}p\theta^3+\frac{1}{4}p\theta^4+\frac{1}{5}p\theta^5+\ldots}$$
$$= e^{\frac{1}{3}p\theta^3+\frac{1}{4}p\theta^4+\frac{1}{5}p\theta^5+\ldots}$$
$$= b_0 + b_1\theta + b_2\theta^2 + b_3\theta^3 + b_4\theta^4 + \ldots,$$

where

$b_0 = 1,\ b_1 = b_2 = 0,\ b_3 = \frac{1}{3}p,\ b_4 = \frac{1}{4}p,\ b_5 = \frac{1}{5}p,\ b_6 = \frac{1}{6}p + \frac{1}{2}(\frac{1}{3}p)^2 = \frac{1}{18}p(p+3)$, etc.

But $\sqrt{p}c_s' = b_s$, therefore

$$c_0' = \frac{1}{\sqrt{p}},\quad c_1' = c_2' = 0,\quad c_3' = \tfrac{1}{3}\sqrt{p},\quad c_4' = \tfrac{1}{4}\sqrt{p},\quad c_5' = \tfrac{1}{5}\sqrt{p},\ \text{etc.}$$

so that
$$c_0 = \frac{1}{\sqrt{p}},\quad c_1 = c_2 = 0,\quad c_3 = \tfrac{1}{3}\frac{1}{p},\quad c_4 = \tfrac{1}{4}\frac{1}{p\sqrt{p}},\quad c_5 = \tfrac{1}{5}\frac{1}{p^2},\ \text{etc.}$$

For numerical purposes these coefficients are much more usefully obtained in the following way:

Let
$$e^{-p\theta-\frac{1}{2}p\theta^2}(1-\theta)^{-p} = b_0 + b_1\theta + b_2\theta^2 + \ldots\ \text{etc.}$$

Take the differential of the logarithms of both sides; then

$$(-p - p\theta + p/1-\theta)(b_0 + b_1\theta + b_2\theta^2 + \ldots + b_s\theta^s + \ldots) = b_1 + 2b_2\theta + \ldots + sb_s\theta^{s-1} + \ldots,$$

i.e.
$$p\theta^2(b_0 + b_1\theta + \ldots + b_s\theta^s + \ldots) = (1-\theta)(b_1 + 2b_2\theta + \ldots + sb_s\theta^{s-1} + \ldots)$$

Equating coefficients of θ^s we have

$$pb_{s-2} = (s+1)\,b_{s+1} - sb_s,$$

i.e.

$$b_{s+1} = \frac{1}{s+1}\{sb_s + pb_{s-2}\} \quad \text{.....................(xii)}.$$

By this difference formula successive b's can be found very quickly if b_0, b_1, b_2 are known and we have already found these.

Now

$$c_s = (\sqrt{p})^{-s}\,c_s'$$

$$= (\sqrt{p})^{-s}\,b_s\,(\sqrt{p})^{-1} = b_s/(\sqrt{p})^{s+1},$$

or

$$b_s = (\sqrt{p})^{s+1}\,c_s.$$

Substituting in (xii)

$$(\sqrt{p})^{s+2}\,c_{s+1} = \frac{1}{(s+1)}\{s\,(\sqrt{p})^{s+1}\,c_s + p\,(\sqrt{p})^{s-1}\,c_{s-2}\}$$

$$= \frac{(\sqrt{p})^{s+1}}{(s+1)}\{sc_s + c_{s-2}\},$$

or

$$c_{s+1} = \frac{1}{\sqrt{p}}\,\frac{1}{(s+1)}\{sc_s + c_{s-2}\} \quad \text{.........(xiii)}.$$

This formula gives us very readily the coefficients of $\phi(-D)$ and thus the expansion is obtained.

We had, Equation (xi),

$$\frac{x^{p-1}e^{-x}}{\Gamma(p)} = \phi(-D)\,\frac{e^{-\xi^2/2p}}{\sqrt{2\pi}} = c_0\tau_1 + c_1\sqrt{2!}\,\tau_2 + c_2\sqrt{3!}\,\tau_3 + \ldots + c_s\sqrt{(s+1)!}\,\tau_{s+1} + \ldots,$$

and all the c's are known since $c_0 = \dfrac{1}{\sqrt{p}}$, $c_1 = c_2 = 0$.

To find the area under the curve (xi) up to abscissa x, remembering that the left-hand side is zero from $\xi = -\infty$ to $-p$,

$$\int_{-p}^{\xi}\frac{(\xi+p)^{p-1}e^{-(\xi+p)}}{\Gamma(p)}\,d\xi = \int_{-r}^{\xi}\phi(-D)\,\frac{e^{-\xi^2/2p}}{\sqrt{2\pi}}\,d\xi,$$

i.e.

$$\int_0^x\frac{x^{p-1}e^{-x}}{\Gamma(p)}\,dx = \sqrt{p}\int_{-r}^{z}\phi(-D)\,\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,dz$$

$$= \sqrt{p}\int_{-r}^{z}(c_0\tau_1 + c_1\sqrt{2!}\,\tau_2 + \ldots + c_{s-1}\sqrt{s!}\,\tau_s + \ldots)\,dz.$$

Now

$$\int_{-\infty}^{z}\tau_s dz = -\frac{\tau_{s-1}}{\sqrt{s}},$$

therefore

$$\int_0^x\frac{x^{p-1}e^{-x}}{\Gamma(p)}\,dx = \sqrt{p}\left[c_0\int_{-\infty}^{z}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,dz - c_1\frac{\sqrt{2!}\,\tau_1}{\sqrt{2}} - c_2\frac{\sqrt{3!}\,\tau_2}{\sqrt{3}} - \ldots - c_{s-1}\frac{\sqrt{s!}\,\tau_{s-1}}{\sqrt{s}} - \ldots\right]$$

$$= \tfrac{1}{2}(1+\alpha_z) - \sqrt{p}\,\{c_1\tau_1 + c_2\sqrt{2!}\,\tau_2 + \ldots + c_{s-1}\sqrt{(s-1)!}\,\tau_{s-1} + \ldots\}\ \text{since}\ c_0 = \frac{1}{\sqrt{p}}.$$

Therefore finally

$$\int_0^x \frac{x^{p-1}e^{-x}}{\Gamma(p)}\,dx = \tfrac{1}{2}(1+\alpha_z) - a_3\tau_3 - \ldots - a_s\tau_s - \ldots \qquad \ldots\ldots\ldots\ldots\text{(xiv)},$$

(since $c_1 = c_2 = 0$) where $a_s = \sqrt{p}\,c_s\sqrt{s}\,!$.

Now $c_{s+1} = \dfrac{1}{\sqrt{p}}\dfrac{1}{s+1}\{sc_s + c_{s-2}\}$ from equation (xiii),

i.e.
$$\frac{a_{s+1}}{\sqrt{(s+1)\,!}} = \frac{1}{\sqrt{p}}\frac{1}{(s+1)}\left\{s\frac{a_s}{\sqrt{s\,!}} + \frac{a_{s-2}}{\sqrt{(s-2)\,!}}\right\},$$

therefore
$$a_{s+1} = \frac{1}{\sqrt{p}}\frac{1}{(s+1)}\{s\sqrt{(s+1)}\,a_s + \sqrt{(s+1)\,(s)\,(s-1)}\,a_{s-2}\}$$

$$= \sqrt{\frac{s}{p\,(s+1)}}\{\sqrt{s}\,a_s + \sqrt{(s-1)}\,a_{s-2}\} \quad\ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(xv)},$$

where
$$a_0 = 1, \quad a_1 = a_2 = 0.$$

The argument of $\frac{1}{2}(1+\alpha)$ and of the tetrachoric functions is $\xi/\sqrt{p}$, which equals $\dfrac{x-p}{\sqrt{p}} = z$, say.

Since the terms τ_1 and τ_2 do not appear one might hope that only a few terms of the expansion (xiv) would be required to obtain a sufficiently accurate result.

$\frac{1}{2}(1+\alpha_z)$ is the ordinary probability integral at z.

Note that if x is less than p, i.e. z is negative, $\frac{1}{2}(1-\alpha_z)$ must be used instead of $\frac{1}{2}(1+\alpha_z)$ and the tetrachoric functions of even order must be taken of opposite sign to those for positive z such as are given in the tables. The odd order functions are the same for positive and negative z:

$$\tau_{2s}(z) = -\tau_{2s}(-z), \quad \tau_{2s+1}(z) = \tau_{2s+1}(-z).$$

Obviously we could get the area of any portion of the curve between $x = x_1$ and $x = x_2$ by subtracting two expressions like (xiv) for z_1 and z_2.

The general expression for $\dfrac{x^{p-1}e^{-x}}{\Gamma(p)}$ is

$$\frac{x^{p-1}e^{-x}}{\Gamma(p)} = \frac{1}{\sqrt{p}}\tau_1 + \frac{1}{p}\frac{\sqrt{4\,!}}{3}\tau_4 + \frac{1}{p\sqrt{p}}\frac{\sqrt{5\,!}}{4}\tau_5$$
$$+\frac{1}{p^2}\frac{\sqrt{6\,!}}{5}\tau_6 + \frac{1}{p^2\sqrt{p}}\frac{\sqrt{7\,!}}{6}\left(\frac{p+3}{3}\right)\tau_7$$
$$+\frac{1}{p^3}\frac{\sqrt{8\,!}}{7}\left\{\frac{7p+12}{12}\right\}\tau_8 + \frac{1}{p^3\sqrt{p}}\frac{\sqrt{9\,!}}{8}\left\{\frac{47p+60}{60}\right\}\tau_9$$
$$+\frac{1}{p^4}\frac{\sqrt{10\,!}}{9}\left\{\frac{p^2}{18}+\frac{19}{20}p+1\right\}\tau_{10} + \frac{1}{p^4\sqrt{p}}\frac{\sqrt{11\,!}}{10}\left\{\frac{5}{36}p^2+\frac{153}{140}p+1\right\}\tau_{11}$$
$$+\frac{1}{p^5}\frac{\sqrt{12\,!}}{11}\left\{\frac{341}{1440}p^2+\frac{341}{280}p+1\right\}\tau_{12}$$
$$+\frac{1}{p^5\sqrt{p}}\frac{\sqrt{13\,!}}{12}\left\{\frac{p^3}{162}+\frac{493}{1440}p^2+\frac{3349}{2520}p+1\right\}\tau_{13}+\ldots,$$

and

$$\int_0^x \frac{x^{p-1}e^{-x}}{\Gamma(p)}\,dx = \tfrac{1}{2}(1+\alpha_z) - \frac{1}{\sqrt{p}}\frac{\sqrt{4!}}{3\sqrt{4}}\tau_3 - \frac{1}{p}\frac{\sqrt{5!}}{4\sqrt{5}}\tau_4$$

$$- \frac{1}{p\sqrt{p}}\frac{\sqrt{6!}}{5\sqrt{6}}\tau_5 - \frac{1}{p^2}\frac{\sqrt{7!}}{6\sqrt{7}}\left(\frac{p+3}{3}\right)\tau_6 - \frac{1}{p^2\sqrt{p}}\frac{\sqrt{8!}}{7\sqrt{8}}\left(\frac{7p+12}{12}\right)\tau_7$$

$$- \frac{1}{p^3}\frac{\sqrt{9!}}{8\sqrt{9}}\left\{\frac{47p+60}{60}\right\}\tau_8 - \frac{1}{p^3\sqrt{p}}\frac{\sqrt{10!}}{9\sqrt{10}}\left\{\frac{p^2}{18}+\frac{19}{20}p+1\right\}\tau_9$$

$$- \frac{1}{p^4}\frac{\sqrt{11!}}{10\sqrt{11}}\left\{\frac{5}{36}p^2+\frac{153}{140}p+1\right\}\tau_{10} - \frac{1}{p^4\sqrt{p}}\frac{\sqrt{12!}}{11\sqrt{12}}\left\{\frac{341}{1440}p^2+\frac{341}{280}p+1\right\}\tau_{11}$$

$$- \frac{1}{p^5}\frac{\sqrt{13!}}{12\sqrt{13}}\left\{\frac{p^3}{162}+\frac{493}{1440}p^2+\frac{3349}{2520}p+1\right\}\tau_{12} - \dots$$

$$= \tfrac{1}{2}(1+\alpha_z) - \frac{\cdot 8164,9658}{\sqrt{p}}\tau_3 - \frac{1\cdot 2247,4487}{p}\tau_4$$

$$- \frac{2\cdot 1908,9023}{p\sqrt{p}}\tau_5 - \frac{1\cdot 4907,1198}{p^2}(p+3)\tau_6$$

$$- \frac{\cdot 8451,5425}{p^2\sqrt{p}}(7p+12)\tau_7 - \frac{\cdot 4183,3001}{p^3}(47p+60)\tau_8$$

$$- \frac{\cdot 3718,4890}{p^3\sqrt{p}}(10p^2+171p+180)\tau_9 - \frac{\cdot 1511,8579}{p^4}(175p^2+1377p+1260)\tau_{10}$$

$$- \frac{\cdot 0569,8743}{p^4\sqrt{p}}(2387p^2+12276p+10080)\tau_{11}$$

$$- \frac{\cdot 0201,0408}{p^5}(560p^3+31059p^2+120564p+90720)\tau_{12} - \dots .$$

(ii) *Laplacian Form of Expansion.*

This is an expansion with regard to the mode or maximum ordinate as origin. The mode of $y = \dfrac{x^{p-1}e^{-x}}{\Gamma(p)}$ is at $x = (p-1)$, so that it will be easier to deal with y in the form

$$y = \frac{x^{p'}e^{-x}}{\Gamma(p'+1)},$$

where $p' = (p-1)$.

Let $x = p' + \xi$, i.e. take the mode as origin. Then as before we require to find $\phi(-D)$ so that

$$\frac{(p'+\xi)^{p'}e^{-(p'+\xi)}}{\Gamma(p'+1)} = \phi(-D)\frac{e^{-\frac{1}{2}\xi^2/p'}}{\sqrt{2\pi p'}} \quad \text{.....................(xvi)},$$

where $D = \dfrac{d}{dz}$ and $z = \dfrac{\xi}{\sqrt{p'}}$.

The introduction of $\sqrt{p'}$ in the denominator simplifies the integration a little.

Proceeding as before:

$$\int_{-p'}^{\infty} \frac{e^{\theta\xi}(p'+\xi)^{p'} e^{-(p'+\xi)}}{\Gamma(p'+1)} d\xi = \int_{-\infty}^{\infty} e^{\theta\xi} \phi(-D) \frac{e^{-\frac{1}{2}\xi^2/p'}}{\sqrt{2\pi p'}} d\xi,$$

i.e.

$$\int_0^{\infty} \frac{e^{\theta(x-p')} x^{p'} e^{-x}}{\Gamma(p'+1)} dx = \int_{-\infty}^{\infty} e^{\theta\sqrt{p'}z} \phi(-D) \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz,$$

or

$$\frac{e^{-p\theta}}{(1-\theta)^{p+1}} = \int_{-\infty}^{\infty} e^{(\theta\sqrt{p'})z} \phi(-D) \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz,$$

and

$$e^{-p\theta}(1-\theta)^{-(p'+1)} = \phi(\theta\sqrt{p'}) \int_{-\infty}^{\infty} \frac{e^{\theta\sqrt{p}z-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz$$

$$= \phi(\theta\sqrt{p'}) e^{\frac{1}{2}p'\theta^2} \int^{\infty} \frac{e^{-\frac{1}{2}(z-\theta\sqrt{p})^2}}{\sqrt{2\pi}} dz$$

$$= \phi(\theta\sqrt{p'}) e^{\frac{1}{2}p\theta^2};$$

therefore

$$\phi(\theta\sqrt{p}) - e^{-p'\theta-\frac{1}{2}p\theta^2}(1-\theta)^{(p+1)} \quad \text{.....(xvii).}$$

Now if

$$\phi(-D) = c_0 - c_1 D + c_2 D^2 + \ldots + (-1)^s c_s D^s + \ldots,$$

$$\phi(\theta\sqrt{p'}) = c_0 + c_1(\theta\sqrt{p'}) + c_2(\theta\sqrt{p})^2 + \ldots + c_s(\theta\sqrt{p'})^s + \ldots$$

$$= c_0 + c_1'\theta + c_2'\theta^2 + \ldots + c_s'\theta^s + \ldots,$$

where:

$$c_s' = c_s(\sqrt{p'})^s \quad \text{or} \quad c_s = c_s'(\sqrt{p})^{-s},$$

$$e^{-p\theta-\frac{1}{2}p\theta^2}(1-\theta)^{-(p'+1)} = e^{-p'\theta-\frac{1}{2}p\theta^2-(p+1)\log(1-\theta)}$$

$$= e^{-p\theta-\frac{1}{2}p'\theta^2+(p+1)\left(\theta+\frac{\theta^2}{2}+\frac{\theta^3}{3}+\ \right)}$$

$$= e^{\theta+\frac{1}{2}\theta^2+\frac{(p+1)}{3}\theta^3+\frac{(p'+1)}{4}\theta^4+}$$

$$= c_0 + c_1'\theta + c_2'\theta^2 + \ldots,$$

where:

$$c_0 = 1,\ c_1' = 1,\ c_2' = 1,\ c_3' = \tfrac{1}{6} + \frac{p'+1}{3} + \tfrac{1}{2} = \frac{p'+3}{3},$$

and generally by differentiating

$$c_s' = c_{s-1} + \frac{p'}{s} c'_{s-3},$$

or

$$c_s(\sqrt{p'})^s = c_{s-1}(\sqrt{p'})^{s-1} + \frac{p'}{s} c_{s-3}(\sqrt{p'})^{s-3};$$

thus

$$c_s = \frac{1}{\sqrt{p'}}\left\{c_{s-1} + \frac{1}{s} c_{s-3}\right\} \quad \text{.....(xviii),}$$

where

$$c_0 = 1,\ c_1 = \frac{1}{\sqrt{p'}},\ c_2 = \frac{1}{p'}.$$

Therefore

$$\frac{(p'+\xi)^p e^{-(p+\xi)}}{\Gamma(p+1)} = \{c_0 - c_1 D + c_2 D^2 - \ldots (-1)^s c_s D^s - \ldots\}\left(\frac{e^{-\xi^2/2p}}{\sqrt{2\pi p'}}\right)$$

$$= \frac{1}{\sqrt{p}}\{c_0\tau_1 + \sqrt{2!}\, c_1\tau_2 + \sqrt{3!}\, c_2\tau_3 + \ldots + \sqrt{(s+1)!}\, c_s\tau_{s+1} +$$

To find the area up to abscissa x we have

$$\int_{-p'}^{\xi} \frac{(p'+\xi)^{p'} e^{-(p'+\xi)}}{\Gamma(p'+1)} d\xi = \frac{1}{\sqrt{p'}} \int_{-\infty}^{\xi} \{c_0\tau_1 + \sqrt{2!}\, c_1\tau_2 + \dots + \sqrt{(s+1)!}\, c_s\tau_{s+1} + \dots\} d\xi$$

$$= \int_{-x}^{z} \{c_0\tau_1 + \sqrt{2!}\, c_1\tau_2 + \dots + \sqrt{(s+1)!}\, c_s\tau_{s+1} + \dots\} dz$$

$$= \tfrac{1}{2}(1+\alpha_z) - c_1\tau_1 - \sqrt{2!}\, c_2\tau_2 - \sqrt{3!}\, c_3\tau_3 - \dots - \sqrt{s!}\, c_s\tau_s - \dots \text{ as } c_0 = 1,$$

i.e.
$$\int_0^x \frac{x^{p'} e^{-x}}{\Gamma(p'+1)} dx = \tfrac{1}{2}(1+\alpha_z) - a_1'\tau_1 - a_2'\tau_2 - a_3'\tau_3 - \dots - a_s'\tau_s - \dots,$$

where
$$a_s' = c_s\sqrt{s!}.$$

Substituting in (xviii) to obtain the difference equation for the a''s we have

$$\frac{a'_s}{\sqrt{s!}} = \frac{1}{\sqrt{p'}} \left\{ \frac{a'_{s-1}}{\sqrt{(s-1)!}} + \frac{1}{s} \frac{a'_{s-3}}{\sqrt{(s-3)!}} \right\};$$

therefore
$$a_s' = \frac{1}{\sqrt{p'}} \frac{1}{\sqrt{s}} \{s a'_{s-1} + \sqrt{(s-1)(s-2)}\, a'_{s-3}\} \quad \text{..............(xix)},$$

and
$$a_0' = 1, \quad a_1' = \frac{1}{\sqrt{p'}}, \quad a_2' = \frac{\sqrt{2}}{p'}.$$

By this formula the a's are readily obtained numerically. It is to be noted that in this case the terms in τ_1 and τ_2 do not vanish, as they did in the expansion from the mean. The argument of $\frac{1}{2}(1+\alpha_z)$ and of the tetrachoric functions is $\frac{x-p'}{\sqrt{p'}}$, and the remarks with regard to sign made above must be again observed.

Coefficients in the expansion from the mode:

$$a_0' = 1, \quad a_1' = \frac{1}{\sqrt{p'}}, \quad a_2' = \frac{\sqrt{2!}}{p'},$$

$$a_3' = \frac{\sqrt{3!}}{p'\sqrt{p'}} \left\{ \frac{p'+3}{3} \right\}, \quad a_4' = \frac{\sqrt{4!}}{p'^2} \left\{ \frac{7p'+12}{12} \right\}, \quad a_5' = \frac{\sqrt{5!}}{p'^2\sqrt{p'}} \left\{ \frac{47p'+60}{60} \right\},$$

$$a_6' = \frac{\sqrt{6!}}{p'^3} \left\{ \frac{p'^2}{18} + \frac{19}{20}p' + 1 \right\}, \quad a_7' = \frac{\sqrt{7!}}{p'^3\sqrt{p'}} \left\{ \frac{5}{36}p'^2 + \frac{153}{140}p' + 1 \right\},$$

$$a_8' = \frac{\sqrt{8!}}{p'^4} \left\{ \frac{341}{1440}p'^2 + \frac{341}{280}p' + 1 \right\}, \quad a_9' = \frac{\sqrt{9!}}{p'^4\sqrt{p'}} \left\{ \frac{p'^3}{162} + \frac{493}{1440}p'^2 + \frac{3349}{2520}p' + 1 \right\},$$

$$a_s' = \frac{1}{\sqrt{p'}} \left\{ \sqrt{s}\, a'_{s-1} + \sqrt{\frac{(s-1)(s-2)}{s}}\, a'_{s-3} \right\}.$$

We note that the coefficients of powers of θ in the functions

$$\phi(\theta) = e^{p\left\{\frac{\theta^3}{3} + \frac{\theta^4}{4} + \frac{\theta^5}{5} + \dots\right\}}$$

and
$$\phi'(\theta) = e^{\theta + \frac{1}{2}\theta^2 + \frac{p+1}{3}\theta^3 + \frac{p+1}{4}\theta^4 + \dots}$$

(in the expansion from the mode we had p' for p in $\phi'(\theta)$) are closely related.

Then if c_n is the coefficient of θ^n in $\phi(\theta)$ and c_n' is the coefficient of θ^n in $\phi'(\theta)$,

$$c_n = \frac{p}{n} c'_{n-2}.$$

(4) In the last expansion it might seem possible to get rid of the terms in τ_1 and τ_2 by breaking away from Laplace and expanding with regard to $e^{-\frac{1}{2}\xi^2/q}$ instead of $e^{-\frac{1}{2}\xi^2/p'}$; then choose q to give us the desired result. In Laplace's form of the modal expansion the exponential term is $e^{\frac{1}{2}\left(\frac{d^2u}{dx^2}\right)_{m_0}\xi^2}$, where $u = \log y$ and $\left(\frac{d^2u}{dx^2}\right)_{m_0}$ means the value of $\frac{d^2u}{dx^2}$ at the mode.

Now
$$y = \frac{x^{p'} e^{-x}}{\Gamma(p'+1)},$$
$$u = \log_e y = p' \log_e x - x - \log_e \Gamma(p'+1),$$
$$\frac{du}{dx} = \frac{p'}{x} - 1,$$
$$\frac{d^2u}{dx^2} = -\frac{p'}{x^2};$$

therefore
$$\left(\frac{d^2u}{dx^2}\right)_{\text{Mode}} = -\frac{p'}{p'^2} = -\frac{1}{p'}.$$

If $\frac{x^{p'} e^{-x}}{\Gamma(p'+1)} = \phi(-D)\frac{e^{-\xi^2/2q}}{\sqrt{2\pi q}}$ where $D = \frac{d}{dz}$ and $z = \frac{\xi}{\sqrt{q}}$, we have to find q, so that either the τ_1 or τ_2 term or both will vanish.

By proceeding as before equation (xvii) becomes

$$\begin{aligned}\phi(\theta\sqrt{q}) &= e^{-p\theta - \frac{1}{2}q\theta^2}(1-\theta)^{(p+1)} \\ &= e^{-p'\theta - \frac{1}{2}q\theta^2 - (p+1)\log(1-\theta)}.\end{aligned}$$

The term in τ_2 will vanish if $q = p' + 1$ which is the square of the standard-deviation from the mean, but τ_1 will still be left. However, it does not seem likely that any advantage will be gained by departing from Laplace's form of the exponential term.

Having found the two expansions from the mean and the mode respectively we shall now proceed to examine the behaviour of the series by numerical calculation, but before doing so we shall endeavour to find a similar series for the Incomplete B-function.

(5) To expand $\int_0^x \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}dx$ in terms of tetrachoric functions about the mean.

The mean is at
$$x = p/(p+q).$$

The standard deviation is
$$\sigma = \frac{\sqrt{pq}}{(p+q)\sqrt{p+q+1}}.$$

Take origin at the mean; then $x = p/(p+q) + \xi$. Let

$$\frac{x^{p-1}(1-x)^{q-1}}{B(p,q)} = \phi(-D)\frac{e^{-\frac{1}{2}\xi^2/\sigma^2}}{\sqrt{2\pi}\sigma} \qquad \text{(xx)},$$

where
$$D = \frac{d}{dy}, \quad y = \frac{\xi}{\sigma}.$$

As in the case of the Incomplete Γ-function multiply each side by $e^{\theta\xi}$ and integrate. The limits of the integral on the left-hand side will be $x=0$ and $x=1$, as we take the value of the integral outside these limits to be zero.

The ξ limits will therefore be $-p/(p+q)$ for $x=0$ and $q/(p+q)$ for $x=1$. Then

$$\int_{-p/p+q}^{q/p+q} \frac{e^{\theta\xi}(\xi+p/(p+q))^{p-1}}{B(p,q)}\{1-(\xi+p/(p+q))\}^{q-1}d\xi=\int_{-\infty}^{\infty} e^{\theta\xi}\phi(-D)\frac{e^{-\frac{1}{2}\xi^2/\sigma^2}}{\sqrt{2\pi}\sigma}d\xi,$$

i.e.
$$\int_0^1 \frac{e^{\theta(x-p/(p+q))}x^{p-1}(1-x)^{q-1}}{B(p,q)}dx=\int_{-\infty}^{\infty} e^{(\theta\sigma)y}\phi(-D)\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}dy,$$

and
$$e^{-\theta p/p+q}\int_0^1 \frac{e^{\theta x}x^{p-1}(1-x)^{q-1}}{B(p,q)}dx=\phi(\theta\sigma)\int_{-\infty}^{\infty}\frac{e^{(\theta\sigma)y-\frac{1}{2}y^2}}{\sqrt{2\pi}}dy$$
$$=\phi(\theta\sigma)\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}(y-\theta\sigma)^2+\frac{1}{2}\theta^2\sigma^2}}{\sqrt{2\pi}}dy$$
$$=\phi(\theta\sigma)e^{\frac{1}{2}\theta^2\sigma^2} \quad\text{......}\ \text{..................(xxi).}$$

Now

$$\int_0^1 \frac{x^{p-1}(1-x)^{q-1}e^{\theta x}}{B(p,q)}dx$$
$$=\int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}\left\{1+\theta x+\frac{\theta^2x^2}{2!}+\frac{\theta^3x^3}{3!}+\ldots+\frac{\theta^sx^s}{s!}+\ldots\right\}dx$$
$$=\frac{B(p,q)}{B(p,q)}+\theta\frac{B(p+1,q)}{B(p,q)}+\frac{\theta^2}{2!}\frac{B(p+2,q)}{B(p,q)}+\ldots+\frac{\theta^s}{s!}\frac{B(p+s,q)}{B(p,q)}+$$

But
$$\frac{B(p+s,q)}{B(p,q)}=\frac{p(p+1)\ldots(p+s-1)}{(p+q)(p+q+1)\ldots(p+q+s-1)},$$

therefore
$$\int_0^1 \frac{x^{p-1}(1-x)^{q-1}e^{\theta x}}{B(p,q)}dx=1+\theta\frac{p}{p+q}+\frac{\theta^2}{2!}\frac{p(p+1)}{(p+q)(p+q+1)}$$
$$+\ldots+\frac{\theta^s}{s!}\frac{p(p+1)\ldots(p+s-1)}{(p+q)(p+q+1)\ldots(p+q+s-1)}+\ldots.$$

From equation (xxi)

$$\phi(\theta\sigma)=e^{-\frac{p\theta}{p+q}-\frac{1}{2}\theta^2\sigma^2}\left\{1+\theta\frac{p}{p+q}+\frac{\theta^2}{2!}\frac{p(p+1)}{(p+q)(p+q+1)}\right.$$
$$\left.+\ldots+\frac{\theta^s}{s!}\frac{p(p+1)\ldots(p+s-1)}{(p+q)(p+q+1)\ldots(p+q+s-1)}+\ldots\right\}\ \text{......(xxii).}$$

Let
$$\phi(-D)=a_0-a_1D+a_2D^2-\ldots(-1)^s a_sD^s+\ldots,$$
$$\phi(\theta\sigma)=a_0+a_1(\theta\sigma)+a_2(\theta\sigma)^2+a_3(\theta\sigma)^3+\ldots+a_s(\theta\sigma)^s+\ldots$$
$$=c_0+c_1\theta+c_2\theta^2+\ldots+c_s\theta^s+\ldots,$$

where
$$c_s=a_s\sigma^s.$$

By equating coefficients of powers of θ in equation (xxii) the coefficients in $\phi(\theta\sigma)$ can be obtained in terms of p and q, for

$$\sigma^2=\frac{pq}{(p+q)^2(p+q+1)}.$$

Obviously

$$c_0 = 1,$$

$$c_1 = -p/(p+q) + p/(p+q) = 0,$$

$$c_2 = \frac{1}{2!}\frac{p^2}{(p+q)^2} - \frac{1}{2}\frac{pq}{(p+q)^2(p+q+1)} + \frac{1}{2!}\frac{p(p+1)}{(p+q)(p+q+1)} - \frac{p^2}{(p+q)^2}$$

$$= \frac{1}{2}\left\{\frac{p}{p+q}\right\}\left\{\frac{p+1}{(p+q+1)} - \frac{p}{p+q} - \frac{q}{(p+q)(p+q+1)}\right\}$$

$$= \frac{1}{2}\left\{\frac{p}{p+q}\right\}\left\{\frac{q}{(p+q)(p+q+1)} - \frac{q}{(p+q)(p+q+1)}\right\}$$

$$= 0.$$

Similarly the other c's can be determined but the work becomes more and more laborious as we go on.

Unfortunately, as far as the numerical work is concerned, we have failed after many attempts to find a relation connecting successive c's, similar to that found in the case of the Incomplete Γ-function. At first it was thought that the following treatment would facilitate the numerical calculation of these coefficients.

Let $$e^{-\frac{p\theta}{p+q} - \frac{1}{2}\sigma^2\theta^2} = b_0 + b_1\theta + b_2\theta^2 + \ldots + b_s\theta^s + \ldots,$$

then $$-p/(p+q)\,\theta - \tfrac{1}{2}\sigma^2\theta^2 = \log_e\{b_0 + b_1\theta + b_2\theta^2 + \ldots + b_s\theta^s + \ldots\}.$$

Differentiate this and then equate coefficients of powers of θ:

$$(b_0 + b_1\theta + b_2\theta^2 + \ldots + b_s\theta^s + \ldots)(-p/(p+q) - \sigma^2\theta)$$
$$= b_1 + 2b_2\theta + 3b_3\theta^2 + \ldots + sb_s\theta^{s-1} + \ldots.$$

Equate coefficients of θ^{s-1}:

$$sb_s = -p/(p+q)\,b_{s-1} - \sigma^2 b_{s-2};$$

therefore $$b_s = -\frac{1}{s}\left[\frac{p}{p+q}b_{s-1} + \sigma^2 b_{s-2}\right] \quad \ldots \quad \ldots\ldots\ldots\ldots\ldots\text{(xxiii)}.$$

This formula enables us to calculate the b's very rapidly on the machine when $p/(p+q)$ and σ^2 have been determined.

From equation (xxii)

$$c_0 + c_1\theta + c_2\theta^2 + \ldots + c_s\theta^s + \ldots = (b_0 + b_1\theta + b_2\theta^2 + \ldots)$$
$$\left\{1 + \theta\frac{p}{p+q} + \frac{\theta^2}{2!}\frac{p(p+1)}{(p+q)(p+q+1)} + \ldots + \ldots\right\}$$

Equate coefficients of θ^s:

$$c_s = b_0\frac{1}{s!}\frac{p(p+1)\ldots(p+s-1)}{(p+q)(p+q+1)\ldots(p+q+s-1)}$$
$$+ b_1\frac{1}{(s-1)!}\frac{p(p+1)\ldots(p+s-2)}{(p+q)(p+q+1)(p+q+s-2)} + \ldots + b_{s-1}\frac{1}{1!}\frac{p}{p+q} + b_s,$$

i.e. $$c_s = \sum_{r=0}^{s} b_r\frac{1}{(s-r)!}\frac{p(p+1)\ldots(p+s-r-1)}{(p+q)(p+q+1)\ldots(p+q+s-r-1)}$$

The b's, having been calculated previously by (xxiii), this last formula gives a fairly rapid way of calculating the c's, at least the earlier c's. Then

$$a_s = \frac{1}{\sigma^s} \sum_{r=0}^{s} b_r \frac{1}{(s-r)!} \frac{p(p+1)\dots(p+s-r-1)}{(p+q)(p+q+1)\dots(p+q+s-r-1)} \quad (a_0 = 1) \dots\text{(xxiv)}.$$

What we require generally is the area represented by $\int_0^x x^{p-1}(1-x)^{q-1}\,dx$:

$$\int_0^x \frac{x^{p-1}(1-x)^{q-1}}{\mathrm{B}(p,q)}\,dx = \int_{-\infty}^{\xi} \phi(-D)\frac{e^{-\frac{1}{2}\xi^2/\sigma^2}}{\sqrt{2\pi}\,\sigma}\,d\xi$$

$$= \int_{-\infty}^{y} \phi(-D)\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}\,dy,$$

i.e. $$\int_0^x \frac{x^{p-1}(1-x)^{q-1}}{\mathrm{B}(p,q)}\,dx$$

$$= \int_{-\infty}^{y}\left\{a_0\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} - a_1 D\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} + a_2 D^2\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} - \dots (-1)^s D^s\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} - \dots\right\}dy$$

$$= \int_{-\infty}^{y} a_0\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}\,dy - a_1\left[\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}\right]_{-\infty}^{y} + a_2\left[D\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}\right]_{-\infty}^{y} - \dots + a_s\left[(-1)^s D^{s-1}\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}\right]_{-\infty}^{y} - \dots$$

$$= \tfrac{1}{2}(1+\alpha) - a_1\sqrt{1!}\,\tau_1 - a_2\sqrt{2!}\,\tau_2 - a_3\sqrt{3!}\,\tau_3 \dots - a_s\sqrt{s!}\,\tau_s - \dots \quad (a_0 = 1)$$

$$= \tfrac{1}{2}(1+\alpha) - a_1'\tau_1 - a_2'\tau_2 - a_3'\tau_3 - \dots - a_s'\tau_s - \dots,$$

where $$a_s' = a_s\sqrt{s!}.$$

Then $$a_s' = \frac{\sqrt{s!}}{\sigma^s}\sum_{r=0}^{s} b_r \frac{1}{(s-r)!}\frac{p(p+1)\dots(p+s-r-1)}{(p+q)(p+q+1)\dots(p+q+s-r-1)} \dots\text{(xxv)}.$$

Now c_1 and c_2 are equal to zero, so that a_1', a_2' are zero. Thus there are no terms in τ_1 and τ_2. The argument of the tetrachoric functions and of $\frac{1}{2}(1+\alpha)$ is y, which is equal to $\frac{\xi}{\sigma} = \frac{x - p/(p+q)}{\sigma}$. On applying the above formula for a_s', we were greatly disappointed to find, that with the b's to 8 decimal places the expression under the summation sign in the examples used commenced with 4 or 5 zeros after the decimal point. As $\sqrt{s!}$ and $\left(\frac{1}{\sigma}\right)^s$ both increase with s $\left(\frac{1}{\sigma}\text{ being in our case} > 1\right)$ accuracy to the seventh place in our a''s could not be obtained. Accordingly the formula actually used was of a different type.

Let $$\frac{x^{p-1}(1-x)^{q-1}}{\mathrm{B}(p,q)} = \overset{\infty}{\underset{1}{S}}\, c_s\tau_s,$$

where the argument of the tetrachoric function is again $\frac{\xi}{\sigma} = \frac{x - p/(p+q)}{\sigma}$.

Multiply both sides by τ_s, weighting by the factor $e^{\frac{1}{2}\xi^2/\sigma^2}$, and integrate from $-\infty$ to $+\infty$, the left-hand side being taken as zero outside $x = 0$ and $x = 1$.

Then $$\int_{-p/p+q}^{q/p+q} \frac{x^{p-1}(1-x)^{q-1}}{\mathrm{B}(p,q)}\,\tau_s e^{\frac{1}{2}\xi^2/\sigma^2}\,d\xi = \int_{-\infty}^{\infty}\tau_s \overset{\infty}{\underset{1}{S}}\, c_s\tau_s e^{\frac{1}{2}\xi^2/\sigma^2}\,d\xi.$$

Since $\int_{-\infty}^{\infty} \tau_s \tau_{s'} e^{\frac{1}{2}\xi^2/\sigma^2} d\xi = 0$ only the term in τ_s^2 will be left on the right-hand side, i.e.

$$\int_{-\infty}^{\infty} \tau_s \overset{\infty}{\underset{1}{S}} c_s \tau_s e^{\frac{1}{2}\xi^2/\sigma^2} d\xi = c_s \int_{-\infty}^{\infty} \tau_s^2 e^{\frac{1}{2}\xi^2/\sigma^2} d\xi.$$

Putting $$\xi/\sigma = y,\ d\xi = \sigma dy,$$

we have
$$\int_{-p/p+q}^{q/p+q} \frac{x^{p-1}(1-x)^{q-1}}{B(p, q)} \tau_s e^{\frac{1}{2}\xi^2/\sigma^2} d\xi = c_s \int_{-\infty}^{\infty} \tau_s^2 e^{\frac{1}{2}y^2} \sigma dy$$
$$= \frac{c_s \sigma}{s\sqrt{2\pi}}$$

i.e. $$c_s = \frac{s\sqrt{2\pi}}{\sigma} \int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{B(p, q)} \tau_s e^{\frac{1}{2}\left(\frac{x-p/(p+q)}{\sigma}\right)^2} dx$$

$$= \frac{s\sqrt{2\pi}}{\sigma} \frac{1}{\sqrt{s!}\sqrt{2\pi}} \int_0^1 \left\{\left(\frac{x-p/(p+q)}{\sigma}\right)^{s-1} - \frac{(s-1)(s-2)}{2.1!}\left(\frac{x-p/(p+q)}{\sigma}\right)^{s-3}\right.$$
$$\left. + \frac{(s-1)(s-2)(s-3)(s-4)}{2^2.2!}\left(\frac{x-p/(p+q)}{\sigma}\right)^{s-5} - \ldots\right\} \frac{x^{p-1}(1-x)^{q-1}}{B(p, q)} dx$$
$$= \frac{s}{\sigma\sqrt{s!}} \int_0^1 \left\{\left(\frac{x-p/(p+q)}{\sigma}\right)^{s-1} - \frac{(s-1)(s-2)}{2.1!}\left(\frac{x-p/(p+q)}{\sigma}\right)^{s-3}\right.$$
$$\left. + \frac{(s-1)(s-2)(s-3)(s-4)}{2^2.2!}\left(\frac{x-p/(p+q)}{\sigma}\right)^{s-5} - \ldots\right\} \frac{x^{p-1}(1-x)^{q-1}}{B(p, q)} dx \qquad \text{(xxvi).}$$

The integral for any particular value of s reduces to a series of B-functions and so c_s is found.

The area up to abscissa x is generally required:

i.e. $$\int_0^x \frac{x^{p-1}(1-x)^{q-1}}{B(p, q)} dx.$$

But $$\int_0^x \frac{x^{p-1}(1-x)^{q-1}}{B(p, q)} dx = \int_{-\infty}^{\xi} \overset{\infty}{\underset{1}{S}} (c_s \tau_s) d\xi$$
$$= \sigma \int_{-\infty}^{y} \overset{\infty}{\underset{1}{S}} (c_s \tau_s) dy.$$

Now $$\int_{-\infty}^{y} \tau_s dy = -\frac{1}{\sqrt{s}} \tau_{s-1}.$$

$$\therefore \int_0^x \frac{x^{p-1}(1-x)^{q-1}}{B(p, q)} dx = \sigma \int_{-\infty}^{y} c_1 \tau_1 dy - \sigma \left\{c_2 \frac{1}{\sqrt{2}} \tau_1\right.$$
$$\left. + c_3 \frac{1}{\sqrt{3}} \tau_2 + c_4 \frac{1}{\sqrt{4}} \tau_3 + \ldots + c_s \frac{1}{\sqrt{s}} \tau_{s-1} + \ldots\right\}$$
$$= \sigma c_1 \int_{-\infty}^{y} \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} dy - c_1\tau_1 - a_2\tau_2 - \ldots - a_s\tau_s - \ldots,$$

where $a_s = c_{s+1} \frac{\sigma}{\sqrt{s+1}}$, $\quad = \sigma c_1 \frac{1}{2}(1+\alpha) - a_1\tau_1 - a_2\tau_2 - a_3\tau_3 - \ldots.$

If we put $s=1$, $s=2$, $s=3$ in the above formula (xxvi) for c_s,

$$c_1 = \frac{1}{\sigma},$$

$$c_2 = \frac{2}{\sigma\sqrt{2\,!}}\int_0^1 \left(\frac{x - p/(p+q)}{\sigma}\right)\frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}\,dx$$

$$= \frac{2}{\sigma^2\sqrt{2\,!}}\frac{1}{B(p,q)}\{B(p+1,q) - p/(p+q)\,B(p,q)\}$$

$$= 0,$$

$$c_3 = \frac{3}{\sigma\sqrt{6}}\int_0^1\left[\frac{1}{\sigma^2}\{x^2 - 2xp/(p+q) + p^2/(p+q)^2\} - \frac{2\,.\,1}{2}\right]\frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}\,dx$$

$$= \frac{3}{\sigma^3\sqrt{6}}\left[\frac{1}{B(p,q)}\right]\left[B(p+2,q) - 2\frac{p}{p+q}B(p+1,q) + \frac{p^2}{(p+q)^2}B(p,q) - \sigma^2 B(p,q)\right]$$

$$= \frac{3}{\sigma^3\sqrt{6}}\left\{\frac{p(p+1)}{(p+q)(p+q+1)} - 2\frac{p^2}{(p+q)^2} + \frac{p^2}{(p+q)^2} - \frac{pq}{(p+q)^2(p+q+1)}\right\}$$

$$= 0,$$

as obtained before by the other method.

The terms in τ_1 and τ_2 do not exist, so that the expansion becomes:

$$\int_0^x \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}\,dx = \tfrac{1}{2}(1+\alpha) - a_3\tau_3 - a_4\tau_4 - \ldots - a_s\tau_s - \ldots,$$

where $$a_s = \frac{\sigma}{\sqrt{s+1}}\,.\,c_{s+1}$$

$$= \frac{\sigma}{\sqrt{s+1}}\,.\,\frac{(s+1)}{\sqrt{(s+1)\,!}}\,\sigma\int_0^1\left\{\left(\frac{x-p/(p+q)}{\sigma}\right)^s - \frac{s(s-1)}{2\,.\,1\,!}\left(\frac{x-p/(p+q)}{\sigma}\right)^{s-2}\right.$$
$$\left. + \frac{s(s-1)(s-2)(s-3)}{2^2\,.\,2\,!}\left(\frac{x-p/(p+q)}{\sigma}\right)^{s-4} - \ldots\right\}\frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}\,dx$$

$$= \frac{1}{\sqrt{s\,!}}\int_0^1\left\{\left(\frac{x-p/(p+q)}{\sigma}\right)^s - \frac{s(s-1)}{2\,.\,1\,!}\left(\frac{x-p/(p+q)}{\sigma}\right)^{s-2}\right.$$
$$\left. + \frac{s(s-1)(s-2)(s-3)}{2^2\,.\,2\,!}\left(\frac{x-p/(p+q)}{\sigma}\right)^{s-4} - \ldots\right\}\frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}\,dx \qquad \text{(xxvii)}.$$

The argument for the tetrachoric functions and for $\frac{1}{2}(1+\alpha)$ is $\dfrac{x - p/(p+q)}{\sigma}$. If this is negative then we must take $\frac{1}{2}(1-\alpha)$.

From the above expression (xxvii) the coefficients of the expansion can be determined both algebraically and numerically, but for the higher coefficients the algebraic work becomes exceedingly heavy. It is to be remembered that

$$\sigma^2 = \frac{pq}{(p+q)^2(p+q+1)}.$$

Suppose $(p+q)=m$; then $\sigma^2 = \dfrac{pq}{m^2(m+1)}$.

The coefficients $a_1, a_2, \ldots$ etc. are given below:

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = \frac{1}{\sqrt{3!}}\, 2!\, \sqrt{\frac{m+1}{pq}}\, \frac{(m-2p)}{(m+2)},$$

$$a_4 = \frac{1}{\sqrt{4!}}\, 3!\, \frac{1}{(m+2)(m+3)} \left\{ \frac{m^2(m+1)}{pq} - (5m+6) \right\},$$

$$a_5 = \frac{1}{\sqrt{5!}}\, 4!\, \sqrt{\frac{m+1}{pq}}\, \frac{(m-2p)}{(m+2)(m+3)(m+4)} \left\{ \frac{m^2(m+1)}{pq} - (7m+12) \right\},$$

$$a_6 = \frac{1}{\sqrt{6!}}\, 5!\, \frac{1}{(m+2)(m+3)(m+4)(m+5)} \left\{ \frac{m^4(m+1)^2}{p^2q^2} \right.$$

$$\left. + \frac{m^2(m+1)}{3pq}(m^2 - 32m - 60) - \tfrac{2}{3}(2m^3 - 41m^2 - 154m - 120) \right\},$$

$$a_7 = \frac{1}{\sqrt{7!}}\, 6!\, \sqrt{\frac{m+1}{pq}}\, \frac{(m-2p)}{(m+2)(m+3)(m+4)(m+5)(m+6)}$$

$$\times \left\{ \frac{m^4(m+1)^2}{p^2q^2} + \frac{m^2(m+1)}{12pq}(7m+15)(m-20) \right.$$

$$\left. - \tfrac{5}{12}\{7m^3 - 59m^2 - 342m - 360\} \right\},$$

$$a_8 = \frac{1}{\sqrt{8!}}\, 7!\, \frac{1}{(m+2)(m+3)(m+4)(m+5)(m+6)(m+7)}$$

$$\times \left\{ \frac{m^6(m+1)^3}{p^3q^3} + \frac{m^4(m+1)^2}{60p^2q^2}\{47m^2 - 853m - 2100\} \right.$$

$$+ \frac{m^2(m+1)}{30pq}\{-251m^3 + 1503m^2 + 9974m + 10920\}$$

$$\left. + \tfrac{1}{60}\{1271m^4 - 1697m^3 - 44512m^2 - 104364m - 65520\} \right\}.$$

An additional coefficient a_9 was calculated for one of our examples, but it was not considered worth while working it out algebraically.

The coefficients in the tetrachoric expansion obtained by this latter method, that is, by using the property of tetrachoric functions as semi-orthogonal functions, are identical with those obtained from the first method, which consisted in equating moments of the functions on both sides of the equation. Thus we are led to the same expansion in both cases.

(6) The numerical results are certainly interesting but from the utility point of view they are not very satisfactory. Tables I—VIII contain these results in a convenient form; the values of the coefficients a_s and a_s', the tetrachoric functions, the successive terms ($-a_s\tau_s$ and $-a_s'\tau_s$) and the values of the series up to the term containing τ_s are given. It is to be noted that the coefficients do not appear in Tables II and IV but as these are the same as in Tables I and III respectively it

was not necessary to repeat them. In all these tables, in the row $s = 0$ we have placed $\frac{1}{2}(1 - \alpha)$ in the column containing the tetrachoric functions and it is only necessary to draw attention to the fact that in the next column the negative sign in $-a_s\tau_s$ does not apply to the first term $\frac{1}{2}(1 - \alpha)$. The tables will then be easily

TABLE I.

$$\int_0^{29\cdot 4} \frac{x^{48} e^{-x}}{\Gamma(49)} dx, \quad z = -2\cdot 8\,^*.$$

		Tetrachoric Functions τ_s	Terms in Series $-a_s\tau_s$	Value of Series up to term τ_s
0	1·0000000	+ ·0025551	+ ·0025551	·0025551
1	0·0000000	+ ·00791545		
2	0·0000000	− ·01567180	−	
3	0·11664237	+ ·02210325	− ·0025782	·0000231
4	0·02499479	− ·02189644	+ ·0005473	·0005242
5	0·00638743	+ ·01259137	− ·0000804	·0004438
6	0·03228531	+ ·00159776	− ·0000516	·0003922
7	0·01785148	− ·01140536	+ ·0002036	·0005958
8	0·00840223	+ ·01000967	− ·0000841	·0005117
9	0·01470566	+ ·00006659	− ·0000010	·0005107
10	0·01282194	− ·00849985	+ ·0001090	·0006197
11	0·00895618	+ ·00711870	− ·0000638	·0005560
12	0·01042333	+ ·00164419	− ·0000171	·0005388
13	0·01079260	− ·00754632	+ ·0000814	·0006203
14	0·00962776	+ ·00418464	− ·0000403	·0005800
15	0·01015854	+ ·00374438	− ·0000380	·0005419
16	0·01102777	− ·00640271	+ ·0000706	·0006125
17	0·01128126	+ ·00094254	− ·0000106	·0006019
18	0·01209893	+ ·00523425	− ·0000633	·0005386
19	0·01345971	− ·00422873	+ ·0000569	·0005955
20	0·01483350	− ·00218561	+ ·0000321	·0006279
21	0·01660082	+ ·00525599	− ·0000873	·0005407
22	0·01901932	− ·00110396	+ ·0000210	·0005617
23	0·02196131	− ·00426227	+ ·0000936	·0006553
24	0·02561864	+ ·00346981	− ·0000889	·0005664
25	0·03033429	+ ·00205905	− ·0000625	·0005039
26	0·03631783	− ·00439701	+ ·0001597	·0006636
27	0·04391748	+ ·00042653	− ·0000187	·0006449
28	0·05371111	+ ·00393217	·0002112	·0004337
29	0·06638572	− ·00244866	+ ·0001626	·0005963
30	0·08285325	− ·00248099	+ ·0002056	·0008018

True value ·0005850.

understood, but, in order that a better appreciation of the results may be obtained, the value of the series up to a certain term has been plotted against the number of that term. A line, drawn across the paper and corresponding to the true value of the integral, shows how much the value of the series is in excess or defect of the true value of the integral. The various points have been joined by continuous wavy lines but, of course, these lines have no real physical meaning. However, by joining the points, the graph will, we think, convey a better idea of the variation

* $z = \frac{t - p}{\sqrt{p}} = \frac{29\cdot 4 - 49}{7} = -2\cdot 8.$

of the values of the series than a set of isolated points would. Figures 1—7 correspond to the data given in Tables I—VII.

Now in the case of the Incomplete Γ-function we obtained two expansions, with respect to the mean and the mode respectively, and the graphs tell us which of these two gives us the better approximation. Figs. 1 and 3 (Tables I and III) show the variations in the values of the series for $\int_0^{29\cdot 4} \frac{x^{48} e^{-x}}{\Gamma(49)} dx$ from the mean and the mode respectively, while Figs. 2 and 4 give us similar information for $\int_0^{42} \frac{x^{48} e^{-x}}{\Gamma(49)} dx$.

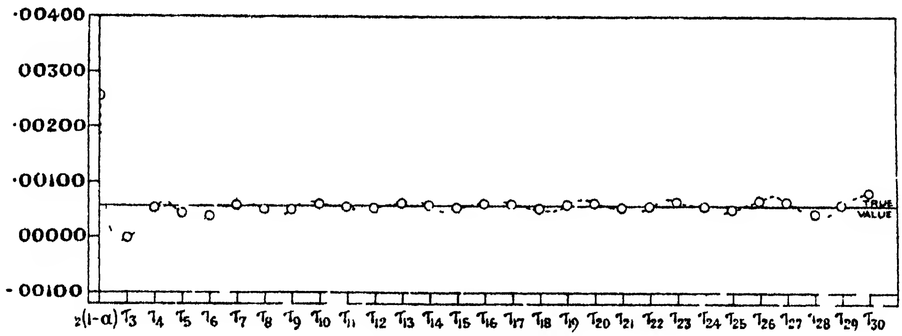


Fig. 1.

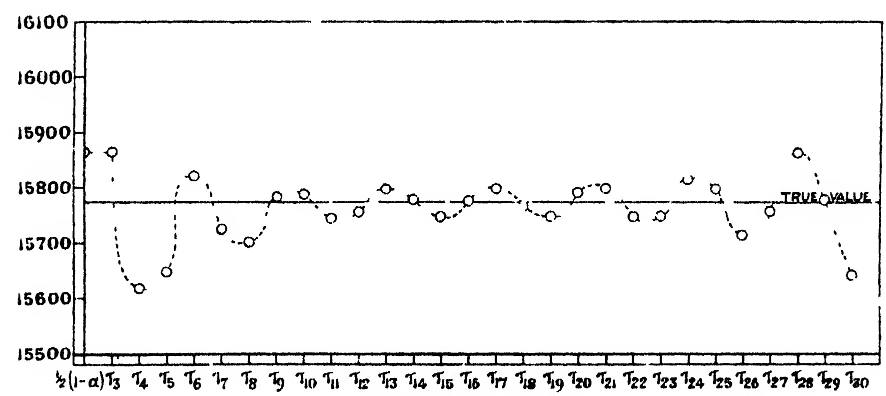


Fig. 2.

It will be seen that in Fig. 1 the points are much closer to the 'true value' line than in Fig. 3 (and similarly in Fig. 2 they are closer than in Fig. 4) so that the expansion from the mean seems to give a better approximation than that from the mode and it has the additional advantage that the terms in τ_1 and τ_2 are missing. Besides, it seems more natural to expand these normal curve functions in terms of the mean and standard deviation. For comparison purposes the graphs are all on the same scale. The graphs for the mode and the mean behave in a very

TABLE II. $\int_0^{42} \frac{x^{48} e^{-x}}{\Gamma(49)} dx, \quad z = -1$*.

s	Tetrachoric Functions τ_s	Terms in the Series $-a_s\tau_s$	Value of Series up to term τ_s
0	+ ·1586553	+ ·1586553	·1586553
1	+ ·24197074	·0000000	—
2	− ·17109916	·0000000	—
3	·00000000	·0000000	—
4	+ ·09878417	− ·0024691	·1561862
5	− ·04417762	+ ·0002822	·1564684
6	·05410632	+ ·0017468	·1582152
7	+ ·05453404	− ·0009735	·1572417
8	+ ·02410087	− ·0002025	·1570392
9	− ·05302190	+ ·0007797	·1578189
10	− ·00355664	+ ·0000456	·1578645
11	+ ·04657133	− ·0004172	·1574474
12	− ·01034833	+ ·0001079	·1575553
13	− ·03814548	+ ·0004117	·1579669
14	+ ·01939964	− ·1001868	·1577802
15	+ ·02921077	− ·0002967	·1574834
16	− ·02483411	+ ·0002739	·1577573
17	− ·02054429	+ ·0002318	·1579891
18	+ ·02755708	− ·0003334	·1576556
19	+ ·01256341	− ·0001691	·1574865
20	− ·02825493	+ ·0004191	·1579057
21	− ·00548187	+ ·0000910	·1579967
22	+ ·02745951	− ·0005223	·1574744
23	− ·00060803	+ ·0000134	·1574878
24	− ·02558848	+ ·0006555	·1581433
25	+ ·00568862	− ·0001726	·1579707
26	+ ·02297227	− ·0008343	·1571364
27	− ·00978859	+ ·0004299	·1575663
28	− ·01987296	+ ·0010674	·1586337
29	+ ·01296514	·0008607	·1577730
30	+ ·01649808	− ·0013669	·1564061

True value ·1577387.

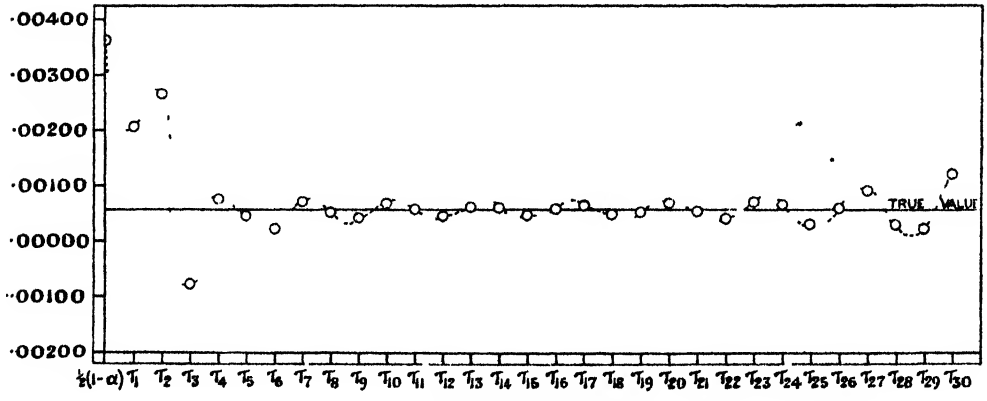


Fig. 3.

* $z = \frac{x-p}{\sqrt{p}} = \frac{42-49}{7} = -1.$

similar manner; for, if we regard the graphs as a wave, it will be noticed that at first the amplitude of the wave is big, decreases gradually up to a term in the neighbourhood of τ_{20} and thereafter increases more and more rapidly. This can be explained fairly easily; as s increases the tetrachoric functions τ_s do not increase or decrease steadily but vary in sign and remain of the same order of magnitude. The coefficients a_s vary in much the same way (except that they are all positive) up to a certain point and then begin to increase very fast. In equation (xv) we had

$$a_{s+1} = \sqrt{\frac{s}{p(s+1)}}\{\sqrt{s}\,a_s + \sqrt{(s-1)}\,a_{s-2}\},$$

i.e. a_{s+1} is of order $\sqrt{\frac{s}{p}}\{a_s + a_{s-2}\}$, so that as s increases there comes a time when $\sqrt{s}$ overcomes the reducing effect of $\frac{1}{\sqrt{p}}$ and then the coefficients will continually

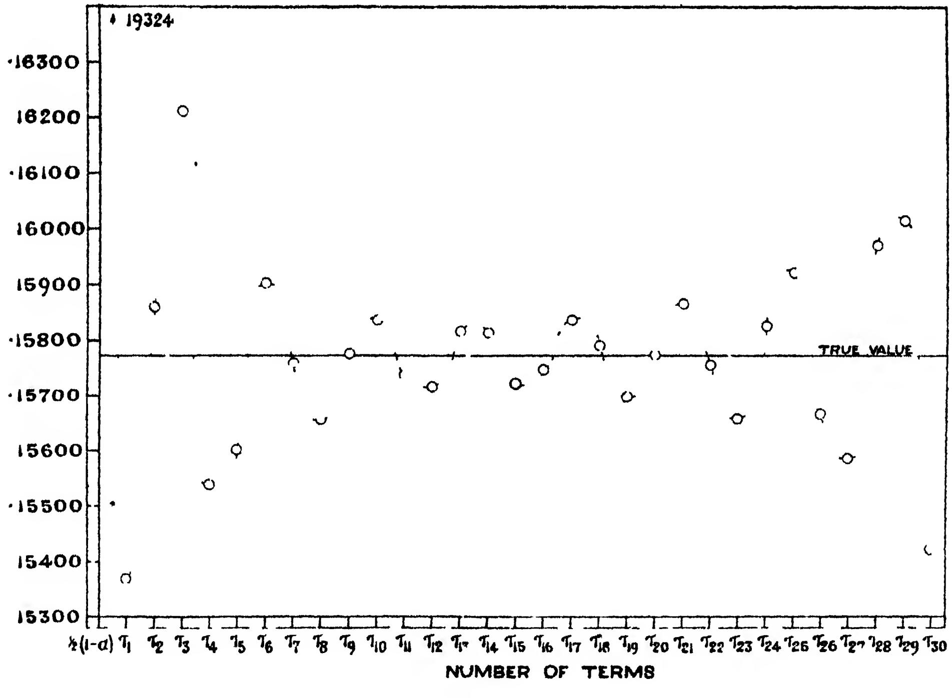


FIG. 4.

increase. For higher values of p this turning point will not be arrived at so soon and the points will hang closer to the 'true value' line for a greater number of terms, but it does not seem likely that the values of the series will tend to a definite limit. The equation for the modal expansion coefficients is a similar one and these coefficients behave in the same way.

Turning our attention to the expansions from the mean, Fig. 1 (and Fig. 3 to a less extent) would seem to suggest that the tetrachoric series gives quite a good approximation to the value of the integral. Although some of the points are very

TABLE III.

$$\int_0^{29\cdot4} \frac{x^{48}e^{-x}}{\Gamma(49)}\,dx, \quad z = -2\cdot6846788^*.$$

	a_s'	Tetrachoric Functions τ_s	Terms in Series $-a_s'\tau_s$	Value of Series up to term τ_s
0	1·0000000	+ ·0036296	+ ·0036296	·0036296
1	0·11433757	+ ·01085979	− ·0015675	·0020621
2	0·02946278	− ·02061573	+ ·0006074	·0026695
3	0·12521683	+ ·02752089	− ·0034461	− ·0007766
4	0·06166251	− ·02503988	+ ·0015140	·0007675
5	0·02648957	+ ·01160193	− ·0003073	·0004601
6	0·04236301	+ ·00557065	− ·0002360	·0002241
7	0·03460284	− ·01460370	+ ·0005053	·0007295
8	0·02288715	+ ·00939504	− ·0002150	·0005144
9	0·02516285	+ ·00363988	− ·0000916	·0004229
10	0·02488684	− ·01101274	+ ·0002741	·0006969
11	0·02136289	+ ·00579095	− ·0001237	·0005732
12	0·02167770	+ ·00509737	− ·0001105	·0004627
13	0·02272772	− ·00713119	+ ·0001621	·0006249
14	0·02256727	+ ·00058475	− ·0000132	·0006117
15	0·02351439	+ ·00599464	− ·0001410	·0004707
16	0·02546065	− ·00455186	+ ·0001158	·0005865
17	0·02739091	− ·00248832	+ ·0000682	·0006547
18	0·02996700	+ ·00573797	− ·0001720	·0004827
19	0·03360181	− ·00121666	+ ·0000419	·0005246
20	0·03803862	− ·00454994	+ ·0001731	·0006977
21	0·04355963	+ ·00382135	− ·0001665	·0005312
22	0·05068120	+ ·00204641	− ·0001037	·0004275
23	0·05968962	− ·00471305	+ ·0002813	·0007089
24	0·07107603	+ ·00066657	− ·0000474	·0006615
25	0·08566837	+ ·00406751	− ·0003485	·0003130
26	0·10443752	·00276906	+ ·0002892	·0006022
27	0·12866399	− ·00240728	+ ·0003097	·0009119
28	0·16018283	+ ·00383980	− ·0006151	·0002969
29	0·20147298	+ ·00036667	− ·0000739	·0002230
30	0·25589543	− ·00382481	+ ·0009788	·0012017

True value ·0005850.

near to the 'true value' line, the approximation is not really a good one. The important question for us is: To how many decimal places does the series give the result correct? On going through the tables it will be found that there is no value of the series up to the *s*th term giving the result correct to more than three or four places. We now come to the real trouble. Suppose a frequency function is expanded in tetrachoric series, how are we to know at what term to stop so as to obtain the most accurate result? If the value of an integral is required, the true value is wanted. In our work we chose integrals of which the value was already known. From Figs. 1—4 it is easily seen that we have as good an approximation at the

* $z = \frac{x - p'}{\sqrt{p'}} = \frac{29\cdot4 - 48}{\sqrt{48}} = -2\cdot6846788.$

TABLE IV. $\int_0^{42} \frac{x^{48} e^{-x}}{\Gamma(49)} dx$, $z = -·8660254$*.

s	Tetrachoric Functions τ_s	Terms in Series $-a_s'\tau_s$	Value of Series up to term τ_s
0	·1932381	·1932381	·1932381
1	+ ·27418875	- ·0395757	·1536624
2	− ·16790564	+ ·0049470	·1586093
3	− ·02798427	+ ·0035041	·1621134
4	+ ·10905792	− ·0067248	·1553886
5	·02346554	+ ·0006216	·1560102
6	− ·07134833	+ ·0030225	·1590328
7	+ ·04145828	− ·0014346	·1575982
8	+ ·04451198	− ·0010188	·1565794
9	- ·01705083	+ ·0011839	·1577634
10	− ·02465039	+ ·0006135	·1583768
11	+ ·04681171	− ·0010000	·1573768
12	+ ·00975248	− ·0002114	·1571654
13	− ·04356976	+ ·0009902	·1581556
14	+ ·00140962	− ·0000318	·1581238
15	+ ·03877058	− ·0009117	·1572122
16	·00966795	+ ·0002462	·1574583
17	- ·03323150	+ ·0009102	·1583686
18	+ ·01562623	·0004683	·1579003
19	+ ·02744360	− ·0009222	·1569781
20	− ·01974338	+ ·0007510	·1577291
21	- ·02171195	+ ·0009458	·1586749
22	+ ·02237974	− ·0011342	·1575407
23	+ ·01622818	− ·0009687	·1565720
24	− ·02382175	+ ·0016931	·1582654
25	− ·01111122	+ ·0009519	·1592173
26	+ ·02431475	− ·0025394	·1566779
27	+ ·00643169	− ·0008275	·1558504
28	·02404492	+ ·0038516	·1597020
29	− ·00222729	+ ·0004487	·1601507
30	+ ·02317774	- ·0059311	·1542196

True value ·1577387.

TABLE V. $\int_0^{·5} \frac{x^{14}(1-x)^4}{B(15, 5)} dx$, $y = -2·6457513$, $p = 15$, $q = 5$, $m = 20$†.

s	a_s	Tetrachoric Functions τ_s	Terms in Series $-a_s\tau_s$	Value of Series up to term τ_s
0	1·00000000	·0040751	·0040751	·0040751
3	− ·19638608	+ ·02950904	+ ·0057952	·0098703
4	+ ·01452267	− ·02602453	+ ·0003780	·0102482
5	+ ·03818545	+ ·01099737	·0004199	·0098283
6	+ ·05515045	+ ·00712711	− ·0003931	·0094352
7	- ·01389639	− ·01561177	− ·0002170	·0092183
8	− ·03609105	+ ·02031787	+ ·0007333	·0099516

True value ·0096054.

* $z = \frac{x - p'}{\sqrt{p'}} = \frac{42 - 48}{\sqrt{48}} = -·8660254.$ † $y = \frac{x - \frac{p}{p+q}}{\sigma} = \frac{·5 - ·75}{·09449112} = -2·6457513.$

TABLE VI.

$$\int_0^{\cdot 5} \frac{x^3(1-x)^{\frac{1}{2}}}{B(4, \frac{3}{2})}\,dx, \quad y = -1\cdot 3010412, \quad p = 4, \quad q = \tfrac{3}{2}, \quad m = 5\tfrac{1}{2}^*.$$

s	a_s	Tetrachoric Functions τ_s	Terms in Series $-a_s\tau_s$	Value of Series up to terms τ_s
0	1·00000000	·0966212	·0966212	·0966212
3	− ·28327885	+ ·04839695	+ ·0137098	·1103310
4	− ·01400852	+ ·05941568	+ ·0008323	·1111633
5	+ ·16688842	− ·06703628	+ ·0111876	·1223509
6	+ ·05349154	− ·00778490	+ ·0004164	·1227673
7	− ·05325140	+ ·05554783	+ ·0029580	·1257253
8	− ·09445982	− ·01930950	− ·0018240	·1239013
9	− ·00063525	− ·03745046	− ·0000238	·1238775

True value ·1188790

TABLE VII.

$$\int_0^{\cdot 1} \frac{x^3(1-x)^{\frac{1}{2}}}{B(4, \frac{3}{2})}\,dx, \quad y = -3\cdot 59087385\dagger.$$

s		Tetrachoric Functions τ_s	Terms in Series $-a_s\tau_s$	Value of Series up to term
0	1·0000000	·0001648	·0001648	·0001648
3	− ·28327885	+ ·00307042	+ ·0008698	·0010346
4	− ·01400852	− ·00458580	− ·0000642	·0009704
5	+ ·16688842	+ ·00530458	·0008853	·0000851
6	+ 05349154	− ·00442734	+ ·0002368	·0003219
7	− ·05325140	+ ·00191632	+ 0001020	·0004239
8	− ·09445982	+ ·00111687	+ ·0001055	·0005294
9	− ·00063525	− ·00291774	− ·0000019	·0005275

True value ·00023603.

5th or 6th term as at the 15th, say, and better than at the 30th. Of course, one might calculate the various terms till the sums became more or less steady, take the mean of these sums after the steady stage is reached and use that as the value required. This process, however, will not give a greater accuracy than three or four decimal places correct and very likely the result will not be so good as that. Besides which it is difficult to give such an arbitrary weighting of terms a theoretical justification. Thus it seems that the tetrachoric series is not at all suitable for the representation of the Incomplete Γ-function.

$$^* \; y = \frac{x - \frac{p}{p+q}}{\sigma} = \frac{\cdot 5 - \frac{8}{11}}{\cdot 17468526} = -1\cdot 3010412.$$

$$\dagger \; y = \frac{x - \frac{p}{p+q}}{\sigma} = \frac{\cdot 1 - \frac{8}{11}}{\cdot 17468526} = -3\cdot 59087385.$$

When we consider the tables and graphs for the Incomplete B-function, the results are certainly no better than in the case of the Incomplete Γ-function. Unfortunately, owing to the lack of a difference formula connecting the successive coefficients, we only calculated a few terms, but the behaviour of the graphs is similar to that of the graphs of the Incomplete Γ-function. Fig. 5 is very like Figs. 1—4 but Figs. 6 and 7 are rather different. In Fig 5 the integral is $\int_0^{\cdot 5} \frac{x^{14}(1-x)^4}{B(15, 5)}\, dx$, where p is of high value and q is of moderate size In Figs. 6 and 7 the integral is $\int_0 \frac{x^3(1-x)^{\frac{1}{2}}}{B(4, \frac{3}{2})}\, dx$, where the upper limits are ·5 and ·1 respectively. Here p is 4 and q is $\frac{3}{2}$. It seems in the incomplete Γ- and B-functions that the points come nearer the 'true value' line for the tail of the integral than if the upper limit is near the mode

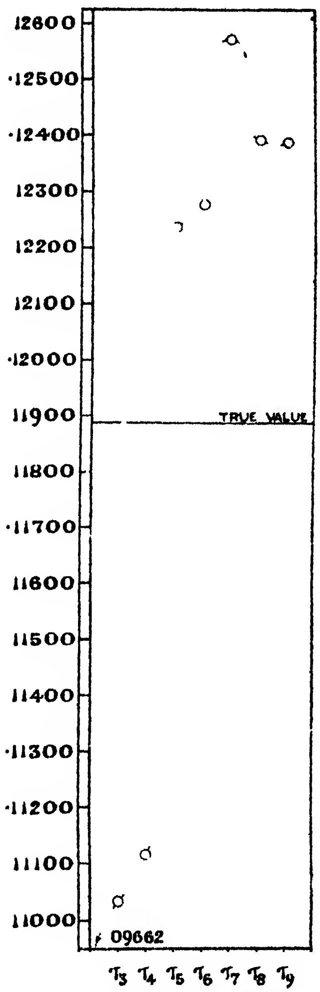


Fig 6.

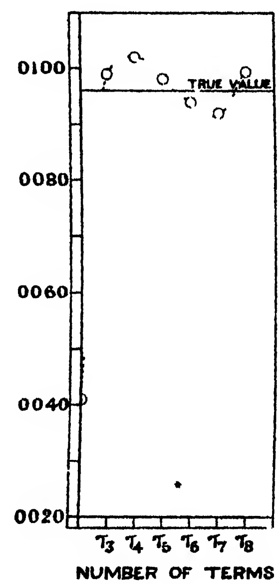


Fig 5.

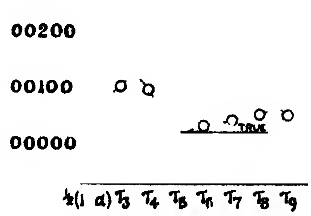


Fig 7.

Table VIII gives the results for $\int_0^{49} \frac{x^{48} e^{-x}}{\Gamma(49)}\, dx$ and, since $z = \frac{x-p}{\sqrt{p}} = \frac{49-49}{7} = 0$ for the expansion from the mean, all the tetrachoric functions of even order vanish. It will be observed that the values of the series vary in a similar fashion to the others and not one of these gives the result correct to more than four decimal places.

TABLE VIII.

$$\int_0^{49} \frac{x^{48} e^{-x}}{\Gamma(49)}\, dx, \quad z = 0^{*}.$$

(Expansion with regard to the Mean)

		Tetrachoric Functions τ_s	Terms in Series $-a_s\tau_s$	Value of Series up to term τ_s
0	1·00000000	·5000000	·5000000	·5000000
3	·11664237	− ·1628675	+ ·0189973	·5189973
	·00638743	+ ·1092549	− ·0006979	·5182994
7	01785148	·0842920	+ 0015047	5198041
9	·01170566	+ ·0695373	− ·0010226	5187815
11	·00895618	− ·0596711	+ ·0005345	·5193160
13	·01079260	+ ·0525526	·0005672	·5187488
15	·01015854	− ·0171442	·0004789	·5192277

True value 5189993

After a careful study of the tables and graphs we are forced to the conclusion that a tetrachoric series is of no practical utility as a representation of skew frequency curves such as $y = y_0 x^{t-1} e^{-x}$ and $y = y_0 x^{m_1-1}(1-x)^{m_2-1}$, and although it may be rash to generalise from our results on these two types it would seem that such a series cannot be generally suitable to represent skew frequency distributions. Moreover, the types, which have been discussed, are of common occurrence and for these the expansion is certainly futile.

The true values of the incomplete Γ-function were taken from *Tables of the Incomplete Γ-function* which will be shortly issued by H M. Stationery Office. The values of the incomplete B-function were determined by direct calculation; the power of $(1-x)$ was expanded and the result readily obtained with the help of the relation

$$B(p, q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}.$$

In his *Vorlesungen über die Grundzüge der mathematischen Statistik* (Hamburg, 1920) Charlier, when dealing with skew frequency curves, gives as the general equation for the skew frequency curves of his Type A

$$\tfrac{1}{\delta} Y = \phi_0 + \beta_3 \phi_0''' + \beta_4 \phi_0^{\text{iv}} + \beta_5 \phi_0^{\text{v}} + \dots$$

* $z = \frac{x-p}{\sqrt{p}} = \frac{49-49}{7} = 0.$

where $\phi_0 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ and ϕ_0''', ϕ_0^{iv}, ϕ_0^{v}, ... are the third, fourth, fifth, etc. differential of coefficients ϕ_0, i.e. Y is really expressed in a series of tetrachoric functions, or

$$Y = 5 \{\tau_1 - \beta_3 \sqrt{4!}\, \tau_4 + \beta_4 \sqrt{5!}\, \tau_5 - \beta_5 \sqrt{6!}\, \tau_6 - \ldots\}.$$

β_3, β_4, β_5, etc. along with M (the mean) and σ Charlier calls the 'characteristics' of the distribution curve. Now he seems to think that generally the coefficients β_3 and β_4* will only be required and so he has tabled $\phi_0(x)$, $\frac{d^3\phi_0}{dx^3}$, $\frac{d^4\phi_0}{dx^4}$ for $x = {}$·00 to 3·00 at intervals of ·01 and also for $x = 4$ (Tables III, IV and V on pp. 123—125) to four decimal places. With the series up to β_4 the theoretical Y-coordinate will be found, according to Charlier, but from our experience of tetrachoric functions we are exceedingly sceptical about the accuracy of such a result. In fact, we feel certain that the approximation will not be a good one. If the frequency curve be little different from the normal then possibly the approximation would not be very bad.

The above investigation was undertaken by me at the suggestion of Professor Pearson and I am indebted to him for several hints. My grateful thanks are due to Miss I M'Learn for her assistance in the preparation of the diagrams.

* Charlier defines the 'skewness' S to be $S = 3\beta_3$ and the 'excess' E to be $E = 3\beta_4$.

MISCELLANEA.

I. On the χ^2 test of Goodness of Fit.

By KARL PEARSON, F.R.S.

In a paper published in the *Philosophical Magazine* for July 1900, pp. 157—175, I dealt with the following problem: A very large population is sampled, say, the population $n_1, n_2, \ldots n_s, \ldots n_p$ with total N, and any individual sample is $m_1, m_2, \ldots m_s, \ldots m_p$, total M. The "probable constitution" is given by:

$$m_1' = \frac{M}{N} n_1, \quad m_2' = \frac{M}{N} n_2, \quad \ldots \; m_s' = \frac{M}{N} n_s, \quad \ldots \; m_p' = \frac{M}{N} n_p.$$

If a large number of samples of size M are taken, what is the distribution of variations from the "probable constitution" in these samples?

I showed that if the distribution of categories were such that no category contained a few isolated units, then the distribution depended on the calculation of $\chi^2 = S_1^p \frac{(m_s - m_s')^2}{m_s'}$, and provided a value for the probability P that samples would not diverge more than any given sample from the "probable constitution." This process is now familiar to statisticians as the χ^2, P test.

The sole limiting conditions were that the samples should be random, and each should be of the same size M.

In some cases the "probable constitution" (m' series) can be found at once because the distribution of the sampled population is known *a priori*. In other cases the values of the m' series have to be approximated to, and such approximations are the general rule in all discussions of probable error.

We say for example that the standard deviation of the mean of a sample taken from an indefinitely large population of size N and standard deviation σ is $\sigma/\sqrt{n}$, where n is the size of the sample.

We say that the standard deviation of second moment-coefficients of samples of size n is

$$\frac{\sqrt{\mu_4 - \mu_2^2}}{\sqrt{n}},$$

where $\mu_2 (= \sigma^2)$ and μ_4 are the second and fourth moment-coefficients of the population sampled. In fact every constant of the sample has a probable error determinable in terms of the constants of the sampled population. All these distributions of deviations from "probable constitution" are true for perfectly general but random samples of size n drawn from our indefinitely large population.

But unfortunately in a considerable number of cases that sampled population is unknown to us; we have no direct means of finding μ_2, μ_4, etc. What accordingly do we do? Why we replace the constants of the sampled population by those calculated from the sample itself, as the best information we have. And the justification of this proceeding is not far to seek. μ_s as found for the sample will only differ from the μ_s of the sampled population by terms of the order $1/\sqrt{n}$; for example if we are not dealing with *small* samples, and σ' be the standard deviation of the sample, σ' differs from σ by terms of the order $\sigma/\sqrt{2n}$ and accordingly the standard deviation of the mean is written $\sigma'/\sqrt{n}$ when it is really $\sigma/\sqrt{n}$. This method of treating probable errors is universal in the case of fair sized samples to-day and scarcely needs justification. In writing the

sample values of the constants for those of the sampled population, we do not in any way alter our original supposition that we are considering the distribution of random samples of size n. We have still $p-1$ degrees of freedom, if we have p categories of frequency.

The process of substituting sample constants for sampled population constants does *not* mean that we select out of possible samples of size n, those which have precisely the same values of the constants as the individual sample under discussion. Clearly the given sample has definite moment-coefficients, and if there be p frequency categories the first $p-1$ moment-coefficients together with the size n of the sample would suffice to fix all the frequencies of the p categories*. Hence no deviations from the "probable constitution" would be possible if we confined our attention to samples of n tied to the constants of the given sample! In using the constants of the given sample to replace the constants of the sampled population, we in no wise restrict the original hypothesis of free random samples tied down only by their definite size. We certainly do not by using sample constants reduce in any way the random sampling degrees of freedom.

What we actually do is to replace the accurate value of χ^2, which is unknown to us, and cannot be found, by an approximate value, and we do this with precisely the same justification as the astronomer claims, when he calculates his probable error on his observations, and not on the mean square error of an infinite population of errors which is unknown to him. The whole of this matter was very fully discussed (pp. 164—7) in my original paper dealing with the χ^2, P test.

The above re-description of what seem to me very elementary considerations would be unnecessary had not a recent writer in the *Journal of the Royal Statistical Society*† appeared to have wholly ignored them. He considers that I have made serious blunders in not limiting my degrees of freedom by the number of moments I have taken; for example he asserts (p. 93) that if a frequency curve be fitted by the use of four moments then the n' of the tables of goodness of fit should be reduced by 4. I hold that such a view is entirely erroneous, and that the writer has done no service to the science of statistics by giving it broad-cast circulation in the pages of the *Journal of the Royal Statistical Society*.

What he would obtain if he placed this restriction on his samples is not the χ^2 for the distribution of samples of size n, but of samples which give definite moments. The absurdity of this manner of approach is at once obvious, if as I have suggested, we consider the p first-moments, as there is no reason why we should not do,—for these are just as much "fixed" as the first four—and the conclusion must be that we can learn nothing at all about variation from our sample; for we have p frequency groups and p-tying conditions

When we wish to find the probable error of a mean or a standard deviation, we do *not* start by fixing down these characters to their values in the individual sample; we suppose them to take all the possible values they could take by sampling, and after we have reached our measure of variation we then put into our formula the sampled values, to give an approximate value to the functions reached, because we are in ignorance of the real values in the sampled population.

The writer in the *Journal of the Royal Statistical Society* speaks as if I applied χ^2 to a contingency table *starting* by fixing the marginal totals. As far as I am aware I am not guilty of this. My conception of contingency is very different from my conception of χ^2. I started my conception of contingency with the idea not of a random sample, but with the idea that some function of frequencies alone without regard to their relation to the measured characters would lead to the value of the correlation. Naturally I started from the deviation of the individual cell contents from the same cell contents on the basis of independent probability, as determined by the marginal totals. There was no question of sampling in the matter. In now fairly usual notation I termed

$$m_{ss'} - \frac{m_{s\cdot}\, m_{\cdot s'}}{M}$$

* This is Thiele's method of representing frequency distributions.

† Vol. LXXXV. p. 87, 1922.

the cell contingency and after playing about with such cell contingencies for a time succeeded in finding a function ϕ^2 of them which for indefinitely fine grouping for a bi-variate normal frequency distribution gave the correlation r as:

$$r = \sqrt{\frac{\phi^2}{1+\phi^2}},$$

where

$$\phi^2 = \frac{1}{M} S \frac{\left(m_{ss'} - \frac{m_{s\cdot} m_{\cdot s'}}{M}\right)^2}{\frac{m_{s\cdot} m_{\cdot s'}}{M}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

I see no reason for confusing this ϕ^2 as a measure of correlation with the χ^2 which is a measure of variability in the samples of constant size drawn from an indefinitely large population. It was different in its origin, as far as I am concerned, and different in its use. It is only when we come to consider the probable error of ϕ^2 that we have to distinguish between (a) the actual marginal totals of the sample and (b) the probable constitution of the marginal totals as deduced from an indefinitely large sampled population.

There are, as those who have read *Biometrika** will recognise, considerable difficulties about determining the probable error of ϕ^2, where

$$1 + \phi^2 = S\left(\frac{m^2_{ss'}}{m_{s\cdot} m_{\cdot s'}}\right),$$

and the determination of the mean ϕ^2 and of the standard deviation of ϕ^2 involves very troublesome analysis.

So laborious is the arithmetic involved that for ordinary statistical use it became doubtful whether it would not be better to define ϕ^2 as the mean squared contingency measured not from the marginal totals of the sample, but from the "probable constitution" of the marginal totals of the sample as deduced from the sampled population. In this case if

$$m'_{ss'} = \frac{M}{N} n_{ss'}, \qquad m'_{s\cdot} = \frac{M}{N} n_{s\cdot}, \qquad m'_{\cdot s'} \quad M$$

$$\phi^2 = S \frac{\left(m_{ss'} - \frac{m'_{s\cdot} m'_{\cdot s'}}{M}\right)^2}{M \cdot \frac{m'_{s\cdot} m'_{s'}}{M}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\beta)$$

or,

$$1 + \phi^2 = S\left(\frac{m^2_{ss'}}{m'_{s\cdot} m'_{\cdot s'}}\right);$$

with this change of definition the probable error and mean of ϕ^2 are more easily obtainable, and in this case *for the first time*, $M\phi^2$ can be looked upon as equivalent to a χ^2.

The form (a) from my standpoint cannot be treated as a χ^2, because it is not the deviation-measure of a given sample from the sampled population. Nor again is (β) the deviation-measure of the sample from the sampled population, unless we assume that population to have zero contingency, i.e. $m'_{ss'} = m'_{s\cdot} m'_{\cdot s'}/M$.

But χ^2 may in the form (β) be treated as a deviation-measure of the actual sample from an artificial sampled population, which differs from the actual population in having no correlation or contingency, but having the same marginal distributions of the two characters.

The moment, however, we assume form (β) for our contingency we are giving, what we clearly must give, absolute freedom to the marginal totals of our samples. The sole limit on our sample is its total size M. But when we come to actually calculating ϕ^2 for the individual sample, or the mean value or the standard deviation (i.e. probable error) of ϕ^2 for a series of samples, we have only one course open to us, if we do not know the constants of the sampled population, we must insert the marginal totals of the individual sample of which we have cognizance in place of the

* Vol. v. p. 191, Vol. x. p. 570, Vol. xi. p. 570, and Vol. xii. p. 259.

unknown values of the sampled population. Thus (α) and (β) provide ultimately the same ϕ^2, but the probable error of ϕ^2 and the mean value of ϕ^2 will be different in the two cases. In the first case we vary our marginal totals with the sample as they obviously would vary in practice. In the second case we define our ϕ^2 to be a deviation from the independent probability of an artificial population, we do not keep the marginal totals of the sample fixed any more than in (α). But if we think in terms of χ^2 (and not ϕ^2) we appear to do so because ultimately we have to take our marginal probabilities as those of the sample in default of a knowledge of any better values.

This point seems to me well illustrated in what my critic in the *Journal of the Royal Statistical Society* has to say on p. 90 of his paper about Messrs Greenwood and Yule's use of χ^2 for a fourfold table. He asserts that they ought to have entered the table of goodness of fit with $n'=2$. The problem before them was whether their fourfold tables could possibly be samples of bi-variate independent probability distributions. Each sample from such a distribution would have perfectly free cell frequencies m_{11}, m_{12}, m_{21}, m_{22}, subject to the sole binding condition that

$$m_{11}+m_{12}+m_{21}+m_{22}=M.$$

The proper χ^2 is given by

$$\chi^2=\frac{\left(m_{11}-\frac{m'_{1.}m'_{.1}}{M}\right)^2}{\frac{m'_{1.}m'_{.1}}{M}}+\frac{\left(m_{12}-\frac{m'_{1.}m'_{.2}}{M}\right)^2}{\frac{m'_{1.}m'_{.2}}{M}}+\frac{\left(m_{21}-\frac{m'_{2.}m'_{.1}}{M}\right)^2}{\frac{m'_{2.}m'_{.1}}{M}}+\frac{\left(m_{22}-\frac{m'_{2.}m'_{.2}}{M}\right)^2}{\frac{m'_{2.}m'_{.2}}{M}} \quad \ldots(\gamma),$$

and this has three degrees of freedom and is what Messrs Yule and Greenwood desired to find, and they properly used the value of P for $n'=4$.

Then like the astronomer, who finding the probable error of his mean to be $\cdot 67449\sigma/\sqrt{M}$ and not knowing the σ of his sampled population, puts it equal to the σ of his observations, so Messrs Yule and Greenwood very properly replaced the marginal totals of their unknown population by those of their sample, but very properly did not replace $n'=4$ by $n'=2$!

But says my critic*, if they had, they would have got the same measure of improbability as if they had compared the difference of percentages! Quite so, and obviously so; for in taking percentages they have actually fixed their marginal totals taking 100 of each class and thus for the first time confined their attention to a limited class of samples, not the random sample of size M, which has not its marginal totals fixed. We have, indeed, reduced our degrees of freedom by two in taking ratios.

When we consider generally the χ^2 for a fourfold table to measure the improbability of a sample we are really comparing the special sample

			with			
	b	$a+b$		a'	b'	$a'+b'$
		$c+d$		c'	d'	$c'+d'$
$a+c$	$b+d$	M		$a'+c'$	$b'+d'$	M

the general population, where in the latter case $a'd'=c'b'$.

Now the mean square contingency of the first of these tables is

$$\phi^2=\frac{1}{M}\left\{\frac{\left(a-\frac{(a+b)(a+c)}{M}\right)^2}{\frac{(a+b)(a+c)}{M}}+\frac{\left(b-\frac{(a+b)(b+d)}{M}\right)^2}{\frac{(a+b)(b+d)}{M}}+\frac{\left(c-\frac{(a+c)(c+d)}{M}\right)^2}{\frac{(a+c)(c+d)}{M}}+\frac{\left(d-\frac{(c+d)(b+d)}{M}\right)^2}{\frac{(c+d)(b+d)}{M}}\right\}$$

$$=\left\{\frac{a^2}{(a+b)(a+c)}+\frac{b^2}{(a+b)(b+d)}+\frac{c^2}{(a+c)(c+d)}+\frac{d^2}{(c+d)(b+d)}-1\right\}$$

$$=\frac{(ab-cd)^2}{(a+b)(a+c)(b+d)(c+d)}.$$

* *Loc. cit.* p. 90.

But the χ^2 is

$$\frac{\left(a-\frac{(a'+b')(a'+c')}{M}\right)^2}{\frac{(a'+b')(a'+c')}{M}}+\frac{\left(b-\frac{(a'+b')(b'+d')}{M}\right)^2}{\frac{(a'+b')(b'+d')}{M}}+\frac{\left(c-\frac{(a'+c')(c'+d')}{M}\right)^2}{\frac{(a'+c')(c'+d')}{M}}+\frac{\left(d-\frac{(c'+d')(b'+d')}{M}\right)^2}{\frac{(c'+d')(b'+d')}{M}}$$

$$=M\left\{\frac{a^2}{(a'+b')(a'+c')}+\frac{b^2}{(a'+b')(b'+d')}+\frac{c^2}{(a'+c')(c'+d')}+\frac{d^2}{(c'+d')(b'+d')}-1\right\},$$

there being *three* degrees of freedom or we must take $n'=4$ in calculating the probability P, this may be written

$$\chi^2=\frac{1}{M}\left\{\frac{a^2}{p'_{\cdot 1}p'_{1\cdot}}+\frac{b^2}{p'_{\cdot 1}p'_{2\cdot}}+\frac{c^2}{p'_{\cdot 2}p'_{1\cdot}}+\frac{d^2}{p'_{\cdot 2}p'_{2\cdot}}-1\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\delta),$$

where $p'_{\cdot 1}$, $p'_{\cdot 2}$, $p'_{1\cdot}$ and $p'_{2\cdot}$ are the four percentage numbers of the marginal categories in the sampled population. Now we do not know these percentages in that population and we do what every physicist, every astronomer, and—till I saw the paper by my critic in the *Journal of the Statistical Society* I should have said—every statistician does, supply the unknown constants from the sample, which leads us to

$$\chi^2=\frac{M(ab-cd)^2}{(a+b)(a+c)(b+d)(c+d)}=M\phi^2$$

as used in my memoir of 1912*.

The problem I had and still have in view is the variability in samples of definite size with no other restriction than sample size. The solution of that problem is absolutely comparable with that of any discussion of the probability of an observed result in the theory of probable errors. We have in the bulk of such cases constants involved which concern the distribution in an unknown population, and we supply those constants from the sample itself.

As I have already noted the probable error of a mean is

$$\cdot 67449\,\frac{\sqrt{\mu_2'-\mu_1'^2}}{\sqrt{M}}.$$

By this we understand that the means of samples restricted solely by their size M from an indefinitely large population of moment-coefficients μ_1', μ_2' about a fixed origin will have a variability determined by the above formula. But when we proceed to give both μ_1' and μ_2' the values determined from the sample we know, we do *not* add in the manner of my Royal Statistical Society critic, "but in doing so the type of samples is reduced to those having the mean and standard deviation of the sample." If we did, this selection of samples would clearly have no variation of mean or standard deviation at all! In fact probable errors would be meaningless, unless we drew our samples from a population already fully known to us, in which case we should not in 99 °/。 of cases want to sample it at all.

In the same way when we use the marginal totals of the sample in formulae like (δ) we do not thereby reduce our samples to those having constant marginal totals, we merely take the best approximation available to the proper value of χ^2, and the fact that χ^2, as found from the sample, is only an approximation to the true χ^2 was fully recognised and discussed in my original memoir in the *Philosophical Magazine*.

It only remains to say that the following sentence of my critic's paper seems to me based upon a fallacious principle and apparently flows from a disregard of the nature of probable errors in general.

"It should be pointed out that certain of Pearson's *Tables for Statisticians and Biometricians*, namely Tables XVII, XIX and XX, together with XXII (*Abac* to determine r_p') are all calculated

* On a novel method of regarding the association of two variates classed solely in alternative categories. *Drapers' Company Research Memoirs*, Cambridge University Press.

on the assumption that $n'=4$ in fourfold tables, and consequently should not be used when, as is almost always the case, the marginal totals are obtained from the data" (*loc. cit.* p. 91).

I hold those tables are quite correctly calculated for $n'=4$, and those who attempt to modify them by assuming $n'=2$ will be dealing with an entirely different problem. Namely, they will be considering not the improbability of the given sample as one of all possible samples of the given size, which it really is, but one of the indefinitely smaller number of samples that have fixed marginal totals. We do not find the probable error of r for a tetrachoric table* on the assumption that the marginal totals are fixed. We find it on the assumption that the marginal totals also vary from sample to sample, and when we have found it, then we substitute in the result the values of not only the marginal totals, but the cell-contents, a, b, c, d of the sample itself for those of the unknown population. With χ^2 we go through an exactly similar process of reasoning. If by this procedure we in some mysterious manner tied our degrees of freedom down to the values of the cell-contents used in our formula and adopted from our sample there could be no probable error for r, for the values of a, b, c, and d are all required and used. I trust my critic will pardon me for comparing him with Don Quixote tilting at the windmill; he must either destroy himself, or the whole theory of probable errors, for they are invariably based on using sample values for those of the sampled population unknown to us. For example here is an argument for Don Quixote of the simplest nature: In the sth category of a population N the frequency is n_s, a sample shows m_s in a total M. The standard deviation of this frequency is

$$\sqrt{M\frac{n_s}{N}\left(1-\frac{n_s}{N}\right)}.$$

But we don't know the population sampled and accordingly obtain an approximate value of the above standard deviation by writing for $\frac{n_s}{N}$, $\frac{m_s}{M}$ and taking for the standard deviation of m_s $\sqrt{m_s\left(1-\frac{m_s}{M}\right)}$. In doing this it is not a question even of using a marginal total, we have used a cell frequency found from our sample. We have therefore according to our critic reduced our possibilities of freedom by selecting out of all possible samples those with m_s in the sth cell—this is exactly parallel to our reducing our freedom by "fixing" marginal proportions or moment-coefficients. But if m_s be fixed, it is ridiculous to talk of a variation of the m_s frequency. Therefore either $m_s=0$ or $m_s=M$, or the usual theory and practice of probable errors are wholly at fault. I think this will illustrate what I mean by Don Quixote and the windmill.

II.

Is Tuberculosis to be regarded from the Aetiological Standpoint as an acute disease of Childhood? By Dr KR. F. ANDVORD (Christiania). *Tubercle*, Vol. III. No. 3, December, 1921.

This paper is, we must confess, unconvincing. The author holds that in a community that has long been subject to tuberculosis the time of infection should be fixed in the infantile years for the great majority of cases and consequently we should protect children for the first three or four years from infection.

As evidence of his views he takes a graph of what he calls a "population frame" which is really the well-known "number living in a stationary population" (l_x) and represents within this graph the numbers dying from tuberculosis and the numbers who have suffered from it at each age. We are doubtful if his graphs for deaths are correctly drawn. They are made to rise suddenly for about a year and then fall till age 7 but we suspect that they should fall from birth till age 7. We cannot justify his chart (No VIII) which gives the whole population and the

* *Phil. Trans.* Vol. 195 A, p. 14.

tubercular population. The non-tubercular found by this chart actually increase after age 17 for many years so that the non-tubercular not only have no mortality but are increased by some process of resurrection! Admittedly the chart is hypothetical but as it stands it calls for amendment.

Dr Andvord's remark that "one would hardly gather from these per-thousand curves," i.e. from rates of mortality for various ages, "that, as is really the case, more persons die from tuberculosis in the first and second years of life than in any subsequent age period" seems to betray an inexperience in matters related to a life table: this weakness is shown elsewhere, e.g. p. 102, where deaths are stated without populations and without reference to age distributions.

Dr Andvord may have other evidence in support of his views but the article under review does not justify them statistically; we think every point he brings out could be explained as well on other hypotheses. He cannot, moreover, completely prove his case till he has studied communities which become subject to infection after having been kept free from it. For if his theory be correct, the measures he proposes would necessarily produce such a community.

W. PALIN ELDERTON.

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A FIRST STUDY OF THE TIBETAN SKULL.

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CONTENTS

1. *Introductory.*

THE present paper continues the craniometric studies which have already been carried out in the Biometric Laboratory and it is supplementary to Miss Tildesley's recent memoir on the Burmese skull. The writer hopes that it may be followed by papers dealing with material from the Himalayan border-lands and Eastern China. These peoples are of peculiar interest because the country which they inhabit is a borderland between Mongolian and Indian races, and in prehistoric and early historic times was probably traversed by nomadic tribes in their wanderings. The general tendency of anthropologists now-a-days is to lay smaller stress on the value of local races and to seek for fundamental human types which may be mingled or hybridised in local races. All the races at present inhabiting the country between India and China and the Chinese themselves are supposed to constitute such a type which is usually termed "mongolian" and is characterised by a brachycephalic skull and a broad, flattened face. But in the Tibetan province of Khams there is a dolichocephalic people with no signs of flattening of the facial bones. They have heavy and capacious crania with strong muscular development and other characters which clearly differentiate them from Chinese, Malayan and Burmese skulls. No relationship can be suggested with the dolichocephalic peoples of India for the crania of the latter are feeble and under-sized. Comparison is more possible with Moriori and Fuegian crania than with those of any of their immediate neighbours east and west of which we have knowledge. That the cranium of the Khams Tibetans

should be so widely differentiated from that of other Orientals is a fact of great suggestiveness and it shows how important the study of local races may be.

In India and Tibet there are two races which are obviously unrelated, and yet the cephalic indices of the two are almost identical. This is but one illustration of the inadequacy of that criterion when used as a measure of racial affinity. The writer has been led to the conclusion that the cephalic index alone is quite incapable of discriminating between fundamental types or of distinguishing relationships between races which are known to be allied. Furthermore, no single character which has yet been suggested can fulfil either of these purposes and it is extremely unlikely that one will ever be found.

The object which the writer of this paper has kept in view has been the means of distinguishing biometrically between various types of man as far as such are represented in greater or less dominance in local races. The two methods thus far adopted by the biometric craniologists have been a comparison of type contours and the admittedly temporary "Coefficient of Racial Likeness." This latter method furnishes a general measure of racial resemblance which will give weight to all craniometric characters, or at least to a considerable number of well chosen characters, and it is intended to supplant the old methods of comparison between single characters such as the cephalic index or the nasal index. There is no great difficulty theoretically in devising a reasonable measure of racial resemblance provided we had an adequate knowledge of:

(i) the means of some 30 to 40 characters in 50 to 100 local races;

(ii) the standard deviations and correlations of these means among themselves—i.e. inter-racial variations and inter-racial correlations.

Unfortunately the material for (i) and therefore (ii) is very inadequate at present.

Therefore we are thrown back for the time being on the "Coefficient of Racial Likeness" which Professor Karl Pearson has suggested. This is a measure of whether any two races can be considered samples of the same population. In actual use it depends on the assumptions:

(i) that there is intra-racially no *high* correlation between the characters selected for comparison;

(ii) that failing intra-racial variability being adequately known—especially for short series—it will be sufficient to assume equal variabilities for all races, and to use for these variabilities those obtained for a really long series of a single race.

Such a method cannot be final or completely satisfying to the statistically trained mind, but it may well serve, as it was originally intended to do, as a stop-gap, until sufficient material has accumulated for its replacement by an ampler procedure.

But the assumptions on which it is based are not so contrary to experience as to lead us to discard what aid we can get from the "Coefficient of Racial Likeness." There is in fact at the present time nothing to supply its place. Our aim should rather be to determine how far it leads us to the same conclusions as we can draw from type contours, and it has this great advantage that it enables us by the very process of its deduction to ascertain in which characters two races significantly

diverge from one another. It may be, as the comparisons already made partly suggest, that there are some characters which differ between race and race far more than others do, and if a selection of the characters which fluctuate most could be made it might be possible to neglect the more constant ones.

The other method of estimating the significance of racial differences is that based on a comparison of type contours. The three type or mean racial contours are constructed from the mean lengths of the individual contours of a racial series and on these three there are no adequate representations of some important regions of the skull. For this reason and also because the conclusions to be drawn from drawings necessarily lack the precision of arithmetical analysis, the contours can never rival in importance the system of direct measurements. Nevertheless graphical methods of comparison appeal to many craniologists who suspect all statistical deductions and one of the chief objects of this paper has been to study how far they are helpful in the comparison of human types. But no very definite conclusions could be arrived at because at present we have type contours for only a few races and there has been a rapid evolution in the technique of representation so that considerably more of the skull is now portrayed than was shown on the earlier drawings. As the problem of human types now stands, we cannot assert that geographical proximity naturally connotes racial affinity. Cultures have spread from one end of the world to the other and the bearers of those cultures have undoubtedly in many cases gone with them. The type contours of palaeolithic man differ widely from those of modern man; those of Khams Tibetans and Moriori show some resemblance. Are we to argue in one case that difference associated with a wide temporal interval is significant, but on the other resemblance with a wide geographical interval can mean nothing? The present paper only touches the fringe of matters when it endeavours to contrast results obtained from the type contours with those suggested by other methods. But it is already clear that no classification of *normae verticales* can be a reliable guide to racial relationships. The *norma lateralis* is a more important aspect, but if considered alone it may lead to quite erroneous conclusions. Contours are also of importance because certain measures of curvature can be derived from them and these could not be easily determined directly.

The last object of this paper has been to develop the biometric methods of studying the human mandible. The few Tibetan mandibles which were available did not themselves justify this purpose, but by dealing with even so few it was possible to discover the difficulties of technique which practice alone can teach. A review of the definitions of mandibular "standard planes" and "points" and the methods of measurement provided in some text-books of anthropometry suggested that they had been drawn up before they were applied in practice, and accordingly after experience it became necessary to modify them considerably. The system of measurements has to be a detailed one if comparison is to be made with the mere fragments of jaws through which alone certain races of fossil man (and apes) are known to us. The type contours of the mandibles of the two Tibetan groups will appear in a subsequent memoir.

The subject of the present paper is a collection of Tibetan skulls made by Colonel L. A. Waddell in the south-western and eastern districts of Tibet and presented by him to the Royal College of Surgeons. With the kind permission of Sir Arthur Keith I have been able to measure these crania.

There are 32 skulls in the collection which is divided into two groups. The first group (Type A) comprises 17 skulls which were obtained from graves in Sikkim and neighbouring regions of Tibet proper; the remaining 15 skulls (Type B) were picked up on a battle-field in the Lhasa district and were believed to be those of native soldiers from the eastern province of Khams. With the exception of one skull of the B type, the parietal bones of which had been badly damaged by a bullet, they were all in a very good state of preservation. There are 16 mandibles of the A series and 12 of the B, but 5 of the latter did not correspond with any of the skulls. All the skulls of the B series were of a heavy and decidedly masculine adult type, but a few of the A had not reached maturity and the sex of several could not be positively determined.

Such material, though no very reliable means can be reduced from it, is of peculiar interest because of the rarity of Tibetan crania: we are only aware of the existence of five specimens from the interior regions of the country in any other European museum*. In spite of the meagreness of our material we ought to be able to discover any striking resemblances or differences there may be between the skulls of the Tibetans and those of neighbouring peoples.

2. *Measurements and Methods of Measurement.*

The following measurements are with certain exceptions those of the *Frankfurter Verständigung* taken in the manner described by C. D. Fawcett† together with those added by subsequent workers in this laboratory.

F = Flower's Ophryo-occipital length. L' = glabellar projective horizontal length. L = maximum length from glabella to occiput in the median plane. B = maximum horizontal breadth on parietal bones. B' = least forehead breadth from one temporal crest to the other. H' = basio-bregmatic height. H = height measured on craniophor, from basion to the point vertically above it (generally a little behind the bregma). OH = auricular height, as measured on the craniophor, that is, the height of the skull above that part of the auricular orifice which rests on the top edge of the ear-plugs‡. LB = length from basion to nasion. Q = transverse arc perpendicular to the horizontal plane, passing through the "apex," and terminating on both sides at the top of the ear-rods, this measurement being taken when the skull is on the craniophor. Q' = a similar measurement taken off the craniophor

* Dr W. L. H. Duckworth has kindly furnished me with a description of two Tibetan skulls in the Anatomical Museum at Cambridge. These were obtained in the southern province of Dokthol and appear to be of the Tibetan A type. There are three skulls from the interior of Tibet and many more from the south-west border-land of the country in the Hodgson collection at the British Museum (Natural History).

† *Biometrika*, Vol. I. pp. 412—419.

‡ The accuracy and significance of this measurement, which has become of peculiar importance since cranial contours have been drawn, will be discussed later.

and terminating at the "auricular points*." S = sagittal arc from nasion to opisthion. S_1 = arc from nasion to bregma. S_2 = arc from bregma to lambda. S_3 = arc from lambda to opisthion. S_3' = chord from lambda to opisthion. U = horizontal circumference measured directly above the superciliary ridges and round the most projecting sagittal part of the occiput, i.e. the most projecting part between the inion and lambda. PH = premaxillary height, measured from alveolar point to tip of anterior nasal spine, if such exists, and, where it does not, to the sagittal point of the lower edge of the pyriform aperture. $G'H$ = upper face height from nasion to alveolar point. GB = face breadth from the lower end of one zygomatic-maxillary suture to that of the other, where the suture crosses the lower front rim of the cheek-bone. J = zygomatic breadth, from the most lateral point of one zygomatic arch to that of the other. NH, R and L = nasal height from nasion to lowest edge, right and left, of pyriform aperture. NB = nasal breadth, greatest breadth of pyriform aperture, wherever it may be. DS = dacryal subtense, being the shortest subtense from the bridge of the nose to the dacryal chord, measured with Mérejkowsky's simometer. DC = chord from dacryon to dacryon. DA = shortest arc over the bridge of the nose from dacryon to dacryon, measured with the tape. SS = simotic subtense, the shortest subtense from the nasal bridge to the simotic chord, as measured by the simometer. SC = minimum chord between the two naso-maxillary sutures. O_1 = greatest breadth of orbit, R and L, from side to side, using Fawcett's curvature method† to determine the margin on the nasal side. O_1' = breadth of orbit from dacryon. This measurement is only taken because it is frequently given by other workers for other races. O_2 = greatest height of orbit, R and L, taken perpendicular to O_1. G_1 = length of palate, from the point of the *spina nasalis posterior* to an imaginary line tangential to the inner rims of the alveoli of the middle incisors. G_1' = similar measurement to G_1 but taken from base of spine. G_2 = breadth of palate between inner alveolar walls at second molars GL = profile length from basion to alveolar point. fml = length of *foramen magnum* from basion to opisthion. fmb = greatest breadth of *foramen magnum*. $P\angle$ = profile angle, or angle made by the line from the nasion to alveolar point with the horizontal, found by means of Ranke's Goniometer when the skull is in the horizontal position on the craniophor, as described by Fawcett (*Biometrika*, Vol. I. p. 414). Various indices are calculated from these direct measurements: occipital ($Oc.\ I$), cephalic ($100\,B/L'$, $100\,H/L$, $100\,H/L'$, $100\,B/H$), facial ($100\,G'H/GB$), nasal ($100\,NB/NH$), orbital ($100\,O_1R/O_2R$, $100\,O_1L/O_2L$), palatal ($100\,G_2/G_1$), foraminal ($100\,fmb/fml$), dacryal ($100\,DS/DC$), simotic ($100\,SS/SC$) The angles ($N\angle$, $A\angle$, $B\angle$) of the fundamental triangle whose angles are the nasion, alveolar point and basion, were calculated from the measured lengths LB, GL, $G'H$ with the aid of Pearson's Trigonometer in the manner described by Fawcett‡. θ_1, the basio-nasal horizontal angle, is obtained by subtracting $N\angle$ from the supplement of $P\angle$; θ_2, the basio-alveolar horizontal angle, by subtracting $A\angle$ from $P\angle$.

* These are defined on p. 201.

† See *Biometrika*, Vol. I. p. 430 and Vol. VIII. pp. 311, 312.

‡ *Biometrika*, Vol. I. p. 418.

Of the direct measurements above L', H', H, OH, Q and $P\angle$ are taken when the skull is in the Frankfurt horizontal position on the craniophor; Q, Q', S, S_1, S_2, S_3 and U are made with the steel tape; F, L, B, B', LB, S_3' and J with Flower's callipers; DS and SS with the simometer and the remainder with small callipers. The small arc from dacryon to dacryon (DA) was not measured with the steel tape but indirectly by marking its extremities on a small slip of paper stretched over the bridge of the nose.

To test the supposed correlation between intelligence or size of skull and height of palate*, some means of measuring the latter is required. The sagittal contours will give the height alone, but by means of an instrument known as the "Uraniscometer," designed by Professor Karl Pearson and shortly to be described by him, it is possible to measure directly both the height and breadth of the palate. The breadth (EB) was taken between the inner alveolar walls at the second molars and the height (EH) from the line joining the extremities of the chord (EB) and in the transverse plane through them. Thus EB is precisely the same measurement as G_2 made with the small callipers and it was only repeated to test the accordance of the two methods of determining it. The greatest difference between them was 0·5 mm., so G_2 need not be taken in future. $100\,EH/EB$ provides a second palatal index.

The only other direct measurements taken were those of the weight and capacity of the skull. The latter was determined by tight packing with mustard-seed, weighing the contents and estimating the capacity indirectly by filling a prepared skull first with seed and then with water. This method is precisely that used and fully described by Macdonell†.

After some preliminary practice a skull (T 27) was packed as tightly as possible with mustard-seed which was then weighed. Repeating this operation 6 times gave the weight of the contents in grams:

(1) 1332·2, (2) 1345·4, (3) 1349·9, (4) 1351·6, (5) 1353·8, (6) 1350·3.

With the exception of the first these results appeared to agree quite as well as we could expect. The weight of seed which each skull would hold was then determined. After an interval of a week this operation was repeated and a fairly uniform increase in weight on the first readings was found, though it never exceeded 15 grams. The skull T 27 held seed weighing 1357·2 grams, an increase of 3·4 on the maximum for the first occasion. There had been no practice in the intervening period and the conditions and method of packing appeared to be the same on both trials. Though showing how large the "personal equation" might be, the difference between the two determinations would be of no consequence if we could feel confident that all the skulls in the second case had been packed with the same degree of tightness. There was good reason for supposing that such had been the case, for the weight of seed held by the "crâne étalon" δ had been found once on every day of the period over which the second weighings extended, giving in grams:

(1) 1062·2, (2) 1060·2, (3) 1056·9, (4) 1061·7, (5) 1058·7.

* See *Biometrika*, Vol. XIII. p. 281. † *Biometrika*, Vol. III. pp. 203—206.

These show that the method of procedure was as nearly the same for all the skulls as we could reasonably hope it would be, and accordingly the second series of weights was used to determine the capacities. Filling the *crâne étalon* with water, of which the volume was afterwards found by pouring it into a measuring cylinder, gave in cm.^{3}:

(1) 1350, (2) 1360, (3) 1365.

Much greater care was taken in filling on the last occasion than on the previous ones and the increase is very significant. Accepting the last volume and taking the weight of seed held by the skull to be 1062·2 grams, gives a weight of 778·2 grams corresponding to a volume of 1000 cm.^{3}. Repeating this process with the *crâne étalon* γ gave a weight of 777·2 grams for the same volume. Using the standard skull δ Macdonell found that 1000 cm^3 contained 764·6* grams of seed and Tildesley's result for the same skull was 779·9 grams. To test the accuracy of the method, the seed that had been packed into the prepared skull δ was turned into a measuring cylinder, as well as being weighed, and pressed down as tightly as possible. The mean capacity found in this way was 1349 cm.^{3}, which shows that the seed could be compressed rather more easily in the glass vessel than in the skull. The difference in the results given by the two methods is small, but the former is almost certainly the more reliable. The final capacities were calculated on the supposition that 1000 grams of seed occupied a volume of 1000/778·2 cm.^{3}.

The only other direct measurement taken was that of the weight of the skull. No attempt was made to make allowance for lost teeth or missing portions of bones.

The occipital index (*Oc. I.*), which is defined to be $100\,\dfrac{S_3}{S_3'}\sqrt{\dfrac{S_3}{24\,(S_3-S_3')}}$, was found with the aid of Miss Tildesley's table of the function, given in the appendix to her Burmese paper.

A few remarks must be made on the methods of measurement used by biometric craniologists. Special care has to be taken when compiling comparative material as a few of the letters denoting measurements have not been used by different workers with precisely the same significance

H and H′. The H' defined above is the H of Macdonell, Benington and Thomson.

Q and Q′. The Q of Macdonell and Benington was taken over the bregma, instead of in the vertical plane Benington's Q' is our Q. We have used the same Q as Thomson, but her Q' was taken from the upper margin of one auricular passage to that of the other.

O_1R and O_1L. We have followed Fawcett, Thomson and Tildesley in using the "curvature" method to determine the width of the orbit. Macdonell and Benington used the "geodesic†."

NH. This measurement is frequently lacking in precision owing to the difficulty of determining exactly where the lowest point of the edge of the pyriform aperture

* The seed used had been kept for a number of years after Macdonell determined capacities with it; it may have become more shrunk during that time, and the *crânes étalons* had been repacked with cement.

† See *Biometrika*, Vol. I p. 431.

is. Thomson measured to the lowest point on the left side and Fawcett, Macdonell and Benington to the lowest point whether right or left. Tildesley was the first to measure both sides and we have followed her in this. The Tibetan crania were apparently more symmetrical in this respect than the Burmese.

DA and DC. These were first measured by Benington and, before Mérejkowsky's simometer was used, the subtense *DS* was obtained from them by calculation.

OH. The error detected by Tildesley in the vertical scale of the Ranke's craniophor used in the Biometric Laboratory was confirmed to be 2·5 mm. and that amount was accordingly subtracted from all the scale readings of *OH*. This correction was made to the means of previous workers before using them for comparative purposes*.

Sex and Age. The skulls were examined by Professor Pearson with the object of sexing them. The Tibetan *B* skulls were more bleached and could be readily distinguished from the others. They were undoubtedly all adult males of quite a massive and heavy type and with well marked glabellas and inions. The Type A skulls were smaller and lighter and could not be sexed so easily. The majority of these were undoubtedly male, but three of them were young, with third molars appearing but not fully erupted, and the sex of one or two was doubtful. A few were certainly female, but the series was such a short one, comprising only 17 skulls, that it was thought best to pool them all to obtain racial means. This would have the effect of making the mean direct measurements slightly less than for an entirely male population, but the indicial and angular measurements would be less definitely affected.

Mandibles. The mandibular measurements made are described in Appendix I to this paper.

3. *On the Difficulties attached to the Location of certain Cranial "Points."*

Inion. No direct measurements terminate in this point but it is marked on the sagittal contours. In the cases where the inionic prominence was clearly marked the "point" was taken to be the meeting place of the *lineae nuchae supremae* and this was found to be very close to the common horizontal tangent of the *lineae nuchae superiores*†: see Plate XIV. The intersection of that line with the median plane determined the inion in less clearly defined cases. But often its position could only be vaguely conjectured.

Basion and Opisthion. These points were not marked on the skull, but the divergencies between the measurements terminating in them, taken on two separate occasions, were no larger than the average. The differences which were observed between the mean foraminal lengths deduced from the sagittal contours and direct measurements were more significant. They would probably have been less if the points had been marked on each skull before the contours were drawn.

* A table of these corrected means is given in *Biometrika*, Vol. XIII. p. 217.

† When determining the position of the inion the writer was especially guided by a discussion of the subject in *Biometrika*, Vol. XIII. p. 184.

Auricular Points. The difficulties of defining clearly the positions of the auricular points have been discussed at considerable length by many craniometricians. It seems best to accept and use in practice two definitions which will give very close but not always coincident points.

The craniophor auricular points are defined to be those at which the skull, when in the Frankfurt horizontal plane, touches the ear-rods of the craniophor. The ear-rods were only inserted a short way into the auricular passages; almost as short a distance as was consistent with stability. The *OH* height is taken from the top of the rods and the divergencies between the two measurements for each skull found on separate occasions had a maximum of 2 mm., but they were only more than 1 mm. in three cases. These craniophor points are the ones shown on the sagittal and transverse vertical contours and though they had not been marked on the skulls, there seemed to be no difficulty in roughly locating them again. A comparison of the three mean *OH* measurements—one by direct measurement and the other two from the contours—showed a maximum difference of 1·0 mm. (see p. 251 below). This is larger than one would like, but it could probably be made less by marking the points on the individual skull while it is still on the craniophor and before drawing the contours.

The auricular points were defined by Fawcett to be "the highest points of the upper rims of the auricular passages*," and these will not always coincide with the points of contact of the ear-rods. It was found that there was a thin lip of bone on nearly every skull, terminating posteriorly in a well marked notch, and coinciding almost exactly with any line that could be supposed to mark the upper rim of the auricular passage. The point of intersection of that lip of bone with the plane bisecting the orifice transversely was taken to be the "auricular point by definition" at which the arc Q' terminates.

4. "*Remarks*†" *on Individual Crania.*

The "remarks" made on out-standing cranial anomalies are similar to those of previous biometric craniometricians. The number of skulls (32) with which we were dealing was, for statistical purposes, so small that no reliable conclusions at all could be drawn as to the racial significance of the proportions of the total population having or lacking any particular character.

The following characteristics and anomalies were looked for and, if no mention of one of these anomalies has been made, it is to be definitely understood that it did not exist in the skull.

Age and signs of age. Where no comment is made the skull is adult with sutures joined but not obliterated; basilar synchondrosis noted, also any teeth in process of coming through or lost during life and consequent absorption of alveolus; falling-in or thinning of calvaria, due to old age, looked for.

Teeth. Undeveloped or imperfectly developed third molars; extreme attrition of teeth and any irregular formations round palate; diseased teeth. The teeth of

* *Biometrika*, Vol. I. p. 413.

† These are given, together with the individual measurements, in Appendix II to this paper.

the majority of the Tibetan skulls, and particularly those of the B type, were in a very good state of preservation and hardly any showed signs of disease or wear.

Palate. Each was examined for the existence of bony ridges across the palatine grooves leading from the pterygo-palatine canals*. When such a one is thrown over the inner groove on either side, it is described as an inner palate bridge, left or right; over the outer groove as an outer palate bridge. One inner palate bridge was found and there were no cases of an outer one. But on several of the palates there were small spikes of bone on both sides of the canals which might well have been the broken extremities of what was originally a complete bridge. The palate was also examined for palatine torus, a thickening of the bone on both sides of the median suture.

Precondyles. The existence of a pair of small, distinct precondyles was noted in the case of two skulls and there was also one in which the precondyles had become completely fused: see Plates XIII and XIV.

Tympanic perforation. Perforation of the tympanic plate was recorded in all cases.

Base of the Pyriform Aperture. This feature has been indicated by the letters P.B., and the description has reference first to that part of the floor of the nasal cavity which lies immediately behind the edge of the aperture, and secondly to the edge itself. The floor is described as "flat"—in which case it may be assumed to be approximately horizontal as well—or "sloping upwards and outwards," or "downwards and outwards"; the edge itself is described as sharp, blunt or rounded, and if it is double with a groove between its two parts, that is also mentioned. In some cases the floor of the cavity is so rounded as to form part of an almost unbroken curve with the rounded lip: such a case is described shortly as P.B. rounded, or V. rounded. The following frequencies were found for the Tibetan skulls:

Type of Edge	Type A	Type B
Sharp ...	12	7
Blunt ...	4	7
Rounded ...	1	1

There were three skulls having a double edge to the pyriform aperture.

Asymmetry. Each skull is marked SR or SL according as the Sylvian depression is greater on the right or left side, and, in the same way, JR or JL has reference to the greater size of the right or left jugular foramen. The frequencies of these characters were:

	Sylvian depression	Jugular foramen
Greater on right side	18	26
Equal	13	2
Greater on left side	1	4

The balance in favour of the right side shown here is similar to that which was observed in the case of Burmese crania†. All cases of asymmetry of other parts of the skull are noted; only 4 skulls were at all markedly asymmetrical.

* See Le Double, *Variations des Os de la Face*, p. 266.

† M. L Tildesley, "A First Study of the Burmese Skull." *Biometrika*, Vol. XIII. pp. 253, 255.

Malar bones. There were no cases of a complete horizontal suture across the malar bone, although in several there were slight traces of it.

Metopism. Six of the 32 Tibetan skulls had a persistent frontal suture. this is a very unusually high percentage.

Ossicles in the sutures. All ossicles of any considerable size have been noted. If one side of the lambdoid or coronal suture is not indicated, it is to be understood that where wormians are mentioned they were present on both sides. One fairly marked case of wormians of the lambdoid suture is figured Plate XII.

Conformation at the pterion. All cases of epipteric bones have been mentioned; also instances of the thrusting of a process of the squama temporalis between the parietal and sphenoid bones to join the frontal.

*Interparietal bones**. There was one case of a simple interparietal bone, two of the bipartite (rather different from that figured by Benington, *Biometrika*, Vol. VIII. p. 328) and no cases of the tripartite interparietal: see Plates XI and XII.

Other features. All cases of flattening of the obelion have been recorded together with any other singularities worthy of comment.

5. *Comparative Material.*

The greater part of Tibet is inhabited by seminomadic pastoral tribes.

"With the exception of the extreme northern and northeastern portions .....of Tibet, the population belongs essentially to one race, and the purest representatives of this stock are to be found among the pastoral tribes, or Drupa, which, whether found around the Kokonor, in eastern, western, or central Tibet, offer a uniform type which may be called the *Drupa type*†."

These people are said to be short and brachycephalic with high cheek bones, "but not so high as with the Mongols," noses usually narrow and strong but irregular teeth. This supposed primitive and pure type only persists among the nomadic tribes, while a hybrid one occupies the places of permanent habitation.

"It (the latter) is a mixed race, becoming more Chinese as one goes toward China, or more Indian (Nepalese or Kashmiri) as one travels southward or westward. The reason of the very pronounced departure of this portion of the Tibetan population from its original type is easily accounted for in the custom of foreign traders, soldiers, pilgrims, or officials inhabiting the country, of never bringing their wives into Tibet, but taking native concubines, a custom common in most parts of Asia. In as small a population as that of Tibet, which does not probably exceed 3,000,000, where the principal centres of population are and have been inhabited by comparatively large numbers of foreigners at least, this profound alteration of the primitive type is easily accounted for in this manner‡."

Our crania of Type A are, in all probability, those of such people, for they come from the line of the trade route from Bengal through Sikkim to Lhasa along

* See *Biometrika*, Vol. III. p. 220.

† W. W. Rockhill, "Notes on the Ethnology of Tibet" (1895), p. 673. From the *Report of the U.S. National Museum for* 1893.

‡ Rockhill, *op. cit.* p. 674.

which all but the nomadic tribes live. Indian traders do not, as a rule, themselves proceed beyond the confines of Tibet; their merchandise is carried to Lhasa by the Nepalese and Bhotanese. It is to these peoples that the supposed hybrid natives are probably racially akin, but unfortunately there are at present no data which can be used to test such a theory. It must not be forgotten that a few of the Type A crania come from Lhasa where the floating population is predominantly Chinese*.

Of the inhabitants of Khams, as the eastern portion of Tibet is called, W. W. Rockhill writes†, "there is absolutely nothing Mongol about this people, who are good representatives of old Tibetan civilisation." There are few Chinese traders, soldiers or officials in most parts of the country, but their number increases rapidly as the thickly populated boundary of western China is approached. We may hope then that the Type B crania are representative of the aboriginal inhabitants of the country and we should expect them to differ significantly from the hybrid Type A.

For comparative purposes we are able to use the table of means of several Asiatic races compiled by Tildesley from various scattered sources‡. This material is scanty and in some cases of very doubtful homogeneity, but together with the three Burmese series with which it has already been compared, it is, at present, all that is available and has been measured by methods sufficiently like ours to be comparable. The table of means is based on measurements given for the following races. We need only consider male skulls.

Malayans. These means are based on 78 skulls contained in a single collection.

Chinese. Measurements were collected from various sources and, although the part from which some of the skulls had been obtained was not known, the means may be taken to represent most nearly the South Chinese type.

Hindus. Only a portion of this racially heterogeneous people is represented. The material considered relates to Bengal and north-east India, all Dravidian tribes being excluded.

Dravidians. These are of the Maravar tribe from the Madras Presidency.

Burmese. Finally, there are the three Burmese series measured by Miss Tildesley and these were compared with the above racial material she had collected. The A series was of Burmans proper, the B of supposed hybrids and the skulls of the C series were thought to be of Karen origin.

Aino. These means, which were reduced in the Biometric Laboratory in 1900 from Koganei's tables, are quoted in *Biometrika*, Vol. I. p. 426.

The standard deviation of the cephalic index (B/L) may be conveniently taken to be an approximate measure of the homogeneity of a series of skulls, but it must not be forgotten that there is some evidence to show that advanced races are more variable than primitive ones.

* See L. A. Waddell, *Lhasa and its Mysteries* (1905), pp. 344—346.

† *The Land of the Lamas* (1891), p. 188 (footnote).

‡ *Biometrika*, Vol. XIII. p. 239.

Race	S.D. of B/L	Number of Crania
Aino*	$2{\cdot}41 \pm {\cdot}18$	88
Naqada*	$2{\cdot}80 \pm {\cdot}17$	130
Whitechapel English† ...	$3{\cdot}26 \pm {\cdot}20$	131
Tibetan A (Hybrids?) ...	$3{\cdot}87 \pm {\cdot}66$	17
Burmese A (Burmans) ...	$4{\cdot}30 \pm {\cdot}46$	44
Chinese	$4{\cdot}52 \pm {\cdot}29$	73
Tibetan B (Khams) ...	$5{\cdot}00 \pm {\cdot}94$	15
Malayans	$5{\cdot}13 \pm {\cdot}28$	77

It is curious, though no stress could be laid on the point without more complete evidence, that the variabilities in this index of the Oriental races, with the exception of the Aino, are significantly greater than those given for Western peoples‡. The probable errors of the standard deviations of the two Tibetan types are, of course, very large, but we may reasonably conclude that the collections are as homogeneous as those of the other Asiatic races with which they are to be compared. Table I (p. 206) gives a full list of mean measurements for the two Tibetan series and the other Asiatic ones with which comparison is to be made.

6 *The Coefficient of Racial Likeness.*

If an adequate comparison of the mean characters of two races is to be made, we require a knowledge of inter-racial standard deviations and correlations (i.e. of those of different races). The comparative material at present available is too meagre to provide even an approximation to these. Professor Karl Pearson has suggested that a single coefficient, known as the Coefficient of Racial Likeness (C. L.), which takes into account a large number of mean characters and requires intra-racial standard deviations (i.e. those of individuals all of one race) only, may be used provisionally.

Let m characters be measured and let the sth character in the first race have M_s for mean and σ_s for standard deviation, these two constants being based on n_s individual measurements. Let the corresponding characters for the second race be σ_s' and n_s'.

Then, if the two races were really samples of the same population, the following expression would be sensibly zero:

$$\frac{1}{m}\left\{S\frac{(M_s - M_s')^2}{\frac{\sigma_s^2}{n_s}+\frac{\sigma_s'^2}{n_s'}}\right\} - 1,$$

and the probable error of its deviation from zero will be ${\cdot}67449/\sqrt{2m}$.

The value of this expression computed from a number of mean characters of two races is the Coefficient of Racial Likeness between them and it is thus a measure of the probability of the two being random samples from the same population. It is not a true measure of absolute divergence, and must not for a moment be considered as such, but nevertheless we shall speak of it, for convenience, as if it were an absolute measure of racial affinity. When it is said that

* Quoted by Fawcett, *Biometrika*, Vol. I. p. 440. † Macdonell, *Biometrika*, Vol. III. p. 224.

‡ The variabilities of the nasal and orbital indices of the Oriental races are also significantly greater than those of the Egyptians and Whitechapel English.

TABLE I. *Comparative Table of Means* (all Male). Asiatic Races.*

Character	Tibetan A	Tibetan B	Burmese A	Burmese B	Burmese C	Malayan	Chinese	Maravar	Hindu	
Capacity	1438·0 (17)	1537·7 (14)	1406·9 (27)	1415·0 (4)	1442·2	[1424·1 (76)]	[1467·6 (46)]	[1289·7 (17)]	[1319·9 (34)]	[1462
W. in grs.	525·4 (17)	673·1 (15)	656·8 (38)	595·6 (7)	578·0		--			
F	174·4 (17)	184·5 (14)	172·4 (44)	172·4 (8)	175·5		179·6 (16)	175·6 (17)	175·9 (45)	
L'	175·2 (17)	185·7 (14)	174·2 (44)	176·7 (7)	177·2		179·2 (4)		178·2 (9)	
L	174·8 (17)	185·5 (14)	173·5 (44)	173·8 (8)	176·7 (8)	174·7 (78)	177·1 (84)	175·6 (21)	175·4 (33)	
B	139·4 (17)	139·4 (14)	143·7 (45)	141·1 (7)	140·4 (8)	142·2 (77)	139·5 (102)	131·4 (38)	132·8 (69)	
B'	92·6 (17)	94·3 (15)	91·3 (44)	89·7 (8)	90·4 (8)	93·4 (77)	93·9 (49)	93·2 (21)	92·4 (10)	
H	132·0 (17)	134·8 (15)	136·8 (43)	136·6 (7)	140·1 (8)	136·2 (76)	136·6 (37)	132·5 (38)	132·1 (48	
H'	131·2 (17)	134·1 (15)	136·0 (43)	134·7 (7)	139·1 (8)	137·4 (76)	136·9 (69)		131·5 (10	
OH	113·2 (17)	115·5 (15)	117·7 (44)	116·7 (7)	116·9 (8)	118·1 (77)	119·2 (38)	-	111·4 (10	
LB	95·7 (17)	99·2 (15)	98·5 (43)	98·8 (8)	100·5 (7)	99·5 (76)	99·1 (66)	98·8 (38)	99·2 (44)	
Q	310·5 (17)	312·1 (15)	323·7 (44)	319·2 (7)	317·8 (8)	–				
Q'	310·9 (17)	312·9 (14)	325·8 (43)	319·5 (7)	319·4 (8)	319·0 (77)	321·2 (33)		302·9 (10)	
S	360·9 (17)	378·6 (14)	363·7 (44)	367·0 (7)	367·4 (8)	365·7 (73)	370·3 (78)	—	363·5 (33)	
S_1	125·8 (17)	129·3 (15)	128·6 (44)	124·9 (7)	127·9 (8)	–	126·6 (57)	127·0 (21)	127·1 (33)	
S_2	122·6 (17)	126·1 (14)	124·0 (45)	124·4 (7)	122·5 (8)		126·0 (56)	—	126·5 (33)	
S_3	112·5 (17)	122·7 (15)	111·2 (44)	117·7 (7)	117·0 (8)		116·1 (56)	–	110·1 (33)	
S_3'	94·4 (17)	100·3 (15)	94·7 (44)	98·1 (7)	99·7 (8)	98·0 (73)	98·7 (19)	—	92·3 (9)	
U	503·6 (17)	525·6 (14)	505·7 (44)	497·7 (7)	503·1 (8)	505·7 (77)	508·5 (82)	488·7 (38)	493·5 (69)	
PH	18·9 (15)	22·0 (15)	19·9 (38)	17·2 (7)	20·1 (7)	--	–	—	--	
$G'H$	69·1 (15)	76·5 (15)	71·4 (39)	68·2 (8)	74·5 (7)	70·1 (73)	71·2 (49)	—	63·8 (9)	
GB	98·7 (17)	100·7 (15)	101·9 (40)	100·9 (8)	94·3 (7)	99·3 (76)	99·9 (38)	—	95·2 (9)	
J	130·4 (17)	137·5 (15)	134·0 (40)	131·7 (8)	126·7 (7)	133·2 (75)	131·8 (65)	124·4 (21)	126·8 (32)	
NH, R	51·7 (16)	54·9 (15)	53·4 (41)	52·3 (8)	55·1 (7)	51·9 (75)	53·1 (54)	46·5 (38)	49·0 (45)	
NH, L	51·6 (16)	55·1 (15)	53·5 (41)	52·4 (8)	55·2 (7)					
NB	25·3 (17)	27·1 (15)	28·1 (41)	26·6 (8)	25·5 (7)	26·1 (73)	25·5 (60)	24·0 (38)	24·2 (43)	
DS	8·1 (16)	8·8 (15)	10·3 (42)	10·6 (8)	11·6 (7)					
DC	22·1 (17)	22·7 (15)	22·8 (43)	21·3 (8)	18·7 (7)	--				
DA	28·0 (16)	29·4 (15)	32·6 (41	30·9	30·7 (7)	—				
SS	2·1 (16)	2·6 (15)	3·0 (41	3·5	3·6 (7)					
SC	7·4 (16)	7·7 (15)	9·0 (42)	7·8	7·2 (7)	--				
O_1R	41·7 (17)	44·0 (15)	44·2 (41)	44·3 (8)	44·4 (7)			—	42·6 (10)	
O_1L	41·7 (17)	43·4 (15)	43·6 (41)	43·2 (8)	44·1 (7)					
O_2R	35·0 (17)	36·5 (15)	35·0 (41)	33·6 (8)	35·6 (7)	33·5 (74)	(54)	31·7 (38)	32·4 (45)	
O_2L	35·0 (17)	36·7 (15)	35·0 (41)	34·3 (8)	35·9 (7)					
O_1'	39·5 (17)	41·2 (15)	39·6 (41)	39·1 (8)	39·4 (7)	39·1 (73)	38·3 (54)	36·6 (17)	37·6 (46)	
G_1	47·7 (17)	51·5 (13)	49·9 (37)	49·2 (8)	50·0 (7)	—	54·5 (4)	50·5 (21)	46·4 (5)	
G_1'	44·5 (17)	47·7 (13)	45·4 (37)	44·8 (8)	46·2 (7)	47·3 (69)	45·8 (32)	—	45·2 (7)	--
G_2	41·0 (15)	43·6 (14)	39·6 (40)	40·1 (6)	40·6 (7)	41·2 (70)	39·2 (35)		35·9 (9)	38·2
GL	91·5 (17)	97·2 (15)	96·6 (39)	93·8 (8)	95·9 (7)	98·2 (76)	97·7 (58)	96·0 (36)	95·1 (39)	104·9
EB	11·1 (15)	43·7 (14)	–							
EH	11·8 (15)	14·2 (14)	—	--						
fml	35·7 (17)	37·4 (15)	36·7 (43)	35·4 (8)	37·3 (8)	35·5 (73)	36·0 (23)	33·9 (21)	35·5 (8)	
fmb	30·1 (17)	31·4 (15)	31·7 (42)	30·6 (8)	30·8 (8)	30·0 (74)	30·9 (23)	28·2 (21)	27·4 (8)	
$100\,B/L'$	79·6 (17)	75·2 (14)	82·4 (44)	80·0 (7)	79·4 (8)		77·8 (13)	--	76·2 (9)	
$100\,H/L'$	75·2 (17)	72·3 (14)	78·2 (43)	76·3 (7)	78·7 (8)	--	—		75·5 (9)	
$100\,B/L$	79·8 (17)	75·3 (14)	82·9 (44)	80·4 (7)	79·5 (8)	81·7 (77)	78·9 (73)	74·6 (21)	75·8 (33)	76·5
$100\,H/L$	75·5 (17)	72·4 (14)	78·5 (43)	76·7 (7)	78·9 (8)	78·2 (76)	77·4 (21)	75·2 (21)	75·6 (9)	
$100\,B/H$	105·7 (17)	104·0 (14)	105·8 (43)	104·9 (7)	101·0 (8)	104·6 (76)	103·2 (37)	{99·1 (21)}	100·0 (43)	
$100\,H'/L$	75·1 (17)	72·1 (14)	{78·4 (43)}	{77·5 (7)}	{78·7 (8)}		77·4 (35)	-	75·8 (10)	
$100\,B/H'$	106·3 (17)	104·5 (14)	{105·7 (43)}	{104·8 (7)}	{100·9 (8)}	–	102·4 (46)	–	99·8 (10)	
$100(B-H')/L$	4·7 (17)	3·5 (14)	{4·5 (43)}	{2·9 (7)}	{0·8 (8)}	-	{1·5 (69)}	–	{0·0 (10)}	
$100\,G'H/GB$	70·4 (15)	76·0 (15)	69·8 (38)	67·8 (8)	79·0 (7)	70·7 (73)	71·1 (36)		68·0 (8)	
$100\,NB/NH, R$	49·3 (16)	49·5 (15)	52·8 (41)	51·2 (8)	46·4 (7)	52·7 (41)	51·0 (8)	46·3 (7)	50·4 (73)	
$100\,NB/NH, L$	49·3 (16)	49·3 (15)	52·7 (41)	51·0 (8)	46·3 (7)					
$100\,O_2/O_1, R$	84·3 (17)	83·0 (15)	79·1 (41)	75·9 (8)	80·3 (7)					
$100\,O_2/O_1, L$	84·2 (17)	84·6 (15)	80·0 (41)	79·4 (8)	81·5 (7)	-			77·2 (9)	
$100\,O_2/O_1'$	88·7 (17)	88·6 (15)	{88·4 (41)}	{87·7 (8)}	{91·1 (7)}	86·0 (73)	88·4 (54)	84·4 (17)	86·3 (45)	
$100\,fmb/fml$	84·3 (17)	84·1 (15)	83·8 (42)	86·6 (8)	82·9 (8)	84·6 (73)	85·9 (23)	83·3 (21)	77·3 (8)	
$100\,DS/DC$	37·0 (16)	39·1 (15)	45·4 (42)	49·8 (8)	61·9 (7)	--	—	-	—	
$100\,SS/SC$	31·5 (16	34·6 (15)	32·7 (41)	45·2 (8)	54·0 (7)	—	—	—	—	
$100\,G_2/G_1$	86·5 (15)	84·1 (12)	79·8 (37)	82·2 (6)	81·3 (7)		78·4 (4)	—	77·8 (5)	
$100\,EH/EB$	28·7 (15)	31·7 (14)	—	—	--		—	-	—	
$P\angle$	87°·4 (15)	85°·7 (14)	86°·0 (39)	85°·9 (8)	84°·0 (7)		84°·9 (17)	-	86°·1 (9)	82°·0 (67)
$N\angle$	64°·7 (16)	65°·5 (15)	66°·8 (38)	65°·2 (8)	64°·4 (7)	{68°·2 (73)}	{67°·3 (49)}	—	{67°·2 (9)}	{70°·2 (69)}
$A\angle$	71°·8 (16)	68°·8 (15)	70°·5 (38)	73°·4 (8)	71°·2 (7)	{70°·4 (73)}	{70°·1 (49)}	-	{74°·8 (9)}	{71°·2 (69)}
$B\angle$	43°·5 (16)	45°·7 (15)	42°·6 (39)	41°·4 (8)	44°·5 (7	{41°·4 (73)}	{42°·6 (49)}	—	{38°·0 (9)}	{38°·6 (69)}
θ_1	28°·0 (15)	28°·5 (14)	27°·2 (38)	28°·8 (8)	31°·7 (7					
θ_2	15°·5 (15)	17°·7 (14)	15°·4 (38)	12°·9 (8)	12°·8 (7					
$Oc.\ I.$	61·7 (17)	59·2 (15)	62·8 (44)	61·5 (7)	62·8 (8		{62·1 (19)}		{60·5 (9)}	

* The capacities in square brackets were found by methods which would only give results approximately the same as those given by the method described above (p. 198). The indices enclosed in curled brackets give the ratio of the two mean measurements and are therefore only approximate values. The angles in curled brackets were determined with the Trigonometer from the mean lengths of the sides of the fundamental triangle.

a low coefficient between two races A and B indicates a closer relationship than a higher coefficient between, say, A and C, what is meant always is that it is more probable that A and B are random samples from the same population than that A and C are.

Intra-racial correlations between characters not connected in some obvious way, such as the various measurements of head length, are known to be small; otherwise the coefficient could not theoretically be used. The measurements chosen should therefore show little correlation among themselves and at the same time be as representative as possible of all regions of the head. Further, the same list should as far as possible be used for computing all coefficients.

The standard deviations σ_s and σ_s' cannot be determined with at all sufficient accuracy from the short series of skulls with which, as a rule, the craniometrician is concerned. Accordingly the assumption has to be made that they are equal to each other and to the standard deviation of the longest homogeneous series of crania available. This supposes that the races considered are alike both in variability and homogeneity. This latter condition may seldom be closely fulfilled, but we are obliged to postulate it as long as the inter-racial variabilities are unknown. At least we have some indication from the standard deviation of the cephalic index that all the modern Asiatic races we shall deal with are of a similar degree of heterogeneity and this makes the coefficients derived from them more comparable.

If $\sigma_s = \sigma_s'$ then:

$$\text{C. L.} = \frac{1}{m} S \left\{ \frac{n_s n_s'}{n_s + n_s'} \left(\frac{M_s - M_s'}{\sigma_s} \right)^2 \right\} - 1 \pm \frac{\cdot 67449}{\sqrt{2m}},$$

and we shall use the coefficient in this form

Several series of sufficient length for which the variabilities of characters are given were found. These were:

(i) An unpublished series (E) of 800 male Egyptian crania; (ii) Naqada Prehistoric Egyptians measured by Fawcett (*Biometrika*, Vol. I. p. 438); (iii) Whitechapel English—Macdonell (*Biometrika*, Vol. III. p. 222); (iv) Moorfields English—Macdonell (*Biometrika*, Vol. V. p. 92); (v) Congo Negroes—Benington (*Biometrika*, Vol. VIII. p. 298).

A table of standard deviations compiled from these sources is given below.

The variabilities of the 5 races shown in Table II are in fairly close agreement and the Egyptians and Negroes do not seem to differ appreciably from the English. This fact to a certain extent justifies the invariable use of the standard deviations of one race only in computing Coefficients of Racial Likeness, but this use is still open to criticism if any of the data compared are suspected to be less homogeneous than the average. The standard deviations of the Egyptian Series E were used in calculating all Coefficients of Racial Likeness since the probable errors are smaller than those of the other races.

The need has been noted above of using, for the computation of the Coefficient of Racial Likeness, only characters which will be fairly independent of each other.

TABLE II.

Standard Deviation of Characters. All Male.

Race	B/L'	H/L'	B/H	$Oc.I$	$G'H/GB$	$O_2/O_1\,(R)$	$NB/NH\,(R)$	fmb/fml	G_2/G_1	$\hat{P}$	$\hat{N}$
Egyptians (E) ...	2·67 ± ·04	2·94 ± ·05	4·30 ± ·06	3·30 ± ·05	4·96 ± ·08	5·05 ± ·08	3·82 ± ·06	5·79 ± ·09	6·79 ± ·13	3·24 ± ·05	3·31 ± ·06
Naqada	2·88 ± ·13	2·77 ± ·13	4·53 ± ·19	—	4·52 ± ·25	5·00 ± ·27	4·18 ± ·23	—	7·36 ± ·43	2·87 ± ·17	3·77 ± ·21
Whitechapel English	2·97 ± ·17	2·67 ± ·15	5·14 ± ·23	—	5·39 ± ·35	4·66 ± ·27	4·58 ± ·26	—	6·40 ± ·39	3·92 ± ·24	3·52 ± ·20
Moorfields English	3·27 ± ·37	3·13 ± ·36	—	—	4·77 ± ·61	4·16 ± ·48	4·13 ± ·46	6·64 ± ·54	5·59 ± ·74	3·99 ± ·49	3·54 ± ·41
Congo Negroes ...	2·88 ± ·13	—	4·55 ± ·31	—	4·82 ± ·34	5·12 ± ·36	4·91 ± ·33	5·96 ± ·41	6·27 ± ·55	3·74 ± ·26	—

Race	$\hat{A}$	F	B	B'	OH	LB	Q	S	U	$G'H$	J
Egyptians (E) ...	3·46 ± ·06	5·73 ± ·09	4·76 ± ·08	4·05 ± ·06	4·12 ± ·07	3·97 ± ·06	9·75 ± ·16	12·51 ± ·20	13·77 ± ·22	4·15 ± ·07	4·57 ± ·08
Naqada	4·16 ± ·23	6·03 ± ·26	4·60 ± ·19	4·82 ± ·19	4·46 ± ·18	4·85 ± ·22	10·11 ± ·45	11·91 ± ·52	13·00 ± ·57	4·11 ± ·21	5·22 ± ·34
Whitechapel English	3·41 ± ·20	6·17 ± ·25	5·28 ± ·22	4·20 ± ·17	4·28 ± ·18	4·13 ± ·18	11·40 ± ·51	13·69 ± ·57	15·02 ± ·63	3·86 ± ·21	5·57 ± ·40
Moorfields English	3·71 ± ·43	5·90 ± ·42	5·31 ± ·37	4·12 ± ·29	4·69 ± ·33	4·57 ± ·37	12·54 ± 1·06	12·01 ± ·90	14·45 ± 1·13	4·08 ± ·43	4·65 ± ·84
Congo Negroes ...	—	6·55 ± ·44	5·00 ± ·34	3·85 ± ·26	4·05 ± ·28	4·58 ± ·32	10·29 ± ·72	10·73 ± ·72	17·24 ± 1·16	4·14 ± ·28	6·90 ± ·57

Race	NH, R	NB	O_1R	O_2R	G_1	G_2	fml	fmb	C
Egyptians (E) ...	2·92 ± ·05	1·77 ± ·03	1·67 ± ·03	1·91 ± ·03	3·33 ± ·06	2·63 ± ·05	2·47 ± ·04	2·15 ± ·03	113·51 ± 1·97
Naqada	3·00 ± ·05	1·98 ± ·10	2·14 ± ·11	2·32 ± ·13	3·62 ± ·20	3·75 ± ·21	—	—	106·68 ± 5·42
Whitechapel English	2·60 ± ·14	2·16 ± ·12	2·02 ± ·12	2·22 ± ·13	2·74 ± ·16	2·85 ± ·17	—	—	122·37 ± 6·88
Moorfields English	2·60 ± ·28	1·90 ± ·21	1·42 ± ·16	2·12 ± ·24	4·25 ± ·49	2·36 ± ·29	2·86 ± ·23	1·82 ± ·15	132·18 ± 13·44
Congo Negroes ...	2·83 ± ·19	2·05 ± ·14	1·64 ± ·12	1·95 ± ·14	4·19 ± ·39	3·95 ± ·28	2·39 ± ·16	2·30 ± ·16	126·57 ± 8·81

For example, if the sagittal arc S is employed, the arcs S_1, S_2, S_3 and the chord S_3' may not be for they are significantly correlated with S; two angles of the fundamental triangle may be included but not three. The correlations between the arcs and lengths of the skull not connected in any such close way and those between an index and its component lengths are mostly small, although some are quite significant, and so provisionally we may use the 31 measurements, for which the standard deviations are given in Table II*. The correlations between these, if sensibly differing from zero, are fairly small and it does not seem possible to advantageously add to the list. If the coefficients computed by different workers are to be comparable, some such list must be adopted and adhered to. Unfortunately all these mean measurements will not be given for some comparative material, but the majority are standard ones adopted by most craniometricians. Our most meagre data are those for the Maravar crania for which there are only 19 of the 31 chosen characters. For all the Asiatic material the transverse arc was given only measured to the auricular points (Q) and not to the ear-rods (Q). But we were only provided with the standard deviation of Q and this was used instead of that of Q'; the difference between these two variabilities would certainly be small. In comparing the Tibetan means with those of unallied races Q and not Q' was used. Besides the Coefficient of Racial Likeness computed from the 31 characters, we have computed the coefficient for indices and angles alone, using the 12 given in Table II.

The results for the two Tibetan and other races are given below (Table III).

We shall first consider the relations between the two Tibetan series and other Asiatic races as shown by the Coefficients of Racial Likeness computed from all the characters - lengths, arcs, indices and angles. There is clearly a remarkable difference between the two types: the coefficients with Type B are in every case larger and in most cases approximately twice as large as those with Type A. Our supposed hybrids (A) are seen to be very similar to the Burmese hybrids and more akin to all the Burmese, the Malayans and the Chinese than to their own countrymen of Khams†. This is distinct evidence in support of the hypothesis which would make Type A a mixed race. To the aboriginal Dravidian (Maravar), who rarely leaves the confines of his own land, the western Tibetan has no relationship but the closer affinity to the Hindu is, perhaps, due to or symptomatic of contact through trade. Our Hindu data are representative of the Bengali, a supposed eastern people, and not of migrants from the west. There is little direct contact in these days between western Tibet and Burma and certainly much less than between China and Tibet, so our figures are quite consistent with the philological and historical evidence which tells of Tibeto-Burmese invasions from north-west China and they show that all these oriental peoples are very similar in type.

The Tibetan Type B is most nearly, though not closely, related to the Burmese

* These 31 characters were chosen so that no pair would be highly correlated and there was no other method of selection. They include all the more important measurements of the skull.

† For the Moorfields and Whitechapel English measured by Macdonell I find C. L. $= 2{\cdot}05 \pm {\cdot}09$ using 26 characters and $1{\cdot}73 \pm {\cdot}16$ for indices and angles only (9 characters), so the coefficient for Tibetan A and Burmese B must be supposed to indicate quite a close relationship.

TABLE III.

*Coefficients of Racial Likeness. All Male**.

		Tibetan, Type A						Tibetan, Type B (Khams)					
		All Characters			Indices and Angles only			All Characters			Indices and Angles only		
		C. L.	Probable Error	Number of Characters	C. L.	Probable Error	Number of Characters	C. L.	Probable Error	Number of Characters	C. L.	Probable Error	Number of Characters
Asiatic Races	Burmese B (Hybrids)	·80	± ·09	31	·33	± ·14	12	4·83	± ·09	31	4·11	± ·14	12
	Burmese C (Karens)	2·36	± ·09	31	2·27	± ·14	12	3·46	± ·09	31	3·67	± ·14	12
	Malayan	2·88	± ·10	21	1·13	± ·21	5	11·17	± ·10	21	15·99	± ·21	5
	Chinese	3·23	± ·10	24	·82	± ·18	7	6·37	± ·10	24	2·21	± ·18	7
	Burmese A (Burmans)	5·71	± ·09	31	4·87	± ·14	12	10·96	± ·09	31	14·07	± ·14	12
	Tibetan B	6·46	± ·09	31	3·44	± ·14	12	—	—	—	—	—	—
	Hindu	6·56	± ·09	28	5·71	± ·16	9	18·34	± ·09	28	4·78	± ·16	9
	Maravar	12·34	± ·11	18	10·96	± ·21	5	25·72	± ·11	18	3·65	± ·21	5
	Tibetan A ..	—	—	—	—	—	—	6·46	± ·09	31	3·44	± ·14	12
	Aino	13·82	± ·10	23	9·84	± ·17	8	13·61	± ·10	23	11·94	± ·17	8
Other Races	Congo Negroes ...	7·23	± ·09	29	11·55	± ·15	10	21·61	± ·09	29	18·68	± ·15	10
	French	9·12	± ·10	25	9·47	± ·17	8	22·96	± ·10	25	16·96	± ·17	8
	Moriori...	12·78	± ·09	30	7·38	± ·14	11	4·85	± ·09	30	4·74	± ·14	11
	Whitechapel English...	14·59	± ·10	24	11·97	± ·16	9	11·93	± ·10	24	5·32	± ·16	9
	Naqada (Egyptians) ...	17·64	± ·09	27	23·57	± ·15	10	18·32	± ·09	27	11·57	± ·15	10

* The references to the tables of means for the races other than Asiatic are:

Congo Negroes. Biometrika. Vol. VIII. p. 298. This is the series which was measured by Dr Benington.

Whitechapel English and French (the latter quoted by Macdonell). *Biometrika,* Vol. III. p. 208.

Moriori. Biometrika, Vol. XI. p. 93.

Naqada and Aino (quoted by Fawcett). *Biometrika,* Vol. I. p. 426.

hybrids and Karens. It has equal affinities with the Chinese and Tibetan A. The coefficient with the Maravar is the highest recorded in the table, for both Asiatic and western races, and it must indicate a very fundamental difference in type. The Tibetan B is markedly differentiated, too, from the Hindu, Malayans and Burmans. If, as the close relationships of our Type A with these peoples suggested, they had all a common primitive origin then the Type B must belong to a radically different stock. Though adjacent to the Chinese it is much less like that people than are the inhabitants of Sikkim 600 miles away. The fact that the Tibetans of Khams resemble none of their neighbours suggests that they belong to a well differentiated and possibly pure stock.

We turn now to the coefficients between Tibetan A and the races other than Asiatic. These all appear to be smaller than we should have anticipated when compared with the earlier ones considered. It is surprising to find that two Tibetan peoples are hardly more alike in cranial type than one of them is to a race of African negroes and that Frenchmen have heads that more closely resemble those of a Tibetan people than the latter do those of Indians living not a 1000 miles from them.

Turning to the coefficients with Tibetan B, the outstanding feature is the lowness of that with the Moriori. Even among Asiatic races there are only two, Burmese B and C, smaller than it. Only more abundant comparative material can show what the significance of this somewhat unexpected similarity between two primitive peoples is*.

The primitive Aino are almost as far removed from the two Tibetan types as any of the races.

If mere size is to have no part in determining racial affinities then we must base judgments on indices and angles only. To test the accordance of such a method with the more general one, which considers both size and form, the Coefficients of Racial Likeness were computed without including lengths and arcs and these are given in Table III.

In most cases these coefficients were lower than those previously found—there being 21 such cases out of a total of 28—and many of the differences were very considerable. Looking first at the coefficients for Tibetan A and other Asiatic races: every one of these deduced from indices and angles only is smaller than the corresponding one for all the characters and the races are now arranged in a much more suggestive order. After the Burmese hybrids, the Chinese have cranial indices and angles most similar to those of the Tibetan A type and the resemblance is very close indeed. It is noteworthy, that the corresponding coefficient with Tibetan B is the lowest of all with that type, so the Chinese skull is very similar in shape if not in size to that of both our series. The coefficient for Tibetan A and Tibetan B is lessened quite considerably by taking fewer characters, but there is a far greater reduction than that in the case of the Hindu and Maravar with Tibetan B. There would seem to be a great difference between the Tibetan B

* Our coefficients are unfortunately not comparable with those given by Miss Tildesley in her Burmese paper as I have not used precisely the same characters as she did.

type and those of the two Indian peoples in size, the Tibetan being much larger than the others, but comparatively little in shape of head as characterised by angles and indices. But in one case the coefficient was calculated from only 9 characters and in the other there were only 5 available. We are dealing here with small samples and no stress can be laid on conclusions deduced from them. It may be that there are some characters which usually differ more significantly between race and race than other characters do and if these are neglected when a coefficient is being calculated from a small number of means we may arrive at very misleading results.

If the Coefficients of Racial Likeness between the two Tibetan and non-Asiatic races are compared, the values given for indices and angles only are not, with one exception, viz. that of the Whitechapel English and Tibetan B, smaller than the coefficients for all characters and in this respect there appears to be a very real difference between the allied and unallied races. Further evidence is needed to establish definitely the theory, but it seems at present to be highly probable that differences in size are of relatively little importance; resemblance between the shapes of heads is the real criterion of relationship and this we are able to measure with angles and indices.

7. *Comparison of the Tibetan mean direct Measurements with those of other Races*

The means of samples taken even from a very skew frequency distribution are known to have themselves a distribution which is nearly normal, so that if the difference of two mean characters divided by the probable error of that difference (i.e. approximately $\dfrac{M_s - M_s'}{\cdot 67449\sqrt{\dfrac{\sigma_s^2}{n_s} + \dfrac{\sigma_s'^2}{n_s'}}}$, with the notation of p. 205) is between 0 and 2 then the means might very probably have been random samples from the same population; if between 2 and 3, the inference is probable but uncertain; while a value over 3 denotes great improbability (of the order ·002). Hence if the value of $\left\{\dfrac{(M_s - M_s')^2}{\dfrac{\sigma_s^2}{n_s} + \dfrac{\sigma_s'^2}{n_s'}}\right\}$ is between 0 and 2·7 a verdict can be given in favour of common origin; between 2·7 and 6·1 it is uncertain and greater than 6·1 very improbable. After making the assumption that the two standard deviations may be considered equal, this function, i.e. $\dfrac{n_s n_s'}{n_s + n_s'}\left(\dfrac{M_s - M_s'}{\sigma_s}\right)^2$ which we will call α, becomes one which has already been evaluated for the individual characters of the Tibetan and other races, when computing the Coefficients of Racial Likeness. We have C.L. $= \dfrac{1}{m} S(\alpha) - 1$. From the values of α we can see at once in which characters the two races compared differ most essentially and in which they are most alike. Tables IV A and IV B give these values of α.

All the Burmese types, the Malayan and the Chinese are more closely related to the Tibetan A race than the Tibetan B is. In what ways do these two differ

TABLE IV A.

Values of $\frac{n_s n_s'}{n_s + n_s'} \left(\frac{M_s - M_s'}{\sigma_s} \right)^2$ for Tibetan A and other Races.

		Asiatic Races									Other Races				
		Burmese B (Hybrids?)	Burmese C (Karens?)	Malayan	Chinese	Burmese A (Burmans)	Tibetan B	Hindu	Maravar	Aino	Congo Negroes	French	Moriori	Whitechapel English	Prehistoric Egyptians (Naqada)
Indices	B/L	·11	·03	8·55	3·34	13·19	20·82	9·54	32·94	19·21	1·55	13·24	15·28	37·03	97·16
	H/L	69	7 71	—		12·69	7 48	·06	·00	·26	·06	12·69	5·25	36·38	6·71
	B/H	·17	6·79	90	3·94	·01	1·23	21·39	22·13	36·67	2·99	11·28	·88	·28	28·32
	$Oc. I.$	·02	·60	—	–	1·36	1·57	—	–	—	58·06	—	—	–	—
	$G'H/GB$	1·43	14·35	05	·21	·16	9·56	1·22	—	1·99	5·63	6·94	5·80	12·36	·03
	$O_2/O_1(R)$	9·01	3·11	—	—	14·44	·53	11 63	—	·55	·02	4·60	·04	23·23	48·14
	$NB/NH(R)$	29	3·00	1·09	·25	9·66	·02	·13	1 45	1·78	28·76	13·64	21·73	2·58	2·94
	fmb/fml	·86	·32	·04	·19	10	·01	7·96	·28	–	·24		3·16	—	–
	G_2/G_1	1·72	2·82		1·49	10·40	·83	7 65	—	8·44	20·41	10·51	39 94	–	56 67
Angles	$P\angle$	·50	·37	—	·00	1·25	1·99	·77	—	9·82	4·78	2·89	·04	1·82	·29
	$N\angle$	·07	·04			4·53	·46	–	—	—	—	–	·01	·30	4·32
	$A\angle$	1·14	·15	—		1·59	5·82	—		—	—	—	00	2·78	1·10
Lengths	F	66	·20	—	6 79	1·50	23·87	·85	·37		1·12	10·78	20·15	77·98	41·89
	B	·63	·24	1·80	00	10·06	·00	30·29	33·11	2·04	·44	9·19	1·99	1·12	13·51
	B'	2·60	1·61	·54	1·30	2·16	1·40	·01	·21	11·90	18·57	10·30	5·09	26·78	2·08
	OH	4·67	5·64	23 88	29·19	18·04	2·48	·62	—	36·50	1·06	·03	12 91	3·20	7·50
	LB	3·31	8·48	12·70	9·90	6·04	6·18	9·50	7·14	84·85	4·97	13·21	70·30	32·77	12·06
Arcs	Q' or Q*	3 75	4·02	9·34	12·85	27·66	·31	4·12	—		·32	·22	2·90	—	6·81
	S	1·18	1 47	2 03	7·89	·61	15·37	·48	–	12·60	·06	2·16	1·52	25·23	13·92
	U	·91	·00	·32	1·79	·16	19·60	7·34	13 75	26·84	·05	11·66	22·03	34·01	4·29
Face	$G'H$	·41	7·22	·36	2·16	2·52	21·98	10·25	—	·11	24·01	·69	30·35	·46	2 41
	J	·44	3·25	5·20	1·27	7·40	19·21	6·83	16·19	31·21	8·17	·05	26·56	·05	11·18
Nose	$NH(R)$	·22	6 99	·07	2·84	3·90	9·29	10·09	35·69	2·25	28·61	·37	40·36	·40	12·50
	NB	2·94	·06	2·83	·17	30·10	8·25	4·71	6·34	·40	1·99	20·16	·00	4·36	·17
Orbit	O_1R	13·18	12·96	—	—	26·92	15·11	1·83		3·20	1 60	57·26	29 90	8·24	4·07
	O_2R	2·87	·49	8 52	5·11	·00	1·91	22·85	35·05	·04	·55	10·33	16·42	9·56	36·21
Palate	G_1	1·10	2·37	—	—	5·08	9·59	·59	6·64	35·10	6·62	1·43	29·50	--	81·99
	G_2	·50	·11	·07	1·91	3·09	7·07	21·11		14·13	7·21	21·69	6·63	31·15	·89
For. Mag.	fml	·08	8·76	·10	·15	1·48	3·78	·04	4·99	—	·09	—	·29	--	—
	fmb	·29	·88	·03	1·36	·94	2·92	8·58	7·34		·03	—	1·20	—	
Capacity	C	·08	·02	·08	1·15	·56	6·53	11·23	13·55	·90	7·69	1·62	·11	2·06	2·99
C. L.		·80	2·36	2·88	3·23	5·71	6·46	6·56	12·34	13·82	7·23	9·12	12 78	14 59	17·64

* See p. 209.

TABLE IVB.

Values of $\dfrac{n_s n_s'}{n_s + n_s'}\left(\dfrac{M_s - M_s'}{\sigma_s}\right)^2$ *for Tibetan B and other Races.*

		Asiatic Races									Other Races			
		Burmese C (Karens[1])	Burmese B (Hybrids?)	Chinese	Tibetan A	Burmese A (Burmans)	Malayan	Aino	Hindu	Maravar	Moriori	Whitechapel English	Prehistoric Egyptians (Naqada)	Congo Negroes
Indices	B/L'	12·60	15·08	6·39	20·82	77·22	69·70	2·86	·77	·43	2·35	·00	10·78	12·03
	H/L'	24·14	8·64	—	7·48	42·54	—	15·22	6·49	8·17	·93	5·12	1·15	12·05
	B/H	2·49	·21	·36	1·23	1·85	·23	17·66	9·12	10·90	·13	3·57	12·07	·10
	Oc. I.	6·20	2·31	—	4·57	13·30	—	—	—	—	—	—	—	96 63
	$G'H/GB$	1·75	14·26	10·33	9·56	15·64	11·21	30·20	13·58		1·53	·12	14·84	38·07
	$O_2/O_1\ (R)$	1·37	5·75	—	·53	6·54	—	2·61	7·43		·40	13·54	32·23	·53
	$NB/NH\ (R)$	3·35	1·03	·29	·02	8·19	·70	1·25	·03	3·57	22·36	2·92	2·20	25 55
	fmb/fml	·22	98	·88	·01	·03	·10	—	7·20	·17	3·32	—	-	13
	G_2/G_1	·75	·31	2·12	·83	3·63	—	32·13	3·04	—	22 86		32·86	11·56
Angles	$P\angle$	·28	3·53	2·11	1·99	7·69		1·59	4·39	—	99	9 17	1·56	·16
	$N\angle$	·53	·05	—	·46	1·63		-	—	—	·47	·10	1 37	—
	$A\angle$	2·30	9·22	—	5 82	2·55		—	-	—	7·83	21·78	16·64	—
Lengths	F	12·57	22·73	5·47	23·87	47·41	—		24·08	18·54	1·91	3·25	·10	22 12
	B	·22	·59	·01	·00	8·70	4·10	1·73	25·81	28 81	1·75	9·43	11·35	39
	B'	4·84	6·44	·11	1·40	·00	·94	2·82	1·32	·65	·64	11·25	8·45	7·20
	OH	1·11	·81	11·64	2·48	4·80	7·09	13·94	4·57	--	1·58	·12	·28	·81
	LB	·51	·05	·01	6·18	·34	·08	31·18	·00	11	26 98	4·86	01	72
Arcs	Q' or Q^*	2·20	2·08	7 25	·31	17·97	4·28	—	5·96	-	·91		9·67	1 43
	S	4·08	4·01	·63	15·37	15·07	6·60	2·72	14·37	-	6·09	·18	2·51	19 72
	U	13·59	19·16	18·44	19·60	22·19	24·75	·62	63·15	73·46	·41	·11	14·07	26 17
Face	$G'H$	1·11	20·89	18·75	21·98	16·37	29·59	32·48	52·81	—	·01	28·53	58·71	114·58
	J	26·66	8·40	18·95	19·21	6·39	11·06	·02	60·88	71·85	·00	29 13	79·22	59·70
Nose	$NH\ (R)$	·05	4·13	4·46	9·29	2·90	13·19	28·62	45·91	88·92	7·09	20·22	54·34	79·83
	NB	3·90	42	9·82	8·25	3·51	3·97	9·05	21·69	33·00	10·88	29·46	16·32	4·47
Orbit	O_1R	·27	·17		15·11	·16	–	43·42	4·11	—	·60	4·42	8·90	34·09
	O_2R	1·06	12·02	23·44	4·91	6·76	30·76	8·84	51·81	67·95	1·82	32·44	71·96	11·19
Palate	G_1	·92	2·36	-	9·59	·22		2·25	8·47	·72	2·15	—	18·47	1·18
	G_2	6·07	7·43	27·98	7·07	23·98	9·71	49·61	46·95	--	31·66	77·18	18·57	34·39
For. Mag.	fml	·01	3·42	2·91	3·78	·89	7·37	—	3·09	17·57	7·46	—	-	4·22
	fmb	·41	·73	·49	2·92	·12	5·29	—	18·08	19·39	·81	—	-	4·85
Capacity	C	2·61	3·63	4·09	6·53	12·24	11·77	5·26	36·51	36·64	9·68	3·36	23·02	31·44
C. L.		3·46	4·83	6·37	6·46	10·96	11·17	13·61	18·34	25·72	4·85	11·93	18·32	21·61

[1] See p. 209.

most? Of the 31 values of α given there are 15 greater than 6·1. The largest of these are for the characters B/L', F, S, U and $G'H$, the B type being in every case the larger. Its greater size, weight and capacity are thus chiefly due to its excess length; in breadth and height, as shown by the low values of α for B, B', OH and Q, the two are not significantly different. Another very noticeable difference between them is for the size and shape of the face ($G'H$, J and $G'H/GB$), the B type being again the larger. The orbits, nose and palate differ in the same way in all dimensions, though they are remarkably similar when measured indicially. In fact these two races apparently resemble each other in both size and shape only in the neighbourhood of the transverse section through the auricular points and in the *foramen magnum*. The Eastern Tibetan is altogether a bigger type than its compatriot one of the West.

We turn now to compare the races more like the latter with it, and we take first the Burmese B. This series only consisted of 8 skulls and no great reliance can be placed on deductions drawn from its mean values, but it seems to be remarkably closely related to the Tibetan A. Both these, it is interesting to notice, have been supposed to represent hybrid stocks Only for one direct measurement do they differ by more than the amounts we should expect from random sampling. That is in the width of orbit for which the Burmese exceeds the other by 2·6 mm. The three Burmese types are very alike in that particular. The C is distinguished from the Tibetan by a greater upper face height ($G'H$), giving a very different facial index ($G'H/GB$), and OH height (affecting the indices H/L' and B/H). The excess in alveolar-basion and foraminal lengths is also noticeable.

The Burmese A series is the only one with which reliable comparison can be made since the others are based on such short series. The mean breadth of the calvaria and face and the OH height are appreciably greater than for the Tibetans, though the index B/H is almost the same for both and the measurements made in the sagittal plane are very similar. Thus the discrepancies between the lengths and arcs of the brain-box are almost the exact reverse of those between the Tibetan B and A types. The Burmese A differs most significantly from the latter with its much smaller nasal breadth which gives the very high value $\alpha = 30·1$. The difference in palate index is well marked.

The characters for which the α's with the Malayan type are greater than 6·1 are B/L', LB, Q', O_2R and most notably OH, the Tibetan mean being 5·4 mm. less than the other; and with the Chinese we find F, LB, Q', S and OH, the last showing again the highest value. We should expect to find the Malayans and Chinese quite closely related since they appear to resemble and differ from the Tibetans in very much the same way. The comparison of the Chinese with our A type proves particularly interesting. For all the indicial measurements and angular measurements and for those of the face, nose, orbit (though O_1R is not available), palate and *foramen magnum* the value of α is less than 6·1 and for 9 of the 15 of these it is less than 2·7. All the significant differences are in the lengths of the calvaria. In breadth the two only differ by 0·1 mm. but in length (F) the Chinese exceeds the Tibetan by 5·2 mm., for LB 3·4 mm., for Q' 10·3 mm., and for S 9·4 mm.

The lesser breadth of the Hindu skull is the feature which distinguishes it most from that of its northern neighbour. In size, as shown by the longer lengths and arcs, the two do not differ greatly, but all the facial measurements are characteristic of each. The Maravar is evidently a much more divergent type than any we have considered yet. Its points of greatest difference and similarity with the Tibetan appear to be scattered fortuitously throughout the short list. The characteristic nose of the Dravidian finds no counterpart here.

The smallness of many of the mean measurements of the Tibetan A type has been repeatedly noticed in comparing it with these other Oriental peoples and that has accounted for many of its greatest divergencies from them. Consulting the table of means (Table I), we find that it is smaller than all the others in the measurements LB, S, and O_1R (but not O_2R), and if the Hindu and Maravar are excluded the following can be added: OH, Q', NH, NB, G_1 and B (this last being equal to the Tibetan B breadth). The smallness of the transverse vertical section should be most apparent. The Indian peoples appear to resemble the Tibetan A most in absolute size. The Coefficients of Racial Likeness between the Tibetan A and other Oriental races computed for indices and angles alone were found to be all smaller than those derived from the complete list of characters; the reductions were greatest in the case of the Malayans and Chinese. Now it has been shown by comparing the individual characters that the Tibetan only differs significantly from these two in having a smaller calvaria and to show how similar the other measurements are I have calculated the coefficients from them alone*. These are given in the following table:

TABLE V.

Comparison of Coefficients of Racial Likeness for various sets of characters. Tibetan A with other Races.

	All Characters		Angles and Indices only		All characters except those of calvaria	
	C. L.	Number of Characters	C. L.	Number of Characters	C. L.	Number of Characters
Burmese B (Hybrids?)	·80 ± ·09	31	·33 ± ·14	12	1·14 ± ·11	18
Burmese C (Karens?)	2·36 ± ·09	31	2·27 ± ·14	12	2·74 ± ·11	18
Malayan	2·88 ± ·10	21	1·13 ± ·21	5	·67 ± ·14	11
Chinese	3·23 ± ·10	24	·82 ± ·18	7	·80 ± ·13	13
Burmese A (Burmans)	5·71 ± ·09	31	4·87 ± ·14	12	5·86 ± ·11	18
Tibetan B	6·46 ± ·09	31	3·44 ± ·14	12	5·74 ± ·11	18
Hindu...	6·56 ± ·09	28	5·71 ± ·16	9	6·26 ± ·12	16
Maravar	12·34 ± ·11	18	10·96 ± ·21	5	12·00 ± ·16	9

The Chinese, Malayans and Western Tibetans all have similar skulls, but the calvariae differ more than the faces. In spite of their very close resemblance the Tibetan A is smaller for nearly all the direct measurements than the other

* That is to say, the measurements used were $G'H/GB$, O_2/O_1 (R), NB/NH, fmb/fml, G_2/G_1, $\hat{P}$, $\hat{N}$, $\hat{A}$, $G'H$, J, NH (R), NB, O_1R, O_2R, G_1, G_2, fml, fmb—18 in all.

two; it exceeds the Chinese only in the lengths G_2 and O_2R and the Malayans in L, O_2R, fml, fmb and capacity. To produce the Chinese or Malayan skull from the Tibetan we should have to make the latter increase in all directions at the same time, the rate of increase being, perhaps, greater in proportion for the calvaria than for the face.

There are 25 indicial and direct measurements given for the Maravar (Dravidian) race in the Comparative Table of Means (Table I) and if these are compared with the Tibetan A and Hindu (Bengali) means the latter will be found, for the majority of the characters, to be intermediate in size between the other two. There are 8 exceptions to this rule, viz. for H/L, fmb/fml, F, B', LB, S_1, G_1 and fmb. The so-called Hindu is apparently more akin to the Dravidian than to the Tibetan type.

Turning to Table III we find that every single Coefficient of Racial Likeness of the Tibetan B type with any Asiatic race other than the Aino is higher than the corresponding one of the Tibetan A, but again the Burmese series B and C have the lowest values. For the latter of these α is over 6·1 for the characters B/L', H/L', $Oc\ I.$, F, U and J and for the former, B/L', H/L', F, U, J, $G'H/GB$, $A\angle$, B', $G'H$, O_2R and G_2. They both evidently differ from the Tibetan in much the same way. Their lengths (F) are very significantly less, but the breadths are much the same for all three types.

All the greatest differences between the Tibetan B and Chinese are in the facial region, the most marked being between the face heights ($G'H$) and zygomatic breadths (J), the heights of the orbits (O_2R) and the lengths of the palate (G_2). The Chinese is the smaller in all these cases as it is in horizontal circumference (U), but this order is reversed for the characters OH and Q. The Tibetan B type has a longer head than any of the Asiatic races we are able to compare it with, but, at the same time, it is unusually low in proportion to its length so that its indices H/L', H/L, and H'/L are considerably lower than any of the others. Its cephalic index, too, is, with the exception of that of the Maravar, the lowest shown.

The Burmese A type appears to diverge much more widely from the Tibetan B than any we have yet compared with the latter. Of the 31 values of α given there are 17 greater than 6·1 and 5 of these are greater than 20. The difference in length (F), which is reflected in the indices B/L' and H/L', is extreme and actually far greater than any between the Tibetan and quite unallied races. The greater differences between the Hindu and Maravar and the Tibetan B are confined almost entirely to the direct measurements. When angles and indices only are considered the Coefficients of Racial Likeness are lowered in a very remarkable way (see Table III) and we cannot yet say definitely what the significance of such a change is. The difference in size appears to be relatively greater for the face than for the calvaria.

In absolute size the Tibetan A crania were shown to be smaller than those of any other of the Asiatic races, with the possible exception of the Hindu and Maravar. The B type is at the other extreme, being, without doubt, larger than

any of the others. It exceeds them for the characters F, L', B' (with Burmese A), S_1, S_1', S_3, S_3', U, $G'H$, J, O_2R, G_2, fml, fmb and C.

A comparison of the Tibetan means with those of the races other than Asiatic shows that the high and low values of α are scattered more or less at random throughout the list of characters. Two columns require particular mention. The first is that showing the relationships between the Tibetan A people and the Congo Negroes; α has a value greater than 6·1 for the measurements *Oc. I.*, $NB/NH\ (R)$, G_2/G_1, B', $G'H$, J, $NH\ (R)$, G_1, G_2 and C. The chief differences are thus between the noses, palates and face breadths, but far more significant than any of these is the occipital index. That character is given for so few of the races that we can, at present, infer very little from it. But except for the lower occipital region, the calvariae of the Negroes and Tibetans are very similar both in size and shape.

The Coefficient of Racial Likeness for the Tibetan B type and Moriori is remarkably low. The characters giving values of α over 6·1 are $NB/NH\ (R)$, G_2/G_1, LB, NH, NB, G_2, fml, and C which show that all the significant differences are confined to the nasal regions and the base of the skull from opisthion to alveolar point.

Having considered the vertical arrays of Tables IV A and IV B we may ask if the horizontal ones are of any interest. These may show, if we consider both the Asiatic and the other races, what measurements are most likely to characterise the racial skull and in what ways all races resemble each other most closely. Or, we may hope to discover, if there are any characters which are alike for all the members of a family of races, while differing from unallied races. But far more ample and reliable comparative material than we have at present will be needed before any definite answers can be given to such questions.

We will consider first the relationships of the Tibetan A mean measurements with those of other races. There seems to be no indicial or angular measurement here which has not high and low values of α for both of the two racial groups (Asiatic and other races). The palatine index (G_2/G_1) is the only one which affords any clear distinction between them, being, apparently, fairly consistently lower (i.e. in the value of α) for the Asiatics. The corresponding array of Table IV B confirms this idea. The following table of the index shows that the squarer palate of

Values of the Palatine Index for Various Races.

Race	$100\ G_2/G_1$	G_2	G_1
Prehistoric Egyptians (Naqada)	71·9	40·1	55·8
Aino	72·1	38·2	53·0
Modern French	72·2	37·2	46·6
Moriori	73·2	38·9	53·1
Modern German	75·4	33·2	44·3
Whitechapel English	76·3	36·8	48·3
Congo Negroes	76·8	38·9	50·3
Hindu (Bengali)	77·8	35·9	46·4
Chinese	78·4	39·2	54·5
Burmese A	79·8	39·6	49·9
Burmese C	81·3	40·6	50·0
Burmese B	82·2	40·1	49·2
Tibetan B	84·1	43·6	51·5
Tibetan A	86·5	41·0	47·7

all the Asiatics but the Aino, does distinguish them from most other races. The index itself provides a much more suggestive order than either of its components (G_2 and G_1) alone do.

The only absolute measurement which appears to be at all likely to differentiate Asiatics from the other races is the forehead breadth (B'). The latter, with the exception of the Naqada Egyptians, are all larger than the others in that particular. B' does not seem to be at all highly correlated with the zygomatic breadth, J. The Aino, again, show no resemblance to other Asiatic peoples.

Values of B' and J for Various Races.

Race	B'	J
Modern German . . .	103·7	135·0
Whitechapel English	98·0	130·4
Congo Negroes	97·5	126 5
Aino	96·2	137·3
Modern French	96·2	130·7
Moriori	95·3	137·4
Tibetan B	94·3	137·5
Burmese A	94·3	134·0
Chinese	93·9	131·8
Malayan	93·4	133·2
Maravar	93·2	124·4
Tibetan A	92·6	130·4
Hindu	92·4	126·8
Prehistoric Egyptians (Naqada)	91·1	125·6
Burmese C	90·4	126·7
Burmese B	89·7	131·7

There seems to be no other character which at all rigidly distinguishes the Asiatic from the other peoples. Table IV B indicates, indeed, that the Tibetan B type differs markedly from all the Asiatics (being larger than they are) for several of the long head measurements though it finds its counterpart in every case in one of the unallied races. This is notably so for head length (F), horizontal circumference (U), face height ($G'H$) and face breadth (J).

The races of Tables IV A and IV B have been arranged from left to right in order of their Coefficients of Racial Likeness with the Tibetans. No two of the horizontal arrays of indices, angles, or lengths are in the same, or in any very similar, order. Evidently none of these characters alone can give a measure of Racial Affinity even in the roughest way.

A comparison of the skull measurements has shown that the people of the southern provinces of Tibet—and though to a certain extent hybrid they are probably typical of the bulk of the population of the country—belong to the family of races which now inhabit Southern China, Burma, the Malay Peninsula and, most likely, the line of the Himalayas from Assam to the western border of Nepal. The races inhabiting this vast tract of country appear to be all more or less inter-related, although a more detailed study of their physical anthropology will, no doubt, disclose many sub-types among them. The Khams Tibetans do not appear to belong to this stock and it may not be unprofitable to compare them with other aboriginal peoples. The Moriori, Maori, Aino, Fuegians and Eskimo are races of man as

geographically remote from each other as any which could be named and it may be thought idle to seek for affinities between them. But they have this in common, that they are border-land peoples who have been pushed out to the far extremities of the great land masses of the world and having once arrived at their present habitats they were probably undisturbed by the great race upheavals which might exterminate or hybridise other races. Hence these primitive peoples appear to have preserved a more or less pure type since very early times. It may be that the history of the Khams Tibetans has been analogous to that of these other races; isolation in this case being due to the inaccessibility of Eastern Tibet. The importance of such a conception will be obvious to those who put their faith in types rather than in races of mankind.

For comparative purposes we are able to use the Moriori means given by Thomson* and the Maori means given by Scott†. The Fuegian data are more slender, but means are given for a number of characters in Thomson's paper (p. 98). The individual measurements were collected from various sources and reduced in the Biometric Laboratory‡.

It is probable that the Aino of Japan and the Tibetans of Khams are the modern representatives of two distinct aboriginal races of Eastern Asia, but their origins and relationships cannot yet be determined from a craniometric standpoint because there is no adequate data relating to the skulls of the inhabitants of Mongolia and the greater part of Siberia. The Eskimo to the east, however, have attracted more attention, their great importance in the study of ethnography has long been insisted on and consequently several collections of their skulls have been measured and described. We are in a position to determine whether there is any affinity between the Eskimo and either the Aino or the Tibetans.

Though several numerical descriptions of series of Eskimo skulls have been published, there is no single one which we can use for comparative purposes, since they deal with small numbers or give insufficient measurements. Consequently it became necessary to compile a table of means for male skulls from the data given by several craniometricians. Measurements were taken from the following sources:

(*a*) The German Anthropological Catalogues: there are six Eskimo in the Göttingen (1874) portion, one in the Freiburg (1878) and three in the Leipzig (1886) portion.

(*b*) Flower's catalogue of 18 specimens in the Museum of the Royal College of Surgeons (1879).

(*c*) Virchow, *Archiv für Anthropologie*, Bd. IV. (1870), p. 89 (three skulls).

(*d*) Duckworth, *Journal of the Anthropological Institute*, Vol. XXV. (1895), p. 73, and Vol. XXX. (1900) (24 skulls in all).

(*e*) Quatrefages and Hamy, *Les Crânes des Races Humaines* (1882), p. 440, and p. 441 (footnote (1)). Measurements of 11 skulls are given.

* E. Y. Thomson, "A Study of the Crania of the Moriori." *Biometrika*, Vol. XI. pp. 82—135.

† Scott's means for the Maori are quoted on p. 98 of Thomson's memoir.

‡ In place of some of the Fuegian means given on p. 98 of Thomson's paper which are based on only two skulls I have used the Hultkrantz means of six skulls (p. 100).

(*f*) Bessels, *Archiv für Anthropologie*, Bd. VIII. (1875), pp. 116—118* (68 skulls).

The descriptions of the measurements taken by some of these writers are very vague and incomplete, but none are included which had not been determined by methods the same as or very similar to ours.

All the skulls were said to be those of pure Eskimo, but since the sources consulted had been so various it became desirable to test in some way the homogeneity of the whole. This was done by dividing the material into two groups; an eastern one including all the Greenlanders and a western comprising the skulls from Labrador, Baffin Land and the Arctic Archipelago. No skull had been obtained from a locality west of Melville Island so that the Eskimo of Alaska were quite unrepresented. The mean characters of the two groups appeared to be very similar and the Coefficient of Racial Likeness between them was $1{\cdot}66 \pm {\cdot}11$ for 19 characters and $0{\cdot}21 \pm {\cdot}21$ for 5 indices. These low values showed, in my opinion, that the two groups might be combined to give a population which would be as homogeneous as the racial ones with which the craniologist usually has to deal. There can have been little if any European or Indian admixture in the western group—the large majority of the skulls having been obtained from the environs of Smith Sound—so it is probable that the skulls from Greenland represented a pure type also. A comparison of cephalic indices will illustrate this point in a way which will appear more satisfactory to craniologists who do not trust figures far. The following mean values of $100\, B/L$ were given for the Eskimo of Greenland by the various authorities cited: 72·2 (3)†, 72·0 (5), 71·8 (11), 71·6 (3), 70·8 (15), giving a weighted mean 71·4 (37). For Labrador and the islands to the west of Greenland the values are: 72·3 (3), 72·0 (4), 71·7 (11), 71·3 (67), 69·4 (4) and the weighted mean of these is 71·3 (89). These records appeared to be in excellent agreement for this and other characters, but a discordant series was found in a paper by Brierley and Parsons‡. This gives measurements of 13 supposed male Eskimo skulls which had been procured from the west coast of Greenland, and it is said that care was taken to include none that were not of the pure native type. One of these, however, has a cephalic index of 92·6 and the authors suggest that the skull was that of a European sailor. Excluding this one the mean index is 74·6 (12) and this can hardly be reconciled with the values given above. Since some of the other measurements

* Bessels gives measurements of 101 Eskimo skulls, but unfortunately they are not sexed, although he supposed that there were 28 females among them. The collection included 65 which had previously been measured and sexed (52 ♂ and 13 ♀) by Otis and a few of his means are quoted by Quatrefages and Hamy (*op. cit.* p. 441). I have not been able to see the original paper by Otis. By considering three characters—length, breadth and capacity—and supposing that the distribution of Bessels' 101 individuals could be represented by two normal and partly overlapping curves, the means of which were the values given by Otis, I was able, by approximate methods, to pick out 33 skulls which with great probability were all female. The means of the remaining 68 were calculated, supposing them to be all males.

† The numbers in brackets are of the number of skulls on which the means were based.

‡ *Journal of the Anthropological Institute*, Vol. XXXVI. (1906). The table of measurements is given on p. 118. It may be noticed that these skulls were collected some years after the majority of the others we have considered.

were also not in agreement with the general type, we should not feel justified in including the means of Brierley and Parsons with the others. The Eskimo skull differs markedly from those of all our Asiatic races in having a very low cephalic index and in being greater in basio-bregmatic height than in breadth.

The mean measurements for the Khams Tibetans and the various primitive peoples we have mentioned are given in the following table and the Coefficients of Racial Likeness in Table VII. The indices in square brackets are the ratios of mean measurements and not the means of the individual indices. In the case of the Aino and Eskimo it is not stated which orbit was used, but for the other

TABLE VI.

Comparative Table of Means (Male) of Primitive Races.

Character	Tibetan B	Aino	Eskimo	Fuegians	Moriori	Maori
L	185·5 (14)	185·8 (88)	188·2 (148)	192·0 (34)	186·9 (35)	185 5 (43)
B	139·4 (14)	141·2 (88)	134 1 (146)	145·0 (34)	141·4 (34)	140·1 (43)
B'	94·3 (15)	96·2 (88)	94 9 (20)	97·1 (34)	95·3 (35)	95 7 (43)
OH	115·5 (15)	119·3 (88)			117·1 (34)	
H'	134·1 (15)	138·7 (88)	140·0 (56)	141·0 (34)	135·9 (34)	137·6 (43)
LB	99·2 (15)	105·4 (88)	101·9 (39)	106 0 (6)	105·6 (34)	103·9 (43)
Q	312 1 (15)	328·5 (84)	306 0 (3)	316·1 (6)	316·1 (34)	307·5 (43)
S	378 6 (14)	372·8 (77)	373·7 (29)	381 (6)	368·8 (34)	378·1 (43)
S_1	129·3 (15)		127·2 (26)		126·0 (34)	—
S_2	126 1 (14)	—	126 5 (26)		112·4 (34)	-
S_3	122·7 (15)	—	119·4 (26)		121·1 (34)	—
U	525·6 (14)	522 5 (88)	523·4 (145)	(34)	522·8 (34)	518·0 (43)
$G'H$	76·5 (15)	69 8 (73)	72 1 (25)		76·4 (35)	71·4 (43)
J	137 5 (15)	137 3 (74)	136 4 (101)	143·6 (34)	137·4 (34)	136·8 (43)
NH, L	54 9 (15)	50 5 (79)	53 5 (33)	54·6 (6)	57·3 (35)	53·8 (43)
NB	27 1 (15)	25 6 (78)	23 5 (50)	25 3 (6)	25·3 (35)	25·6 (43)
$O_1 L$	43·4 (15)	40·9 (79)	39 8 (26)	40 0 (6)	14·1 (35)	40·5 (43)
$O_2 L$	36 7 (15)	34·9 (79)	35·5 (35)	35·3 (6)	37·3 (34)	35·0 (43)
G_1	51·5 (13)	53·0 (75)	—		53·1 (33)	52·4 (43)
G	47·7 (13)	—	47 0 (3)		50·1 (34)	—
	43·6 (14)	38·2 (74)	40 3 (3)		38·9 (34)	37·3 (43)
GL	97·2 (15)	104·9 (69)	104 2 (23)	106 0 (6)	100·9 (34)	100·8 (43)
GB	100·7 (15)	102·1 (76)	100 4 (29)	110·0 (34) (?)	101·2 (34)	—
fml	37·4 (15)		38 1 (14)	35·3 (6)	35 3 (33)	35·2 (43)
fmb	31·4 (15)		29 1 (14)	31 0 (6)	30·8 (34)	30·8 (43)
C	1537·7 (14)	1462 (76)	1472·2 (109)	1474 (34)	1422·1 (28)	1476 (43)
DC	22 7 (15)		21·8 (14)	—	—	—
$100\ B/L$	75·3 (14)	76 5 (88)	[71 4] (145)	76 6 (34)	76·1 (34)	75·4 (43)
$100\ H'/L$	72·1 (14)	74 7 (88)	[74 2] (55)	73·1 (34)	72·8 (34)	74·7 (43)
$100\ B/H'$	104·5 (14)	101 0 (88)	[95 9] (40)	[102·8] (34)	104·5 (34)	[101·8] (43)
$100\ NB/NH$	49 5 (15)	50 7 (78)	[42·6] (23)		43·9 (34)	47·9 (43)
$100\ O_2/O_1, L$	84·6 (15)	85 3 (80)	89·4 (31)	87 4	84·0 (33)	86·1 (43)
$100\ G_2/G_1$	84·1 (12)	72·1 (72)			73·2 (34)	[71·2] (43)
$100\ G'H/GB$	76·0 (15)	68 4 (65)	74 0 (3)		74·1 (34)	
$100\ fmb/fml$	84 1 (15)	-	[76·4] (14)		87·4 (34)	
$P\angle$	85°·7 (14)	82°·0 (67)		83°·6 (34)	84°·7 (34)	
$N\angle$	65° 5 (15)				64°·8 (34)	
$A\angle$	68°·8 (15)				71°·8 (34)	

races the measurements given are of the left orbit. In most cases the nasal heights were measured on the left side.

TABLE VII.

Coefficients of Racial Likeness. All Male.*

		Tibetan B	Aino	Eskimo	Fuegians	Moriori	Maori
Tibetan B	All Characters	—	13·61 ± ·10 (23)	12·28 ± ·09 (26)	4·97 ± ·10 (21)	4·85 ± ·09 (30)	8·14 ± ·09 (26)
	Indices and Angles only	—	11·94 ± ·17 (8)	18·58 ± ·18 (7)	1·17 ± ·21 (5)	4·74 ± ·14 (11)	6·55 ± ·18 (7)
Aino	All Characters	13·61 ± ·10 (23)		26·30 ± ·10 (22)	6·69 ± ·11 (19)	21·80 ± ·10 (25)	8·41 ± ·10 (23)
	Indices and Angles only	11·94 ± ·17 (8)		63·73 ± ·19 (6)	3·72 ± ·21 (5)	17·18 ± ·17 (8)	1·38 ± ·19 (6)
Eskimo	All Characters	12·28 ± ·09 (26)	26·30 ± ·10 (22)	—	19·53 ± ·10 (21)	25·06 ± ·10 (24)	13·25 ± ·10 (24)
	Indices and Angles only	18·58 ± ·18 (7)	63·73 ± ·19 (6)		34·93 ± ·21 (5)	50·72 ± ·19 (6)	31·80 ± ·19 (6)
Fuegians	All Characters	4·97 ± ·10 (21)	6·69 ± ·11 (19)	19·53 ± ·10 (21)		4·56 ± ·10 (21)	5·67 ± ·11 (20)
	Indices and Angles only	1·17 ± ·21 (5)	3·72 ± ·21 (5)	34·93 ± ·21 (5)		0·96 ± ·21 (5)	1·14 ± ·24 (4)
Moriori	All Characters	4·85 ± ·09 (30)	21·80 ± ·10 (25)	25·06 ± ·10 (24)	4·56 ± ·10 (21)		8·48 ± ·09 (26)
	Indices and Angles only	4·74 ± ·14 (11)	17·18 ± ·17 (8)	50·72 ± ·19 (6)	0·96 ± ·21 (5)		5·33 ± ·18 (7)
Maori	All Characters	8·14 ± ·09 (26)	8·41 ± ·10 (23)	13·25 ± ·10 (24)	5·67 ± ·11 (20)	8·48 ± ·09 (26)	
	Indices and Angles only	6·55 ± ·18 (7)	1·38 ± ·19 (6)	31·80 ± ·19 (6)	1·14 ± ·21 (4)	5·33 ± ·18 (7)	

This table shows clearly that the Eskimo skull stands quite apart from all the others and has no affinity with the primitive oriental types. But the inter-relations of the other races considered are of greater interest.

The most significant differences between the Tibetan B and Moriori types are in height and breadth of nose and breadth of palate, these giving very divergent values for the indices NB/NH and G_2/G_1. The calvariae are very similar except that the Tibetan has the shorter base (LB) and the nose is the only portion of the face which distinguishes one from the other. The Maori type differs more markedly from the Tibetan and particularly in its lesser breadths of palate and orbit and smaller face height. But the calvariae again are very similar for all measurements except LB. The Fuegian skull is no further removed from the Tibetan than the Moriori was found to be, but it does not diverge in the same way from it. The greater size of the Fuegian calvaria is the character which distinguishes it; the shape of the two, as judged by indices, is very similar.

If the Coefficients of Racial Likeness for all characters given in Table VII are compared with those of Table II it will be seen that the Tibetan B type resembles the Moriori and Fuegian types as closely as it does any Asiatic type with the exception of the Karens and that was deduced from only 8 skulls so that little reliance

* The figures in round brackets give the number of characters on which the coefficients are based.

can be placed on its accuracy. But none of these relationships are as close as those which were found between the Tibetan A and neighbouring races. It would be inadvisable to lay stress on these apparent racial affinities while our data are so meagre, but perhaps we are warranted in saying now that craniometric evidence may necessitate theories of very extensive prehistoric wanderings of human types.

The inter-relations of the three southern races, for which the Coefficients of Racial Likeness are given in Table VII, appear to be of peculiar interest. It is surprising to find that both the Maori and the Moriori resemble the Fuegians more than they do each other, and it becomes more than ever difficult to believe "that there has not at some time been a race-link between the present denizens of South America and the denizens of extreme Polynesia*." It is a very noteworthy fact that the South American skull differs most significantly from each of the Polynesian ones in very much the same way; the most aberrant characters in the case of the Maori are L, B, J and U and L, B, J and O_1 for the Moriori. If the Moriori and Maori mean measurements are compared (Table VI) it is, at first, not at all evident why the Coefficient of Racial Likeness between them should be so large as it is. The longer lengths and arcs, with the possible exception of Q, and their indices appear to correspond very closely. The characters which are responsible for raising the coefficient are O_1L, O_2L, $NB/NH\,(R)$ and $NH\,(R)$; no others are of any importance. The most influential of these is O_1L, and, if the three orbital measurements (O_1L, O_2L, $O_2/O_1\,(L)$) are not included, the coefficient is reduced from 8·48 to 3·91: this value is of the order we might have anticipated, but it is still high. If nasal measurements are also left out of account, the coefficient, now based on 20 characters, becomes 2·23, which indicates a close relationship†. This emphasises again the importance of obtaining a criterion of racial affinity which takes into account not one or even 10 characters alone, but all the chief arcs, lengths, indices and angles of the skull.

Turning to the Aino, it is surprising to find that the coefficients with the Fuegians and Maori are so small, these denoting closer relationships than that between the Ainos and Tibetans. The low values of the coefficients between the Fuegians and all the other races with the exception of the Eskimo are the really noteworthy feature of Table VII. The lowering of the values when indices and angles only are considered is very significant. With the Aino the most divergent characters are H/L, L, B, J and NH. The very low values for the coefficient for the Maori and Aino when indices and angles only are considered appear to indicate that the two types are very similar in shape, but it is unfortunate that no more than 6 characters are available for comparison. The coefficient for all characters is so much larger than the other owing to the great difference in length between the transverse arcs (Q). If that is left out of account the coefficient becomes 3·04 for 22 characters. The difference between the nasal heights is also very significant

* E. Y. Thomson, "A Study of the Crania of the Moriori." *Biometrika*, Vol. XI. p. 101.

† With regard to this point it must, however, be noted that personal equation is likely to be of much importance in orbital length and nasal height, and there is some evidence that Scott and Thomson did not use the same conventions. See *Biometrika*, Vol. XI. pp. 92—94 and Vol. XIII. p. 345.

and if the three nasal measurements (NH, R, NB and NH/NB) are neglected as well as Q we find C. L. = 1·01 for 19 characters. Without more abundant evidence it would be rash to assign any ethnographic meaning to these results. After noting such a suggestive resemblance it is remarkable to find that the Moriori and Aino are as unlike each other as any two races we have compared in this paper. The Maori it would thus seem stand in an intermediate position between the Moriori and Aino types.

It has frequently been suggested that one or other single measurement of the skull can by itself be used as a racial criterion. While it is now generally recognised that no direct measurement can fulfil this purpose, yet there are certain indices which still are used to support the claim. Are the differences between the mean racial values of these indices at all correlated with the corresponding Coefficients of Racial Likeness? The two methods of measuring relationship are compared in the following tables, which deal with four of the most important indices suggested.

TABLE VIII.

Comparison of Mean Cephalic Indices with C. L.'s. All Males

Race compared	Values of 100 B/L	With Tibetan A		With Tibetan B	
		Difference of 100 B/L	C. L.	Difference of 100 B/L	C.L.
Burmese A (Burmans)	82 9	+3·1	5·71	+7·6	10·96
Malayan	81·7	+1·9	2·88	+6 4	11 17
Burmese B (Hybrids?)	80·4	+0·6	0·80	+5·1	4 83
French	79 8	0·0	9·12	+4·5	22·96
Tibetan A	79·8	—		+1·5	6·46
Burmese C (Karens?)	79·5	−0·3	2·36	+4·2	3·46
Chinese	78·9	−0 6	3·23	+3·6	6 37
Congo Negroes	78·0	−1 8	7 23	+2 7	21·61
Fuegians	76 6	−3·2	18·86	+1·3	4 97
Aino	76·5	−3 3	13·82	+1 2	13·61
Moriori	76 1	−3 7	12·78	+0 8	4·85
Hindu	75 8	−4·0	6 56	+0·5	18 34
Maori	75·4	−4·4	10·55	+0·1	8 14
Tibetan B	75·3	−1·5	6·46		
Maravar	71·6	−5 2	12·34	−0 7	25 72
Whitechapel English	74·3	5·5	14·59	−1 0	11·93
Naqada Egyptians	73 0	−6·8	17·61	−2·3	18 32
Eskimo	71·4	−8·4	24·15	−3·9	12·28

The simotic 100 SS/SC and mesodacryal 100 DS/DC indices are tabulated below in the same way, the values for the Whitechapel English, Congo and Aino being taken from the paper on the nasal bridge by Miss K. V. Ryley and Dr Julia Bell*.

Whether we confine our attention to Asiatic races or consider all given in these three tables, it is clear that there is no close relation between the Coefficients of Racial Likeness and the values of the single indices. The orders in which these criteria would arrange the races appear to have no obvious points of similarity and we must accept the conclusion that none of these indicial characters can be used alone to

* *Biometrika*, Vol. IX. p. 404.

TABLE IX.

Comparison of Mean Racial Values of the Conjoint Index† $100\,(B-H')/L$ *with C. L.'s. All Males*‡.

Race compared	Values of $100\,(B-H')/L$	With Tibetan A		With Tibetan B	
		Difference of $100\,(B-H')/L$	C. L.	Difference of $100\,(B-H')/L$	C. L.
French	+7·2* (56)	+2·5	9·12	+3·7	22·96
Tibetan A	+4·7 (17)	—	—	+1·2	6·46
Burmese A (Burmans)	+4·5* (43)	−0·2	5·71	+1·0	10·96
Whitechapel English	+4·2 (120)	−0·5	14·59	+0·7	11·93
Moriori	+3·5 (33)	−1·2	12·78	0·0	4·93
Tibetan B	+3·5 (14)	−1·2	6·46	—	—
Fuegians	+3·2*	−1·5	18·86	−0·3	4·97
Malayan	+3·2 (76)	−1·5	2·88	−0·3	11·17
Burmese B (Hybrids?)	+2·9* (7)	−1·8	0·80	−0·6	4·83
Congo Negroes ...	+2·6* (48)	−2·1	7·23	−0·9	21·61
Chinese	+1·3 (69)	−3·4	3·23	−2·2	6·37
Aino	+0·9* (88)	−3·8	13·82	−2·6	13·61
Burmese C (Karens?)	+0·8* (8)	−3·9	2·36	−2·7	3·46
Maori	+0·7*	−4·0	10·55	2·8	8·11
Hindu	−0·2 (44)	−4·9	6·56	−3·7	18·34
Naqada	−0·3* (130)	5·0	17·64	−3·8	18·32
Maravar	−0·6* (38)	−5·3	12·34	−4·1	25·72
Eskimo	−2·8*	−7·5	24·15	−6·3	12·28

* Indices marked with an asterisk were calculated from the mean values of $100\,B/L$ and $100\,H'/L$ and not from the individual measurements.

TABLE X.

Comparison of Mean Racial Values of $100\ SS/SC$ *with C. L.'s. All Males.*

Race compared	$100\ SS/SC$	With Tibetan A		With Tibetan B	
		Difference of $100\ SS/SC$	C. L.	Difference of $100\ SS/SC$	C. L.
Congo Negroes	26 (?)	− 6	7·23	− 9	21·61
Tibetan A	32 (16)	—		− 3	6·46
Burmese A (Burmans)...	33 (41)	+ 1	5·71	− 2	10·96
Tibetan B	35 (15)	+ 3	6·46	—	—
Aino	43 (?)	+11	13·82	+ 8	13·61
Burmese B (Hybrids?)...	45 (8)	+13	0·80	+10	4·83
Whitechapel English ...	51 (?)	+19	14·59	+16	11·93
Burmese C (Karens?) ...	54 (7)	+22	2·36	+19	3·46

indicate racial affinity, except perhaps in isolated cases, and we do not believe that any single direct or indicial measurement can perform that function. Apart from

† Professor Karl Pearson has suggested that the index $100\,(B-H')/L$ should be known by this name.

‡ The values of the index are taken from *Biometrika*, Vol. XIII. p. 245. The tables there give, except for the Burmese types, the values of $100\,(B-H')/L$ and not $100\,(B-H)/L$ as stated. We are obliged to use the basio-bregmatic height (H') because the more important vertical height (H) is as yet available for only a few races. It must be carefully noted that our H' is the H of Thomson, Benington and Macdonell.

that, of course, the measurements of isolated characters are still of great intrinsic interest. If the Coefficients of Racial Likeness calculated from indices and angles only are compared with these single indicial characters in the same way, we are forced to accept the same conclusion with regard to the latter.

TABLE XI.

Comparison of Mean Racial Values of 100 *DS/DC with C. L.'s. All Males.*

Race compared	100 DS/DC	With Tibetan A		With Tibetan B	
		Difference of 100 DS/DC	C. L.	Difference of 100 DS/DC	C. L.
Tibetan A ...	37 (16)	—	—	− 2	6·46
Tibetan B ...	39 (15)	+ 2	6·46	—	—
Congo Negroes ...	39 (?)	+ 2	7·23	0	21·61
Burmese A (Burmans) ..	46 (42)	+ 9	5·71	+ 7	10·96
Burmese B (Hybrids?)...	50 (8)	+13	0·80	+11	4·83
Aino	50 (?)	+13	13·82	+11	13·61
Whitechapel English ...	57 (?)	+20	14·59	+18	11·93
Burmese C (Karens?) ...	62 (7)	+25	2·36	+23	3·46

8. *Study of the Contours of the Tibetan Crania.*

Drawings of the Transverse, Horizontal and Sagittal contours were made with the Klaatsch contour tracer in the manner described by Benington in his paper on Cranial Type contours* and the method of constructing the type from them described there was used. The mean measurements of the individual contours determine the points of the type contour.

(*a*) *The Transverse Vertical or Auricular Coronal Section.* This section is drawn from one auricular point† to the other through the apex (A), the latter point being defined as the highest point of the section perpendicular to the Frankfurt horizontal and passing through the auricular points. The vertical axis MA is the perpendicular bisector of the line joining the auricular points—the "auricular line." Lines parallel to the auricular line are drawn dividing MA into ten equal parts. These lines are numbered upwards from 1 to 10 and measurements are taken to left and right of the vertical axis. Another parallel $A\frac{1}{4}$ is drawn at a distance of one quarter of the last section (A 10) from the apex, and yet another ($M\frac{1}{4}$) the same distance from the "auricular line." The points where the zygomatic ridges are crossed ($ZR . R$ and $ZR . L$) are marked on the individual contours and located from M by rectangular coordinates, x (along MA) and y. The upper trough of the ear passage beyond the auricular points was found in most cases to slope forwards towards the face so that little of it could be traced and no attempt has been made to represent it on the type contour.

The mean measurements of the individual section from which the type was

* *Biometrika*, Vol. XI. p. 129.

† These are not the "auricular points by definition" but those on which the skull rested when in the craniophor. See p. 201.

constructed are given in the following table and the types themselves are Figs. 1 and 2.

TABLE XII.

Tibetan Transverse Vertical Contours. Mean Values.

	MA	$1R=1L$	$M\frac{1}{2}R$	$M\frac{1}{2}L$	$2R$	$2L$	$3R$	$3L$	$4R$	$4L$	$5R$	$5L$	$6R$	$6L$
Tibetan A (17)	113·9	59·6	62·0	62·4	63·2	64·1	66·1	67·0	68·3	69·3	68·4	69·8	67·3	68·6
Tibetan B (15)	115·9	61·6	64·2	64·3	65·6	65·7	67·5	67·6	69·0	69·0	69·2	68·8	68·0	67·4

	$7R$	$7L$	$8R$	$8L$	$9R$	$9L$	$10R$	$10L$	$A\frac{1}{2}R$	$A\frac{1}{2}L$	$ZR.R$ y	$ZR.R$ x	$ZR.L$ y	$ZR.L$ x
Tibetan A (17)	65·0	66·4	60·3	61·8	51·3	52·9	36·5	37·0	19·1	17·8	62·4	3·2	62·6	3·1
Tibetan B (15)	65·3	64·5	60·0	59·3	50·7	50·1	36·6	35·1	20·2	17·9	64·8	3·9	64·9	4·1

Drawings of cranial contours were originally proposed for the purpose of comparing and measuring crania individually and thus obtaining mean racial characters which could not be easily found otherwise, but the use afterwards made of them in constructing average or type contours has become of far greater interest than the original purpose. The comparative material available is unfortunately still meagre. It consists of the Congo, Egyptian, Eskimo, Guanche and English sections of Benington*, the Moriori of Thomson† and the recent Burmese data of Tildesley‡. The three sections we have taken are given for all these and tracings accompany the papers. We are only considering male crania and for these the available types are:

I.	Congo (Gaboon) Fernand Vaz, 1864	... ...	50♂ Crania
II.	Congo " " " 1880	... ...	20♂ "
III.	Congo (Batetelu race)	... ...	41♂ "
IV.	Egyptian (B.C. 600–200)	... ...	100♂ "
V.	Eskimo	... ...	31♂ "
VI.	Guanche	... ...	14♂ "
VII.	Seventeenth Century English (Whitechapel)	...	100♂ "
VIII.	Moriori	... ...	33♂ "
IX. Burmese	Type A (Burmans)	... ...	44♂ "
	Type B (Supposed Hybrids)	... ...	7♂ "
	Type C (Supposed Karens)	... ...	8♂ "

and to these we are now able to add

X. Tibetan	Type A (Sikkim region)...	... ...	17♂ "
	Type B (Khams)...	... ...	15♂ "

* *Biometrika*, Vol. VIII. pp. 155—195.

† *Ibid.* Vol. XI. pp. 118—123.

‡ *Ibid.* Vol. XIII. pp. 188—208.

Many of these series are very short and we must be particularly careful to avoid the omnipresent danger of drawing conclusions without paying due regard to the divergencies which would lie within the limits of probable error. Diagrams

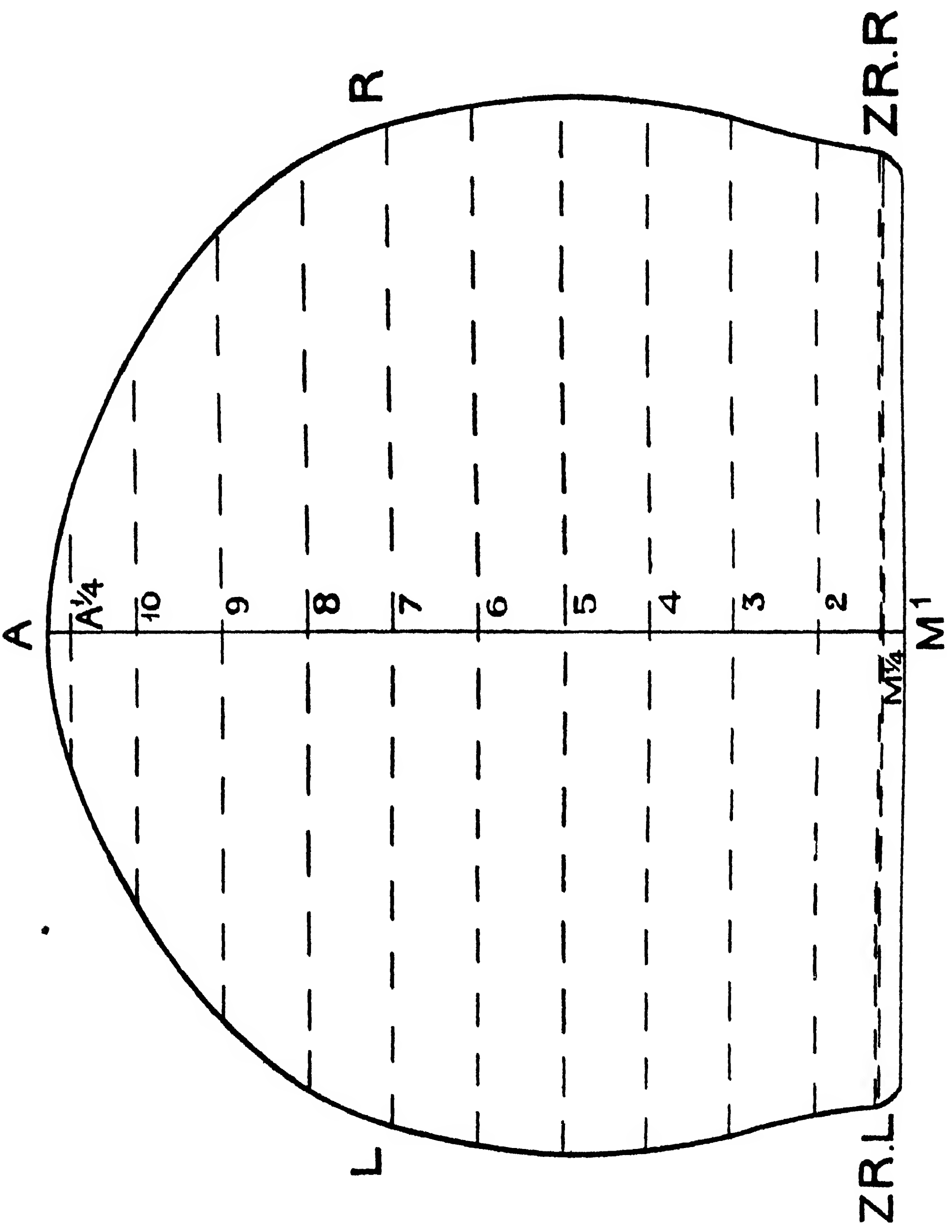


Fig. 1. Tibetan A. ♂. Vertical Type Contour. (17 Crania.)

indicating the range of variation of the contour lines have been given by Benington* for the seventeenth century English series above, the largest we are

* *Biometrika*, Vol. VIII. pp. 143, 145, 147 for the transverse, horizontal and sagittal contours respectively.

dealing with. Points were plotted at distances of twice the probable error of the mean ordinate from each mean ordinate and on either side of it and the outer ring and inner ring of these were splined up to give two diagrams. The outer one is

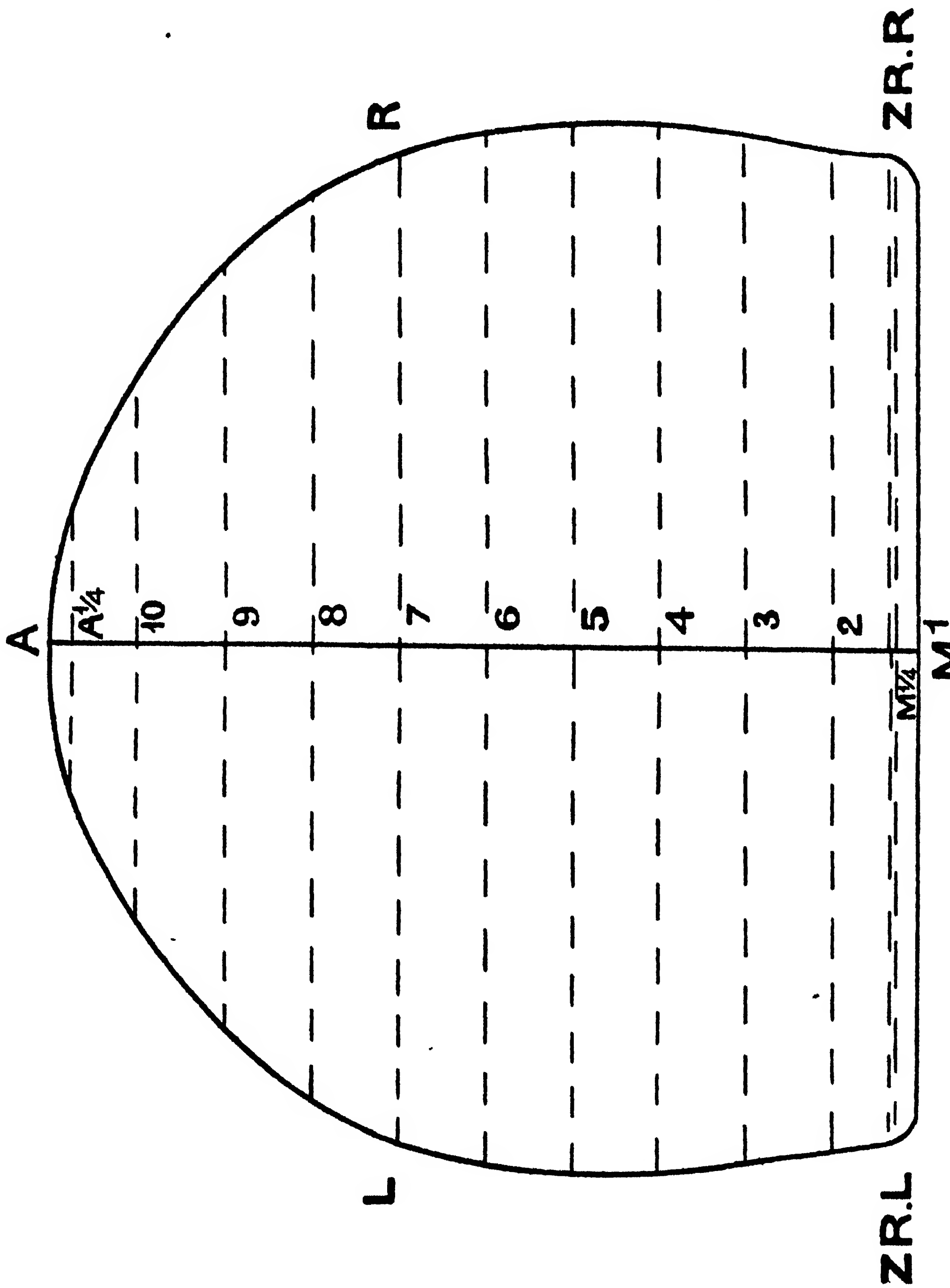


Fig. 2. Tibetan B. ♂. Vertical Type Contour. (15 Crania.)

rather larger than the type and the inner one rather less than it. The space between was termed the type zone and of this it was said, "If two type zones for

two races do not fall wholly or at definite points well outside each other, we suggest that the races represented by these types are either undifferentiated or closely related*."

Now the transverse vertical type zone shown for the English crania is 1·4 mm. wide at the apex and it tapers down to 0·8 mm. on nearing the auricular points Assuming that the standard deviations of English and Tibetan skulls are approximately the same, we calculate, as Tildesley has done for the Burmese series†, that the Tibetan A type zone should have a width of 3·4 mm. at the apex and 1·9 mm. on nearing the base line; the corresponding measures for the Tibetan B series will be 3·6 mm. and 2·1 mm.‡

Not all the transverse vertical contours given for the above races (p. 228) were prepared in precisely the same way. Those of Dr Benington, which we have numbered I to VII, were said to have been drawn from auricular point to auricular point, but the types fall straight down to the auricular line without any sudden bend inwards, such as might have been expected, and this gives them the appearance of terminating at the zygomatic ridges. Neither of the ordinates $M\frac{1}{4}$ or the points $ZR(R)$ and $ZR(L)$ was marked and there was thus no detailed description of the region immediately above the auricular points. These latter appear to have been splined up with the extremities of the parallels 2, 3, 4, etc. I was able to confirm this supposition by examining the individual contours on which the types had been based. In many of them the curve round and below the zygomatic ridge was quite clearly shown and further evidence was afforded by comparing the OH measurements of the transverse and sagittal§ contours of the same skull. These were found to be in close agreement in every case. When comparing the transverse type contours of the Congo, Egyptian, Eskimo, Guanche and English crania then, it must be remembered that they should in reality widen rapidly for a short distance above the auricular points and then rise more vertically than shown to pass through the extremities of the second parallels.

I was also able, through the kindness of Professor Karl Pearson, to examine the individual contours of the Moriori crania drawn by Thomson and they had apparently been made in a somewhat different way. In not a single one of them was there any curve inwards towards the so-called auricular line. In most, indeed, the contour widened at the base in a way that must have been racially peculiar to the Moriori. There can be little doubt that the terminals used were the zygomatic ridges and not the auricular points. This view was confirmed by finding that the height of the transverse type contour was 5·5 mm. less than the auricular height given by the sagittal type‖.

* *Biometrika*, Vol. VIII. p 129. This paper describing Dr Benington's work was prepared for the press by Professor Karl Pearson.

† *Biometrika*, Vol. XIII. p. 226.

‡ The type zones were marked on the tracings of the Tibetan type contours which I used for comparative purposes, but they have not been reproduced because they would confuse the already crowded drawings

§ Though not reproduced in the types, the auricular points had been marked on Dr Benington's individual sagittal contours.

‖ Comparing Figs. I and III, *Biometrika*, Vol. XI. pp. 118 and 115 respectively.

We have used precisely the method described by Miss Tildesley in her study of the Burmese skull; she was the first to measure the ordinates $M\frac{1}{4}$ and to mark the positions where the zygomatic ridges are crossed.

The right and left of the individual contours were interchanged in constructing the type so the *norma occipitalis* is shown. We have followed Thomson and Tildesley in this: Benington's contours show the *norma facialis.* The right and left of Table XII must be interchanged to correspond with the diagrams.

If the auricular lines are superposed the two Tibetan types appear very similar. Between parallels 4 and 9 the divergence between them is less than 0·1 mm. and the larger *B* type only falls outside the type zone of A a short distance in the region of the apex and immediately above the two auricular points. The greater width here, which is especially marked between the points $ZR(R)$ and $ZR(L)$, is quite significant.

The Burmese sections A and B are rather greater in height and very slightly less in breadth at the base than the Tibetan A one. We must hesitate to draw any conclusion from the much more marked narrowness of the lower half of the C contour, because that one represents such very slender data. In height the Tibetan B type is almost exactly equal to the three Burmese, but the greater width of the former between the zygomatic ridges, and of the lower half of the section generally, clearly distinguishes it from them. The curvatures of the vaults, compared by superposing the lines AM and the points A, also differ significantly; the Tibetan B falling away more rapidly from the apex. The Tibetan A section had the more rounded form in common with the Burmese and we can confidently say that these two A types resemble each other more than either does the Tibetan B.

If the line joining the zygomatic ridges of the Tibetan A figure is placed on top of that joining the points marked L. A., R. A., as we suppose they represent the same length, of the Moriori contour the correspondence between the two sections is seen to be remarkably close. The greatest divergence between them for their whole length is 0·8 mm. so that, if nothing was known of their origin, the probability of their representing random samples from the same population would certainly be very high. This correspondence with the Moriori is certainly much closer than that between the Tibetan A and any of the Burmese types. The Tibetan B contour, of course, differs from the Moriori in very much the same way that it differed from the Tibetan A.

The heights and the curves of the vaults of the three Congo series are very similar to those of the Tibetan A type, but the latter differs very appreciably from them in being broader in the middle and for the lower half; the divergence increases as the auricular line is approached. The auricular breadth of the Tibetan is only equalled by that of the Guanche and English and is exceeded by the Eskimo drawings. The English is the only one of these types to which it bears any striking resemblance; in that case the contour falls entirely within the type zone, though the correspondence is not nearly so close as with the Moriori. The vaults of all these

are of a smooth and rounded form and very much alike, except the characteristic one of the Eskimo which is more pointed than the others.

To the Tibetan B we fail to find any counterpart, for its great width at the second, third and fourth parallels distinguishes it from all and in the extreme cases, such as the Congo and Egyptian, it exceeds the other types by as much as 5 mm. It is true that the Eskimo has a greater auricular width than the Tibetan, but immediately above the auricular line the latter widens more rapidly.

When it is said, after comparisons have been made between type contours, that two races appear to be closely related or akin we are speaking only of relationship of form and do not for a moment mean to suggest that any such evidence can provide a criterion of remote consanguinity. Indeed, it will be shown later that two races almost as remote as any imaginable, may have very similar sections*. Even when all possible points have been marked on the sagittal contours there is no indication at all of the orbits, or of the nasal, palatine and foraminal breadths.

The important measurements which can be read off from the transverse contours are the interauricular length and the auricular height. A table of these and the resulting indices has been given by Thomson† and to that we are now able to add the measurements of the three Burmese type contours and the present Tibetan ones. Unfortunately the Moriori lengths cannot be included since the contours were not drawn in the same way as the others.

TABLE XIII.

Measurements on the Transverse Type Section. Males.

Race	Interauricular Length	Auricular Height	Index
Eskimo	128·0	114·2	89·2
English	122·4	111·5	91·1
Guanche	119·6	109·8	91·8
Tibetan B	123·2	115·9	93·3
Egyptian	117·4	110·2	93·9
Tibetan A	119·2	113·9	95·6
Congo, Bantu‡	116·8	113·6	97·3
Burmese A	118·2	116·5	98·6
Burmese C	114·6	115·6	100·9
Burmese B	114·6	115·7	101·0
Cro-Magnon	124·0		

This table shows again that the Tibetan A resembles in this index the three Burmese types, while the Tibetan B differs from all those but is very similar to the English and Guanche.

* See the comparison between Whitechapel English and Tibetan B on p. 248.

† *Biometrika*, Vol. XI. p. 103.

‡ Fernand Vaz, 1864 Series.

(*b*) *The Glabella Horizontal Section.* This section is drawn through the glabella with the inverted skull with the Frankfurt horizontal orientation. The points fixing this are the nasion, a point previously marked above the left auricular passage, and the "gamma" in the occipital region. F and O being points marked on the contours immediately above the nasion and below the lambda respectively, lines dividing FO into ten equal parts and perpendicular to it are measured to left and right of that base line. These are numbered from 2, that being the first tenth division from F, to 10, the nearest to O. Two more parallels, $F\frac{1}{4}$ and $F\frac{1}{2}$, are taken $\frac{1}{4}$ and $\frac{1}{2}$ the distance $F2$ from F with the object of determining the frontal curvature more precisely and another, $O\frac{1}{4}$ being $\frac{1}{4}$ of the same distance from O, serves the same purpose in the occipital region. The points where the temporal lines are crossed, $T(R)$ and $T(L)$, having been marked on the contours are orientated from F by rectangular coordinates, $T(x)$ along FO and $T(y)$.

The mean measurements from which the type contours were constructed (Figs. 3 and 4) are given below.

TABLE XIV.

Tibetan Horizontal Contours. Mean Values.

	FO	$F\frac{1}{4}R$	$F\frac{1}{4}L$	$F\frac{1}{2}R$	$F\frac{1}{2}L$	$2R$	$2L$	$3R$	$3L$	$4R$	$4L$	$5R$	$5L$	$6R$	$6L$
Tibetan A (17)	173·3	27·0	24·0	37·7	36·1	47·7	46·0	51·5	50·9	57·7	57·4	61·4	65·8	68 1	69·9
Tibetan B (15)	183·2	27·6	26·3	39·8	38·7	48·9	48·0	52·6	51·9	59·0	59·2	65·0	66·6	67·7	70·7

	$7R$	$7L$	$8R$	$8L$	$9R$	$9L$	$10R$	$10L$	$O\frac{1}{4}R$	$O\frac{1}{4}L$	$T(R)x$	$T(L)x$	$T(R)y$	$T(L)y$
Tibetan A (17)	68·7	70·5	65·4	67·2	57·7	59·7	44·2	46·5	24·5	25·4	17·6	18·2	48·4	46·7
Tibetan B (15)	68·1	70·6	64·6	67·2	57·2	59·9	44·2	46·6	24·5	24·5	16·9	18·1	49·1	48·6

The type contours of Benington do not show the parallel $F\frac{1}{2}$ or mark the positions where the temporal lines are crossed; these were added by Thomson and have been included by subsequent workers in the Biometric Laboratory. Thus the drawings of the Moriori, Burmese and Tibetan crania have a more detailed and less rounded appearance in the frontal region than the others, though the additional points only make very slight differences to the path of the contour which would have been chosen without their guidance. The *norma verticalis* aspect is shown for all the types, but the right and left of the table above are those of the individual contours taken with the skull placed apex downwards and they must be reversed to correspond with the type contour.

The horizontal type zone given by Benington for 100 English crania has an approximate width of 1·2 mm. all round. The corresponding width for 17 crania will be 2·9 mm., and for 15, 3·1 mm.

In superposing horizontal contours the axis FO is used as the common base line and one or other of the points F and O should also be made coincident.

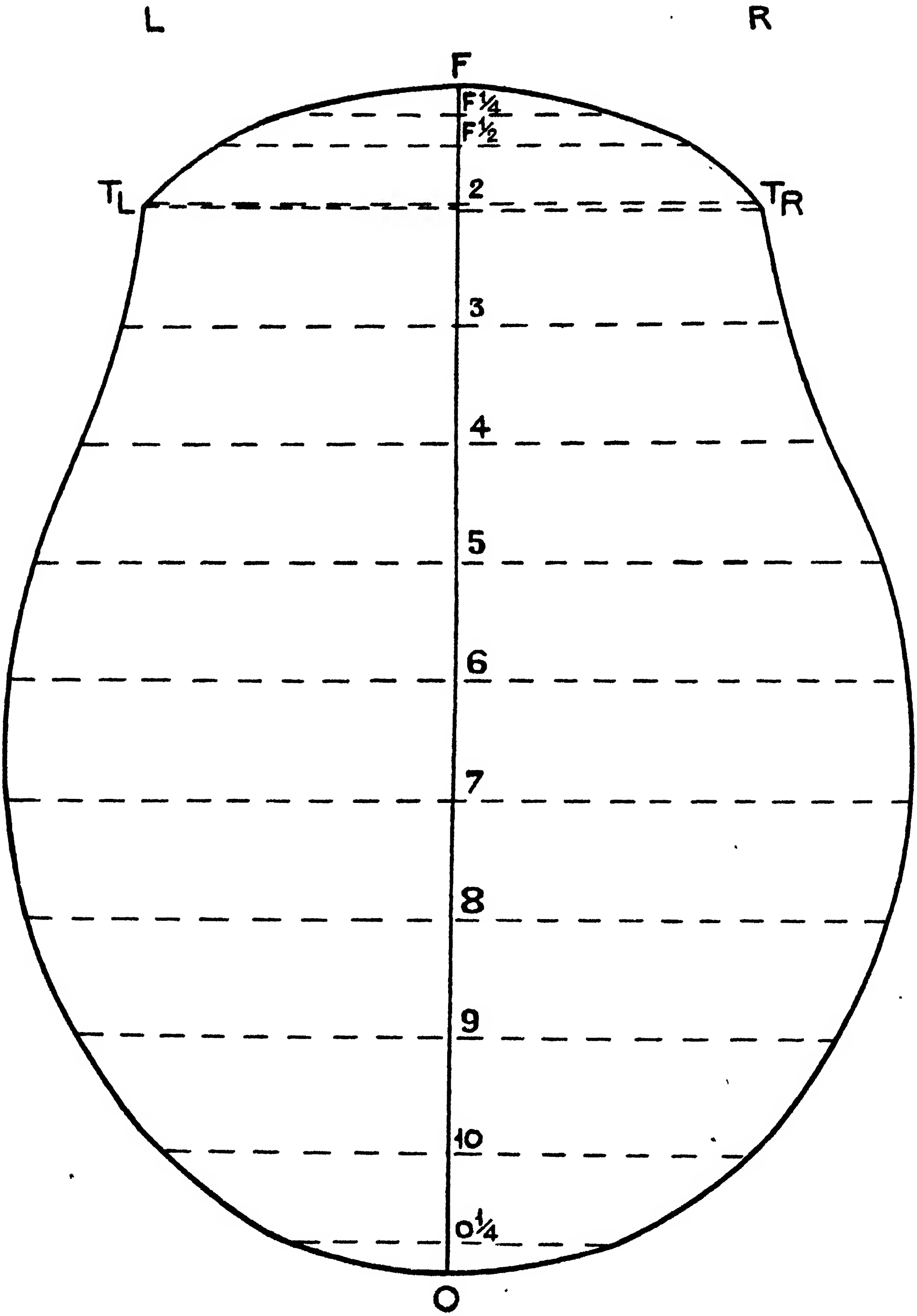


Fig 3. Tibetan A. ♂. Horizontal Type Contour. (17 Crania.)

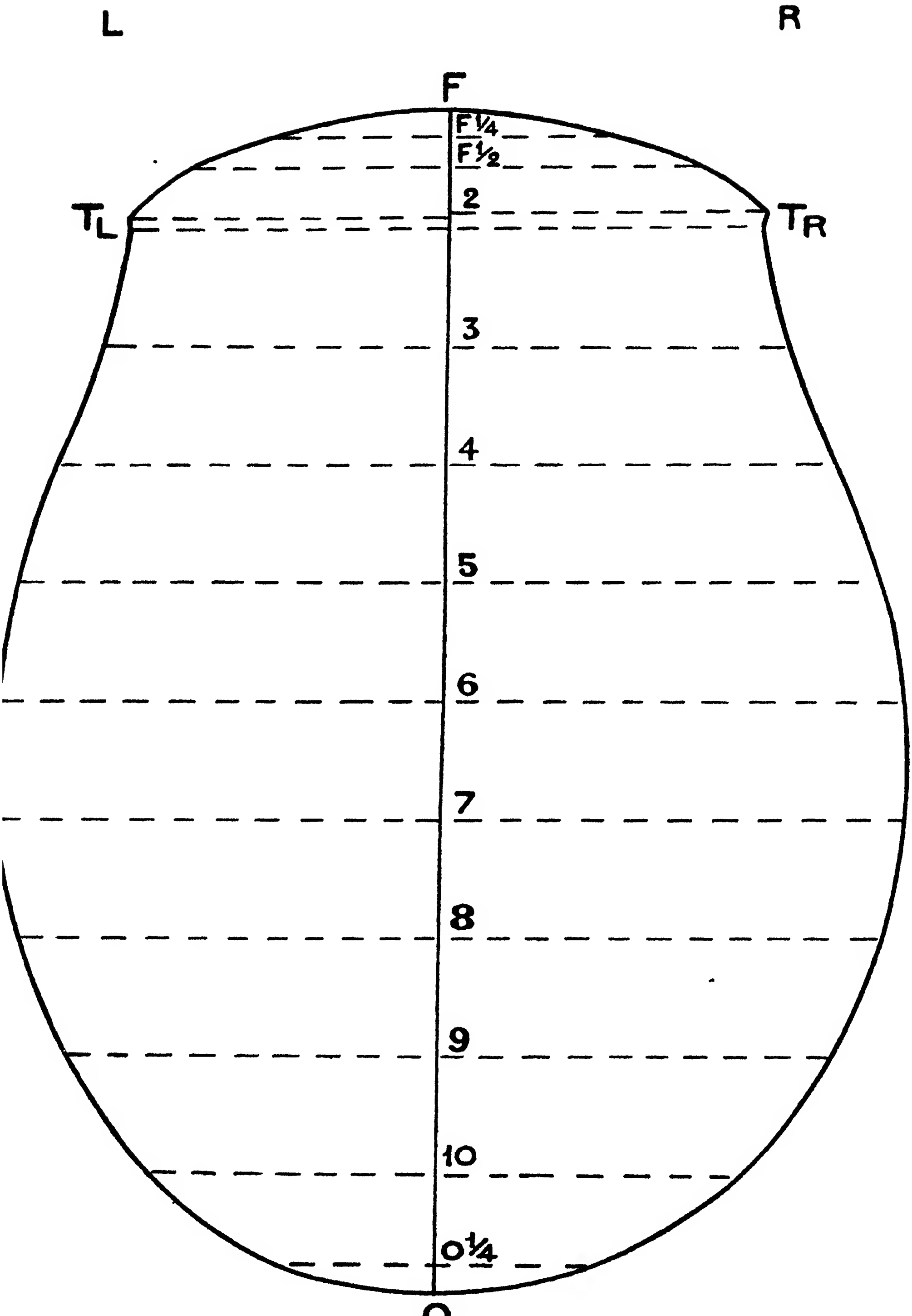


Fig. 4. Tibetan B. ♂. Horizontal Type Contour. (15 Crania.)

Considering first the two Tibetan types and making FO and the points F coincide, on the drawing of Type B and the tracing of type A show that they are of clearly differentiated forms. Type B is very significantly longer, actually by 10·1 mm., and the differences in the curvature of the frontal region are quite marked and particularly on the left side where the point $T(R)$ falls well outside the zone of Type A. In breadth the two are equal and if the tracing is moved down until the points O coincide the shapes of the occipital regions from O to parallel 6 are seen to be very similar. The Tibetan B horizontal section is characterised by very forward and prominent projections of the temporal lines which give an appearance of strength and flatness to the frontal bones. The lines $T(R)y$ and $T(L)y$ are well in front of the second parallel whereas they are usually behind it as with the Tibetan A, Moriori and Burmese types. The Tibetan A section also has the more common rounded form.

The Tibetan A contour is nearly equal in length to the three Burmese ones and very similar to them in all respects. The resemblance is particularly close to the Burmese A type which is nowhere more than 0·2 mm. from the superposed type zone. The correspondence with the Burmese series B and C is not quite so close, though it still suggests near relationship. The Tibetan B type approximates to the three Burmese only in breadth. It is much longer and the frontal regions are significantly wider. We can say without hesitation that the Tibetan A and Burmese A types are much more akin to each other than either is to the Tibetan B.

With a length 12 mm. less and a maximum breadth greater, the Tibetan A type can claim little affinity with the Moriori. The latter hardly falls within the type zone for a quarter of its length and perhaps it differs most characteristically in contracting behind the temporal lines where the Tibetan extends. The well-marked *linea temporalis* of the Moriori was taken to be a criterion of its primitive character*, and nothing so rugged can be found in any other type, but the position of these lines on the horizontal contour differs only slightly from that in which they are usually found and is not nearly so close to F as in the Tibetan B type. Is this prominence also a primitive character? It is unfortunate that the different method of construction obscures this region in the other racial contours. The Moriori skull is clearly differentiated from that of both the Tibetan peoples, but it resembles the B series more than the A.

The Tibetan A is in all cases significantly, and in many much shorter than the three Congo, the Egyptian, the Guanche and the English types; it is broader than all but the Guanche and English and it resembles none in shape except, perhaps, that of the third Congo series (Batetelu race) which falls entirely within the type zone, when the figures are superposed with F and FO coincident, with the exception of the occipital region below parallel 9. The Tibetan B contour is longer, broader and more massive than the three Congo and the Egyptian and shorter than the Eskimo, but it much more nearly resembles the Guanche and English and

* See *Biometrika*, Vol. XI. p. 104.

particularly the former‡. If the Tibetan and Guanche are superposed, with FO and either F or O coincident, the type zone will cover both contours for all but short lengths round the points $T(R)$ and $T(L)$. The difference in manner of construction may account for a small part of the discrepancies here, but, even if allowance is made for such, the characteristic prominence of the Tibetan type will still clearly distinguish it.

Our Tibetan A type then appears, from a comparison of horizontal contours, to be most closely related to the Burmese. There are two indices of interest which can be determined from the lengths of the horizontal type contours; they both give a measure of frontal development. The Temporal Index is formed by measuring the percentage Ordinate 3 is of the length of the section and the other index is Ordinate 3 divided by the minimum frontal breadth. The following table is taken from Thomson's paper with the addition of the Burmese and Tibetan data.

TABLE XV.

Measurements on the Horizontal Type Section. Males.

Race	Temporal Index	$100\,\frac{\text{Ordinate 3}}{\text{Minimum Frontal Breadth}}$
Moriori ..	50·9	99·1
Burmese B ...	54·2	101·8
Burmese C ...	55·7	107·4
Egyptian ..	56·2	112·0
Tibetan B ...	57·0	107·0
Burmese A ...	57·1	102·3
Eskimo ...	57·2	[113·4?]
Congo, Bantu* ...	57·5	104·8
Guanche ...	58·3	[115·6?]
English ..	58·8	111·6
Tibetan A ...	59·1	107·7
Cro-Magnon ...	55·7	114·5†

* Fernand Vaz, 1864 Series.

† The Cro-Magnon minimum frontal breadth was determined from a cast.

The Burmese type has a distinctly low temporal index in spite of the fact that the skull is brachycephalic. The narrowness of the frontal bones behind the temporal lines is very characteristic of the race since it is found associated with a large parietal breadth and this is not a marked feature of the Tibetan A contour. But in spite of this point of difference we still maintain that the latter resembles the Burmese most closely and next the Congo Negroes. The B type resembles none of these, but differs least from the Guanche and then from the English.

Of the horizontal contours generally it may be noted that the divergencies between them are much greater than those observed when considering the vertical section and hence they will be more likely, when a sufficient number have been prepared,

‡ The similarity between the Guanche and the English was insisted on when the type contours were first compared: see *Biometrika*, Vol. VIII. p. 135.

to afford a reliable measure of racial affinity. But the sagittal section is undoubtedly more important than either of the others.

(c) *The Sagittal or Median Section.* With the nasion, bregma and lambda in the same horizontal plane, the sagittal contour is drawn through the alveolar point, nasion, glabella, bregma, vertex, lambda, gamma, inion, opisthion and basion and as much as possible is traced of the palate, nose and the basilar process of the occipital bone to the suture between it and the sphenoid; the vertex (i.e. the highest point in the median plane) and the gamma, or the point in the median plane on a horizontal level with the nasion, having been previously marked on the skull. All these points are dotted on the contour as well as the point where the palatine sutures cross*, the extremity of the *spina nasalis posterior*, the tip of the anterior nasal spine, the projection of the sub-orbital point and the point in the right auricular passage on which the skull must have rested when in the horizontal position on the craniophor—this is called the auricular point of the contour and the line joining it to the sub-orbital point should be parallel to that joining nasion to gamma (the $N\gamma$ line), if the two auricular points are symmetrically positioned. The two lines are very nearly parallel in the type contours. The horizontal $N\gamma$ line is taken as axis and divided into ten equal parts with the proportional compasses, the lines of division being numbered from 0, that through the nasion, to 10 through the gamma. Parallels 8 and 9 are produced below the base line to meet the drawing of the lower portion of the occipital bone and further ordinates are erected at $\frac{1}{4}$ the distance between the nasion and first parallel from the nasion ($N\frac{1}{4}$) and $\frac{1}{4}$ and $\frac{1}{8}$ of the same distance from the γ ($\gamma\frac{1}{4}$ and $\gamma\frac{1}{8}$ respectively). Also the ordinates 0, $N\frac{1}{4}$, I and II are produced below the base line to the palate. From all the points marked, with the exception of that marking the join of the palatine sutures, the basion and the alveolar point, ordinates (y) are dropped to $N\gamma$ and their positions are determined by taking either the nasion or gamma as origin of coordinates. As measures of curvature four maximum subtenses are taken: maximum frontal subtense to nasio-bregmatic chord, maximum occipital subtense to opisthio-lambdoid chord, maximum calvarial subtenses from the nasio-lambda and glabella-inion lines. The point of contact of a vertical tangent to the most projecting point of the occiput determines the occipital point.

The nasal bone of the contour is measured by joining the tip (L) to the nasion; NL giving the nose length and the angle between NL and $N\gamma$ being that of nasal prominence. Some nasal ridges exhibit a double curve, turning downwards towards the tip, so that for a short distance NL either cuts the outline or coincides with it. The point where NL first meets the outline of the ridge is called L' and NL' is also measured: where there is no double curve NL is, of course, equal to NL'. The maximum subtense is drawn from NL to the curve above L' and the point on the nasal bone is located from N with rectangular coordinates, x, along γN, and y.

The position of the alveolar point is determined by measuring its distances from the nasion and basion and the basion is measured from the gamma and the

* When the left and right transverse sutures did not meet in the median one, a point on that line mid-way between the two joins was taken.

nasion. To define more exactly the curvature between the anterior nasal spine and the alveolar point, a vertical tangent is drawn from the most posterior point of the arc. Finally the position of the point of the palate where the sutures cross is located from the alveolar point with rectangular coordinates and the subtense from the mid-point of the chord joining the basion and sphenoidal point is measured from the basion along that line.

These are all the measurements used in the construction of the type contour but a few others are taken with the object of comparing their mean values with the corresponding lengths and angles of the type diagram and also because some of them are wanted for comparative purposes. They are the lengths $N\beta$, $\lambda Op.$, $N\lambda$, GI, the subtenses from $N\lambda$ and GI and the angles $\phi = \angle \beta GI$, $\phi' = \angle \beta NI$, $A \angle$, $N \angle$, $B \angle$, $P \angle$ and $fm \angle$ or the angle which the basio-opisthion line makes with the horizontal*.

Table XVI gives the mean values of the measurements made on the individual contours from which the types are constructed. The abbreviations used here and on the diagrams (Figs. 5 and 6) are: N = Nasion, G = Glabella, *Bas.* or B = Basion, β = Bregma, V = Vertex, A or *Alv.* = Alveolar point, *Sub-Orb.* = Infra-orbital point (left orbit), *Aur.* = Auricular point (right) on which the skull rested when in the Frankfurt horizontal position in the craniophor, λ = Lambda, γ = Gamma, I = Inion, *Op.* = Opisthion, *Sp.* = the Sphenoidal point, i.e. the point of intersection of the median plane and the suture between the basilar process of the occipital bone and the sphenoid bone, P = the point of intersection of the palatine sutures, P' = the extremity of the *spina nasalis posterior*, NS = the extremity of the anterior nasal spine, L = the tip of the nasal bone, L' = the point at which the line NL first meets the nasal bone, $A \angle = \angle NAB$, $N \angle = \angle ANB$, $B \angle = \angle NBA$, $P \angle = \angle$ between AN and γN, $\theta_1 = \angle$ between BN and $N\gamma$, $\theta_2 = \angle AB$ makes with the horizontal.

It should be noted here that when, as was occasionally the case, a point supposed to lie in the median plane (as the nasion, bregma, lambda, etc.) was either slightly below or above it, the pointer of the tracer was temporarily depressed or raised to pass through the point. It was seldom necessary to do this as most of the skulls were fairly symmetrical.

The comparative material for sagittal contours is the same as that for the other two type contours, but in this case many additional details have been added to the drawings since Benington's pioneer diagrams were made. The series we have numbered I to VII (p. 228) give the contour only from nasion to gamma and without indicating the position of any maximum subtense or of the vertex. This latter point was defined by Thomson in the Moriori paper, though not marked on her drawings, while she added the occipital arc from γ to opisthion, the inion, basion, alveolar, auricular and sub-orbital points, so giving the three vertices of the fundamental triangle. It then became possible to compare the angular measurements of the contours. Tildesley, for the Burmese crania, first drew the contour of the nasal bone and included the points P and V. We have indicated besides these the form of the palate and the arcs from alveolar point to anterior nasal spine and

* These measurements are given on p. 252.

from basion to sphenoidal point. It would hardly be possible to add further details without over-crowding and confusing the diagram.

In her Moriori paper* Miss Thomson has discussed at considerable length the relative worth of the many base lines which may be chosen as axis of reference when comparing sagittal type contours; alternatives fortunately not present in the case of the other two types. This point is of great importance if any clear meaning is to be given to such expressions as a "more receding forehead" and a "higher palate," or if comparisons between the positions of points such as vertex and bregma are to be made. A verdict was given in favour of the $N\gamma$ line which is used as axis in constructing the type contour. Its terminals are well defined and it represents very approximately the natural slope of the head. In accepting that as the common axis for most comparative purposes the use of other lines, such as $N\lambda$, $N\beta$, NOp, or even $\beta\gamma$, need not be neglected, for the skull must not be considered to be dependent on one line. But no line terminating in the glabella or inion can be satisfactorily used because those points are not sufficiently well defined.

The sagittal type zone given by Benington is unfortunately only drawn from nasion to gamma; it has for the greater part of its length a width of 1·6 mm., tapering down to 0·1 mm. at the gamma and 0·9 mm. at the nasion. For 17 crania the corresponding widths will be 3·9 mm., 1·7 mm. and 2·2 mm., and for 15 crania 4·1 mm., 1·8 mm. and 2·3 mm. The type zones were drawn for the two Tibetan series, but owing to the danger of over-crowding are not shown on the type contours of Figs. 5 and 6.

The most marked characteristics of the Tibetan A median type section are the high forehead, flat glabella, and its smooth, almost feminine, curves. The B is a much more masculine type with a lower forehead than A and a quite distinct, though not prominent, glabella and inion. Superposing a tracing of A on the drawing of B with the $N\gamma$ line and the point N coinciding reveals at once the great difference in size between the two. In height they are very nearly equal, but the occiput of the B type protrudes massively beyond the other at all points and the whole palate is much lower, though very nearly parallel. From the nasion to as far back as the 8th parallel the Type B line though it skirts the Type A zone for some way, is never more than 0·2 mm. away from it and this similarity in form of the frontal bones is made more apparent as far as the bregma if the tracing is moved round until the $N\beta$ lines correspond. But the occiputs still look of very different shapes.

In the three Burmese types the Tibetan A seems much more likely to find a counterpart of itself. Its differences from these are all small and in general appearance they are much alike. The resemblance is most close to the Burmese A and it must be remembered that the B and C are based on such small numbers of crania that a detailed comparison with other types is hardly warranted. Placing a tracing of the Tibetan A over the drawing of the Burmese A type with the points N and the $N\gamma$ line coinciding, shows that the former is somewhat lower in the vault and more protruding in the occiput. But, keeping N fixed, if the tracing is now

* *Biometrika*, Vol. XI. pp. 105 *et seq.*

TABLE XVI.

Means of Contour Measurements used in plotting Sagittal Type Contours.

	$N\gamma$	Ordinates above $N\gamma$													
		$0=N$	$N\frac{1}{4}$	1	2	3	4	5	6	7	8	9	$\gamma\frac{1}{4}$	$\gamma\frac{1}{8}$	$\gamma=10$
Tibetan A (17)	171·7	27·5	41·6	60·3	72·5	79·5	84·4	86·6	86·4	82·7	73·6	55·4	27·3	19·2	5·9
Tibetan B (15)	181·7	23·6	38·3	59·1	71·8	79·2	83·6	86·1	85·5	80·4	70·4	51·5	23·6	15·7	−1·1*

	Ordinates below $N\gamma$								Vertex		β		Glabella	
	0 N	$N\frac{1}{4}$	1	2	8	9	$\gamma\frac{1}{2}$	$\gamma\frac{1}{4}$	x from N	y	x from N	y	x from N	y
Tibetan A (17) ...	68·4	65·4	55·9	53·3	16·4	35·7	28·5	19·9	90·5	87·2	70·6	84·6	2·5	12·9
Tibetan B (15) ..	73·1	68·6	59·7	57·4	51·5	41·7	35·6	26·9	96·4	86·5	75·4	84·0	3·0	12·0

	Occipital Point†		λ		Sub-Orb.		Aur.		Op		Inion		Basion	
	x from γ	y	x from γ	y	x from N	y	x from γ	y	x from γ	y	x from γ	y	from γ	from N
Tibetan A (17)	−0·3	−3·6	6·0	30·6	6·9	27·9	87·9	27·7	51·3	51·0	9·0	29·2	98·9	95·8
Tibetan B (15)	−0·2	−8·1	6·4	28·0	6·4	29·7	95·7	29·4	58·3	57·1	8·6	35·8	107·6	99·3

	Alv.		Nose (16 Crania only for A series)					Palate (14 Crania for B Series)				Frontal Max. Sub. to $N\beta$		Occipital Max. Sub. to λ Op.	
						Max Subt		P'		P					
	from N	from B	NL	NL'	$\angle LN\gamma$	x from N	y	x from N	y	x from Alv.	y	x from N	y	x from λ	y
Tibetan A (17)	70·1	90·9	23·6	21·4	111·4	0·9	9·6	14·2	50·8	33·0	16·9	50·7	26·1	51·3	27·1
Tibetan B (15)	77·2	96·5	23·7	21·1	114·9	1·1	9·4	44·8	55·2	33·9	19·1	52·4	26·1	55·2	31·4

	Max. Sub to $N\lambda$		Max. Sub. to GI		Sp.		Sub from $\frac{1}{2}$ Bas Sp. Chord		N. S.		Max. Sub. of Alv. N. S. Chord	
	x from N	y	x from G	y	x from N	y	x from Bas	y	x from N	y	x from N	y
Tibetan A (17)	87·1	70·5	96·0	98·7	63·2	30·7	12·7	1·9	3·4	47·0	2·2	58·9
Tibetan B (15)	87·8	71·8	97·2	101·2	64·1	33·2	13·5	1·8	4·9	53·6	4·1	56·9

* The negative sign here means that the ordinate 10 meets the contour below and not above the $N\gamma$ line.

† Both occipital points are below the $N\gamma$ line and to the left, with the type contours as shown, of γ.

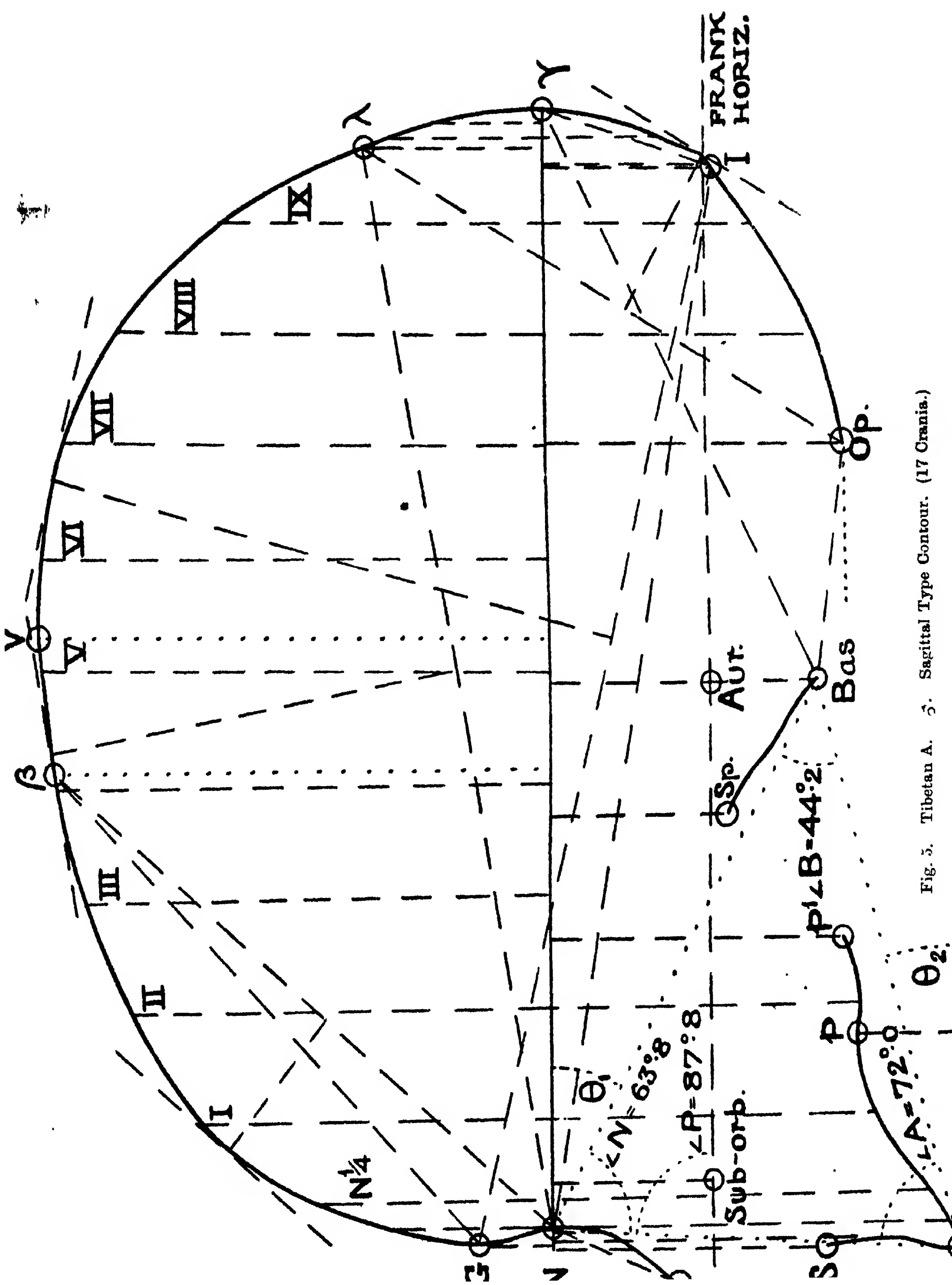


Fig. 5. Tibetan A. ♂. Sagittal Type Contour. (17 Crania.)

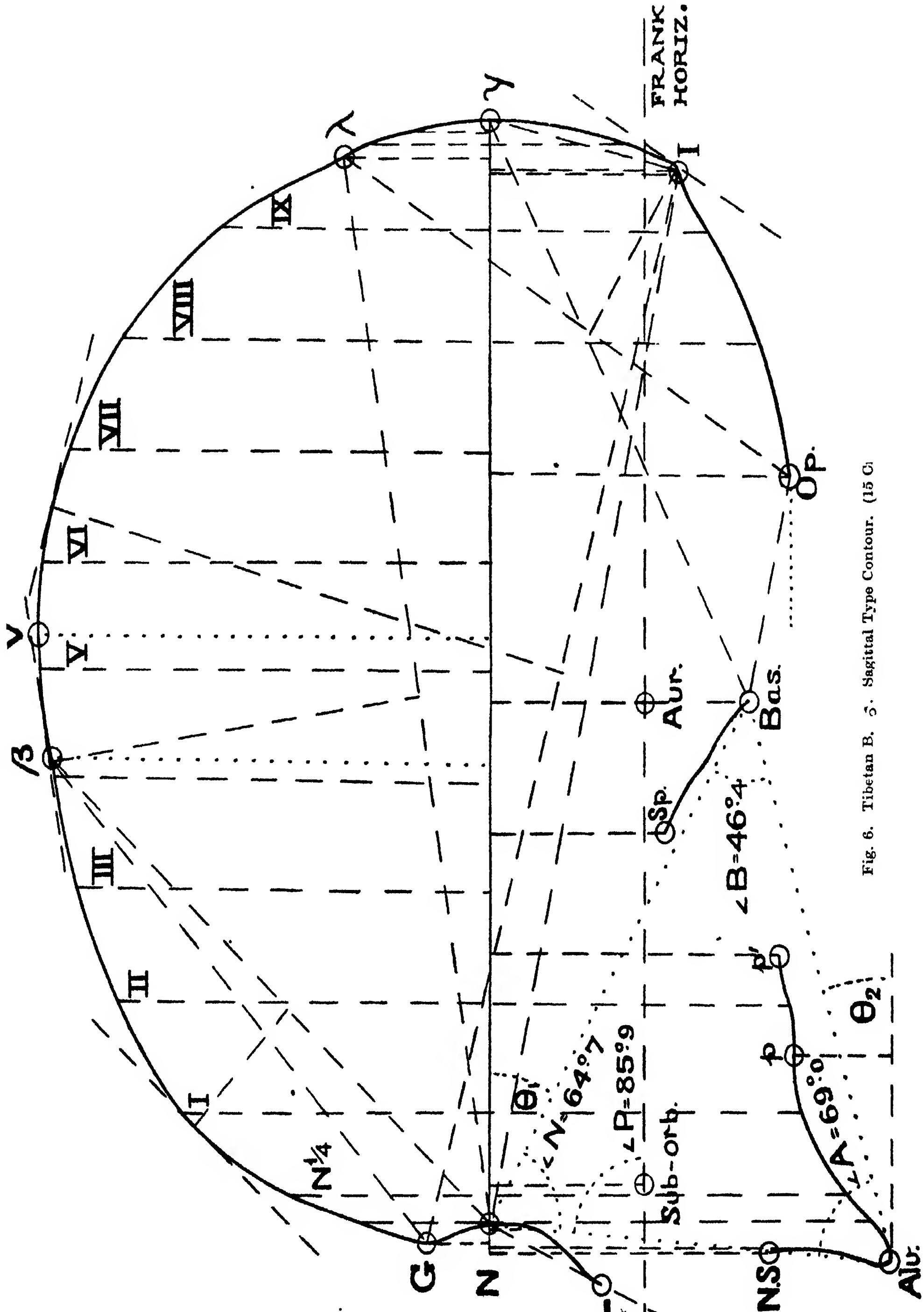


Fig. 6. Tibetan B. ♂. Sagittal Type Contour. (15 C

rotated to make the $N\beta$ lines coincide the resemblance becomes much more striking. From the tip of the nasal bone to the opisthion the type zone, though not shown below γ, must certainly cover both lines except a short distance between the inion and gamma of the tracing. The lines $N\lambda$ and NOp. coincide exactly and the auricular, sub-orbital, and alveolar points, the basion and the point P are all remarkably close. The greatest divergencies are shown between the points gamma and the vertices. The correspondence is so striking and so much more marked in this position that we almost waver in our allegiance to the $N\gamma$ base line. With its lower vault, far greater length and distinctive occiput the Tibetan B is very unlike any of the Burmese types and, though almost equal in length, it has no other points of resemblance with "the depressed and retreating forehead, the bulging glabella and the high sagittal crest*" and the remarkably low γ of the Moriori section.

We have now to compare the Tibetan type contours with those drawn only from nasion to gamma. Our A section is shorter than the three Congo ones along the $N\gamma$ line by 3·9, 6·2 and 4·7 mm. respectively, but almost exactly of the same height and with a frontal bone very similar to theirs. This latter point can be most clearly seen if the $N\beta$ rather than the $N\gamma$ lines are superposed. The correspondence is then very exact, and well within the limits of the type zone, from the nasion to the 6th parallel with the first Congo series, to the 5th parallel with the second series, and the 5th again with the third. That the resemblance is not more general can be seen by making the $N\gamma$ lines and the points γ coincide; the less convex Tibetan occiput clearly differentiates it from the others. There is little to suggest kinship between the Tibetan A and the remaining races. The former is much shorter and at the same time higher than the Egyptian, Guanche and English and not unlike them and the Eskimo posteriorly, but it is most like in the anterior region. This correspondence is again surprisingly close when the $N\beta$ lines of the contours are superposed and much less so when the $N\gamma$ line is the common base. That such should have been the case with all the types compared, with the single exception of the Moriori, is a very noteworthy fact.

The Tibetan A is greater in $N\gamma$ length than any of the three Congo types; in height much the same but differing from them in the higher and more bulging form of its glabella. With $N\gamma$ as axis of reference and the points N coincident, the frontal bones are again seen to have almost identical curves and this correspondence with the first Congo section extends well behind the bregma so that the type zone covers both contours from the nasion to a point not far from the lambda. The discrepancies between each of the three Congo types and the Tibetan B are greater with $N\beta$ than with $N\gamma$ as base line, the latter being unlike its compatriot A type in that respect. If either of these lines is chosen as axis of reference, or perhaps one intermediate between them, the Tibetan B contour is found to have a frontal bone which is hardly distinguishable in shape or size from those of the Egyptian, Eskimo, Guanche and English and this was found with all the other types, with the exception of the Moriori. A parallel statement was made when comparing the

* Thomson, *Biometrika*, Vol. XI. p. 89.

Tibetan A type with these races. The occipital region, on the other hand, is often distinctive and hence more likely to be of use in differentiating racial types. The glabella too is often characteristic, being decidedly higher above the nasion on the Tibetan B type than on those of the Guanche, English and Eskimo. In other respects these correspond very closely and particularly in the form of occiput and vault. The Egyptian, though still very similar, has a frontal bone more unlike that of the Tibetan than any of the others, except the Moriori. The English and Guanche are most like this Asiatic race in the absolute size of the section, but is that to be our sole criterion? Superposing the Tibetan B tracing on the Eskimo with $N\gamma$ as base line does not at first reveal any similarities except in the form and position of the glabella; the contour tracing is entirely surrounded by the drawing. By rotating into the $N\beta$ position, the angles between that line and $N\lambda$ are seen to be identical and one frontal bone is, as far as the bregma, almost entirely covered by the other. Can we now place the one as symmetrically as possible inside the other so as to obtain some appreciation of their similarity in form apart from mere size? This can be effected by placing the nasion of the Tibetan A type 2·2 mm. from that of the Eskimo (the difference in $N\gamma$ length is 4·4 mm.) and keeping that point fixed while the tracing is moved round until the nasion of the tracing and the two lambdas are collinear. The contours are then seen to be very approximately parallel from the N to the γ of the circumscribing one; the greatest distance between them is 2·5 mm. and the least 1·5 mm. Does this denote racial affinity? Not even a tentative answer could be given to such a question without more abundant comparative material; contours for which the regions below the $N\gamma$ line are mapped out being particularly required. The comparison made with the Eskimo type shows, at any rate, how easily a real similarity in shape of head may be overlooked, if attention is to be confined to positions fixed by common base lines and it makes us question again the invariable use of the $N\gamma$ horizontal line for that purpose. Sagittal type contours appear to be potentially of far greater importance than the other two, but at present, owing to the uncertainty attached to the methods of comparing them, they provide a less reliable racial criterion than the horizontal ones.

In Thomson's Moriori paper (pp. 105—112) there is a detailed discussion of the relative values of the nasio-gamma, nasio-lambda and glabella-lambda base lines when each is judged according to its capability of giving an estimate of the flattening of the frontal bone and of physiognomic flatness; a verdict was given in favour of the nasio-gamma line. Accepting that base line, we can now add the Burmese and Tibetan data to the comparative table (*loc. cit.* p. 108) of the subtenses of the calvaria.

The interesting features of Table XVII are seen in columns 7 and 9. The very rounded form of the Burmese and Tibetan A crania, as shown by the high y indices, clearly differentiates them from all the other types, including the Tibetan B. But there seems to be nothing racially distinctive in the positions of the vertex and bregma, as measured by the x coordinates, or in the differences between the indices given in columns 11 and 12. The bregmatic angles are rather more characteristic,

TABLE XVII.

Measurements on the Sagittal Type Section. Males.

Race	$N\gamma$	Max. Subtense		β Subtense		Index Max. Subtense		Index β Subtense		Δ (Ind. Max. Sub. − Ind. β Sub.)		Bregmatic Angle
	B_0	y	x	y	x	y/B_0	x/B_0	y/B_0	x/B_0	y/B_0	x/B_0	
English ...	183·5	84·9	103·7	83·0	77·2	46·3	56·5	45·2	42·1	1·1	14·4	47°·1
Guanche ...	183·4	81·8	99·0	83·0	74·1	46·2	54·0	45·3	40·6	0·9	13·4	48°·1
Egyptian ...	181·9	81·4	106·2	82·5	79·0	46·1	59·5	45·4	43·4	1·0	16·1	46°·2
Tibetan B ...	181·7	86·5	96·4	84·0	75·4	47·6	53·1	46·2	41·5	1·4	11·6	48°·2
Moriori ...	180·2	90·0	103·8	84·9	73·0	49·9	57·6	47·1	40·5	2·8	17·1	49°·3
Congo, Bantu*	175·6	85·1	97·7	83·5	70·6	48·6	55·6	47·6	40·2	1·0	15·4	49°·8
Eskimo ...	186·1	93·9	104·1	89·3	72·1	50·5	55·9	48·0	38·7	2·5	17·2	51°·1
Tibetan A ...	171·7	87·2	90·5	84·6	70·6	50·8	52·7	49·3	41·1	1·5	11·6	50°·1
Burmese C ...	172·3	88·0	98·4	86·0	73·3	51·1	57·1	49·9	42·5	1·2	14·6	49°·6
Burmese B ...	170·9	90·5	100·5	86·5	70·1	53·0	58·8	50·6	41·0	2·4	17·8	50°·7
Burmese A ...	167·4	90·7	91·8	88·2	69·5	54·2	54·9	52·7	41·5	1·5	13·1	51°·8
Cro-Magnon ...	196	100·5	103·2	97·7	68·2	51·3	52·7	49·8	34·8	1·5	17·9	55°·2

but they fail to distinguish between the Oriental races and the Eskimo and Bantu.

Thomson has suggested† as a measure of physiognomic flatness the bregmatic angle for the nasio-gamma line combined with the angle the line joining the nasion to the arc end of the nasio-bregmatic subtense makes with the nasio-bregmatic line. The contributions of each of these to frontal flattening are shown in the following table.

Physiognomic Angle of Flatness.

Race	Frontal Bone Flatness	Rotation of Base of Frontal Bone	Physiognomic Angle of Flatness
Egyptian ...	24°·9	46°·2	71°·1
Moriori ...	22°·3	49°·3	71°·6
English ...	26°·6	47°·1	73°·7
Guanche ...	26°·4	48°·1	74°·5
Tibetan B ...	26°·5	48°·2	74°·7
Burmese C ...	25°·2	49°·6	74°·8
Burmese B ...	24°·2	50°·7	74°·9
Congo, Bantu*	27°·3	49°·8	77°·1
Tibetan A ...	27°·3	50°·1	77°·4
Burmese A ...	26°·7	51°·8	78°·5
Eskimo ...	28°·0	51°·1	79°·3
Cro-Magnon ...	24°·8	55°·2	80°·0

The Physiognomic Angle of Flatness does not appear to distinguish the Tibetan A and Burmese types from the remainder as well as the bregmatic angle does.

* Fernand Vaz, 1864 Series.

† *Biometrika*, Vol. XI. p. 100.

Having made comparisons between the corresponding type contours of various races it is natural to ask how similarities noticed when considering one section are correlated with those adduced from another. And, further, if some judgment of racial affinity can be obtained from all three sections, how will this be correlated with an index based on direct measurements? A tentative answer can be given to the first of these questions. If comparison is confined to members of the same family of races—such a family as the one to which the Oriental races are supposed to belong—then it appears to be probable that all three sections of the two types being compared will be similar if one section is found to be so. But, though one contour of two unallied races may display a striking similarity, it by no means follows that the other two sections will correspond at all closely. The Tibetan A transverse type contour is almost identical with those of both the Moriori and Whitechapel English, but the horizontal and sagittal types are as dissimilar as any.

A comparison of the racial contours has led us to conclude that the Tibetan A and the three Burmese types, particularly the B, are closely related, but the former does not resemble the Tibetans of Kham more than it does the Congo Negro or Whitechapel English type. The Tibetan B appears to be unlike all the Oriental races with which we are able to compare it, but its calvaria is very similar in size and shape to the English calvaria.

We may now ask how the Coefficients of Racial Likeness are correlated with the judgments of racial affinity based on comparisons of the three type contours. Unfortunately we are only able to make this comparison in the case of a few races; these are the two Tibetan races, the three Burmese, the Moriori, Congo Negroes (Batetelu) and Whitechapel English.

Of the 31 direct, indicial and angular measurements from which the Coefficients of Racial Likeness are calculated there are only 18 which are either actually represented, or very nearly represented, on the type contours. These are B, Q, OH and B/H, corresponding to the transverse vertical section; B', U and B/L' to the horizontal and F, LB, S, GH, NB, G_1, fml, $Oc. I.$, $P\angle$, $N\angle$ and $A\angle$ to the sagittal section when all the points we have indicated are marked on that.

There seems at first to be very little correlation between the judgments of racial affinity arrived at by comparing type contours and the corresponding Coefficients of Racial Likeness. The Tibetan A and Burmese B direct measurements appear to be very similar and the coefficient between them is low, but there was no striking resemblance between the type contours. On the other hand, the contour and coefficient methods of comparison both suggest that there is a closer relationship between the Tibetan A and Burmese than there is between the Tibetan A and the Tibetan B or any other of the Oriental races. It was surprising to find that all the Tibetan B and Whitechapel English sections were very similar; the Coefficients of Racial Likeness, however, suggest no affinity. The English sagittal type contour was given only from the nasion to the gamma, so that there was no representation at all of the facial regions and base of the skulls, and of the 31 characters from which the coefficient was calculated there are only 7 lengths,

angles or indices which could be measured on the three contours. The C. L. for these 7 characters alone is 2·96, but when all 31 are considered it becomes 11·93. The great differences between the English and Tibetan B skulls are between the facial and palatine measurements; only the calvariae of the two are similar.

A contour of a skull gives, of course, a more detailed description of the plane in which it is taken and the zones in the immediate neighbourhood of that plane than any number of direct measurements can do, but the three contours cannot give a complete picture; they give no representation at all of the orbits and only a very incomplete one—when all possible points have been marked—of the remainder of the face. The Coefficient of Racial Likeness takes into account all regions of the skull. It would thus seem that a comparison of type contours, though a study of great interest, can never furnish a very reliable measure of racial affinity and it is particularly liable to lead to erroneous deductions when unallied races are compared.

9. *Comparison between Direct Measurements and Contour Values.*

The causes of the discrepancies found between contour and direct measurements have been discussed in detail by Tildesley in her Burmese paper*. We will now compare our results by the two methods.

The defect in the Klaatsch contour tracer, noted by Miss Tildesley and to which she partly attributed the discrepancies found between her measurements, had been remedied before I used the instrument; also the drawing-board on which the

TABLE XVIII.

Comparison of Direct and Contour Mean Measurements†.

		Tibetan A	Tibetan B
Glabellar-occipital Length (L) ...	Direct	175·2 (17)	185·7 (14)
	Sagittal Contours	174·8 (17)	185·2 (14)
Upper Face Height ($G'H$)	Direct	69·4 (17)	76·5 (15)
	Sagittal Contours	70·1 (17)	
Nasion Basion Length (LB) ...	Direct	95·7 (17)	99·2 (15)
	Sagittal Contours	95·8 (17)	99·3 (15)
Basio-Bregmatic Height (H') ...	Direct	131·2 (17)	134·1 (15)
	Sagittal Contours	131·1 (17)	133·9 (15)
Length of Foramen (fml)	Direct	35·7 (17)	37·4 (15)
	Sagittal Contours	36·9 (17)	38·4 (15)
Basion to Alveolar Point (GL) ...	Direct	91·5 (17)	97·2 (15)
	Sagittal Contours	90·7 (17)	96·4 (15)
Opisthion to Lambda (S_3')	Direct	94·4 (17)	100·3 (15)
	Sagittal Contours	94·5 (17)	100·0 (15)

* *Biometrika*, Vol. XIII. Nos. 2 and 3, pp. 210 *et seq.*

† The figures in brackets are of the number of skulls on which the means are based.

tracing was made appeared to be quite level and not in any way warped, at any rate no flaw could be detected by simple tests. After a considerable time had been spent in practising and checking the accuracy of the drawings by comparing several made of the same skull, a fairly reliable method of manipulation was attained.

In some few cases Miss Tildesley deduced the direct and contour means from different numbers of crania one pair differing by 2 and the others by 1. All the series dealt with were small and it was thought best to exclude no measurement that could possibly be made. Fortunately our Tibetans were in a much better state of preservation than the Burmese and all corresponding means have been based on the same numbers.

The contour means in the following table are those reduced from measurements made on the individual tracings and they may not be precisely equal to the lengths of the type contours.

The greatest differences between the means found by the two methods are for the length of the *foramen magnum*, but the lengths from basion to alveolar point and the upper face heights are not in very close agreement. The majority of the discrepancies are so small that we cannot attribute the larger ones to instrumental defects or constant errors in manipulation, for such would affect all the measurements in much the same way. A sufficient explanation seems to be given if we remember the well-known difficulties of determining cranial "points." These were not marked on the skulls when the direct measurements were taken and they may easily have been located differently on the contours. The well-known vagueness of the alveolar point would account for the rather large discrepancies observed for the measurements $G'H$ and GL, since they are both taken from it. The marked discrepancy in the case of the length of the *foramen magnum* is probably due to the fact that the direct measurement was taken further inside the orifice than the contour. On the whole we can feel justified in placing considerable reliance on the accuracy of the type contours.

Why should the two series of Tibetan means be more consistent than those of the Burmese? The greater differences found for the latter must have been chiefly due to the defective tracer and unlevel drawing-board used. Another possible source of error lies in the marked asymmetry of many of the Burmese skulls. In drawing the sagittal contour it was often found necessary to raise or depress the pointer of the tracer in order to make it pass through a point which had to be marked. It was seldom necessary to do this with the Tibetan skulls as there was little asymmetry among them.

Several angular measurements found in the two ways are compared in Table XIX.

The greatest difference here is $0^{\circ}\cdot 8$ so that the agreement is sufficiently close, for practical purposes.

The auricular point was marked on the sagittal contours and a line through it perpendicular to the $N\gamma$ horizontal will give a measure of the height, OH, of the individual contours, if the two auricular points of the skull were symmetrically

TABLE XIX.

Comparison between Mean Angles from Direct Measurements and Mean Angles from Individual Contours.

		$P\angle$	$N\angle$	$A\angle$	$B\angle$
Tibetan A	Direct Measurements	87°·4 (15)	64°·7 (16)	71°·8 (16)	43°·5 (16)
	Sagittal Contours ...	87°·7 (15)	63°·9 (16)	72°·1 (16)	44°·1 (16)
Tibetan B	Direct Measurements	85°·7 (14)	65°·5 (15)	68°·8 (15)	45°·7 (15)
	Sagittal Contours ...	85°·6 (14)	64°·7 (15)	69°·0 (15)	46°·4 (15)

placed, but we should not expect it to agree very closely with the direct measurement. The same length can be found from the individual transverse vertical contours* by dropping a perpendicular from the marked vertex on to the line joining the points which are supposed to have been those in contact with the ear-rods of the craniophor. But here again we should expect no close correspondence.

TABLE XX.

Tibetan Mean Auricular Height, OH.

	Direct Measurements	From Sagittal Contours	From Transverse Contours
Tibetan A (17)	113·2	114·2	113·7
Tibetan B (15)	115·5	115·1	116·0

The conclusion to be drawn from the above comparisons is that direct and contour mean measurements are in reasonably close accordance and should differ by less than 1 mm.

Several measurements are taken from the individual sagittal contours which are not used in the construction of the type and a comparison of the means of these with the corresponding lengths and angles of the type contours is of interest. The following tables give our results.

The greatest difference between these pairs of measurements is 0·4 mm. Several angular measurements can be compared in the same way and the differences will again be found very small. We are dealing with a short series of skulls and the discrepancies would certainly tend to be smaller for a longer series, so we can say confidently that the method of construction of sagittal type contours if special care be used is sufficiently accurate for comparative purposes.

* As much as possible of the auricular passage was traced, though not all could be represented on the type contour.

TABLE XXI.

Comparison between measurements on Type Contours and means of Individual Contour measurements which are not used in Construction of Type Contours.

		Tibetan A (17)	Tibetan B (15)
$N\beta$	Mean Contour Value	110·2	113·2
	Type Contour	110·2	113·5
λ Op	Mean Contour Value	94·4	100·0
	Type Contour	94·0	100·2
$N\lambda$	Mean Contour Value	168·7	177·7
	Type Contour	169·0	177·8
β subt. to $N\lambda$	Mean Contour Value	70·2	71·3
	Type Contour	70·4	71·0
Gl	Mean Contour Value	170·5	182·3
	Type Contour	170·3	182·7
β subt. to Gl	Mean Contour Value	87·5	90·2
	Type Contour	87·1	89·4

TABLE XXII.

Comparison between Angle-measurements on Type Contours and means of Individual Contour measurements.

		Tibetan A (17)	Tibetan B (15)
ϕ	Mean Contour Value	58°·8	58°·1
	Type Contour	59°·0	57°·6
ϕ'	Mean Contour Value	60°·5	59°·9
	Type Contour	60°·7	59°·5
$A\angle$	Mean Contour Value	72°·0	69°·0
	Type Contour	72°·2	69°·3
$N\angle$	Mean Contour Value	63°·8	64°·7
	Type Contour	63°·8	64°·5
$B\angle$	Mean Contour Value	44°·2	46°·4
	Type Contour	44°·0	46°·2
$P\angle$	Mean Contour Value	87°·8	85°·9
	Type Contour	87°·6	85°·8
$fm\angle$	Mean Contour Value	8°·8	12°·7
	Type Contour	8°·3	12°·3

10. *Conclusions.*

It seems reasonable to conclude that:

(i) There are in Tibet at least two distinct races.

(ii) One of these (viz. that from the southern provinces in the neighbourhood of Sikkim—our Type A) is closely allied to the Southern Chinese, Malayans and Burmese.

(iii) The other race (the Khams Tibetans), though showing no marked affinity with any Oriental race, resembles most the Burmese B and C types The skull is, however, very similar to those of both Fuegians and Moriori, and the impression cannot be avoided that we may possibly be dealing with widely scattered fragments of a fundamental primitive human type, with a long-headed, broad-faced, rugous and massive cranium.

I should like to acknowledge here the help of Miss Ida McLearn in drawing the contour diagrams and of Mr E. S Pearson in photographing the crania. I have also to thank Dr W. L. H. Duckworth for several suggestions and corrections to this paper.

APPENDIX I. *On the Measurement of the Mandible.*

These measurements on the Tibetan mandible are placed here with a view to more elaborate future treatment of mandibular characters of various races and especially to provide adequate data for comparisons with the mandible of palaeolithic man.

Of the 32 Tibetan crania which have been dealt with in the first section of this paper 16 of the A series and 7 of the B were provided with mandibles In addition to these the collection included 5 mandibles of the B type which had not been allocated to any of the remaining skulls and it could not be said definitely that they corresponded with any of them. Thus altogether, there were 16 mandibles of the A type and 12 of the B which could be measured. When dealing with the mandibles of any particular race the craniometrician has often to be content with numbers that are not larger than this, while the mandibles are often separated from the skulls and cannot be sexed with any certainty; no reliable determination of type can be adduced from such material. Consequently the measurements of the mandible have been neglected by most workers and when any are given they are invariably scanty.

The following measurements have been taken for the Tibetan series. Many of the lengths and their methods of measurement have been considered by Broca and Rudolf Martin, but some are new.

(*a*) *Definitions of Mandible "Points."*

The Frankfurt horizontal plane is, for obvious reasons, not one to which the measurements of the detached mandible can be conveniently referred. When the mandible is resting freely, with the teeth uppermost, on a horizontal surface its position will usually be unique, but occasionally it may be capable of a slight rocking displacement. Accordingly, the plane of reference to which the majority of heights and angles we shall speak of will be related is defined to be that horizontal plane with which contact is made at three or more points when a vertical pressure is applied to the second left molar tooth or its cavity: this will be known

as the standard basal plane, and when pressed vertically downwards on the second left molar the mandible will be said to be in the "standard basal position*." Usually the three points or regions of contact will be near to the right and left angles and a point on the left corpus somewhere below the canine or premolar teeth, but such is not invariably the case. One or even both angles may not be touching. The basal plane is certainly of a more conventional nature than the Frankfurt horizontal plane, but we are obliged to adopt it because, as far as we can see, no other plane could make the technique of measurement so exact or be less ambiguously defined. An attempt will be made to measure on the individual crania the angles at which the standard basal plane of the mandible and the Frankfurt horizontal plane are inclined ($S\angle$).

Certain mandibular "points" of osteometric rather than anatomical importance have to be defined. The majority of these have been discussed by earlier writers†, but we have found it impossible to accept certain of their definitions without modification. We doubt, indeed, whether they have been tested in practice on any adequately long series.

It must always be understood, unless otherwise stated, that the mandible is in the "standard basal position."

Coronion (c_r). Two coronia are required, one on either ramus. In a perfectly symmetrical jaw one plane could be placed in contact with both condyles and both coronoidal processes, but with the normal asymmetry this is impossible. Accordingly the left coronion is defined by the point of contact of a plane touching the left condyle, left coronoidal process and either right condyle or right coronoidal process as the case may be. The right coronion is obtained by proper interchange. These points, as well as the two condylia, are determined by inverting the mandible on to a flat sheet of duplicating paper so that the points of contact are marked. If there are two summits of equal height the apex of the more forward is taken. When a measurement is taken to only one of the coronia it is always that of the left ramus unless otherwise stated.

Condylion (c_y). This is defined and found in practice in a way similar to that described in the case of the coronion, the capitulum and coronoidal process being interchanged. If a small area of the capitulum instead of a single point is in contact with the above tangential plane then the "centre" or midpoint of that area is termed the condylion and it can be found with sufficient accuracy from the impress left by the duplicating paper.

Gnathion (g_n). The lowest point in the median or symphysial plane when the mandible rests in the standard basal position. This is invariably on, or very close to, the common tangent to the anterior borders of the *fossae digastricae*.

* Unilateral measurements are accordingly taken on the left side, but if, owing to breakage, a measurement has to be taken on the right ramus or corpus, then the second right molar is held to give the standard position.

† See especially, Rudolf Martin, *Lehrbuch der Anthropologie* (1914) (Mandibular "points" are defined, pp. 516--518, and measurements and methods of measurement, pp. 559, 560, 565, 566), and Broca, *Instructions Craniologiques et Craniométriques* (1875), pp. 93—96.

Pogonion A (p_a). The most projecting point of the chin, or, as it is termed by English anatomists, the mental prominence, when the mandible rests in the standard basal position.

Pogonion B (p_b). The most projecting point of the chin when the mandible is *in situ* on the skull adjusted to the Frankfurt horizontal plane.

Infradental (d_f). The point where the sagittal plane of the mandible meets the horizontal tangent to the alveolar rims of the middle incisors.

Intradental (d_t). The tip of the process between the middle incisors.

Gonion (g_o). This is, perhaps, one of the most important points of the mandible and, at the same time, it is the most difficult to define and to determine. A definition of the standard basal plane has been given above. It is now necessary to define the standard rameal plane which we do as follows: Let a tangent plane be taken to the left ramus touching it at the condyle and towards the angle, and let it also touch the right ramus where it can (if the mandible were symmetrical it would touch at the right condyle and at a point towards the right mandibular angle). This tangent plane is the standard rameal plane as defined from the left ramus. It may also be defined from the right ramus. The intersection of the standard basal and rameal planes is the gonional axis. The nearest points on the borders of the angles to the gonional axis are the gonia. The gonia are found in practice with the aid of the mandible board described below. The mandible is held on the second left molar and adjusted so that the rameal wing of the board is in contact with two points on the left ramus and one on the right. Then the zero line of the board corresponds with the gonional axis and the points on the two angles nearest to it are the right and left gonia respectively*.

* The gonion was first defined and named by Broca (*op. cit.* p. 93). In practice the point can be found easily and without ambiguity and it appears to fulfil the ideal condition, that it should be the point of intersection of the posterior line of the ramus and the inferior line of the corpus, as well as any point could. Martin (*op. cit.* p. 517) has proposed quite a different method to determine it. The mandible is inverted so that the angles are uppermost and so that the left ramus and corpus make equal angles with the horizontal. Then the gonion is the highest point of the angle. I marked this point on several mandibles and found that in all cases it was in front of the gonion determined by Broca's method and that for several the distance between was as much as 8 mm. The personal equation of Martin's method must certainly be large as neither of the operations by which the point is determined can be performed with any precision.

We are unable, also, to accept Martin's definitions of some other "points." The pogonion and gnathion, for example, are defined without indicating any plane of reference. They are marked on two figures (*op. cit.* pp. 505, 517) showing the *norma lateralis* and *norma sagittalis* of the complete skull with the mandible in its natural position. It appears from this that the pogonion has reference to the Frankfurt horizontal plane and not to one on which the free bones would rest, so it corresponds to our pogonion B. Martin suggests taking many measurements from this point and defines no other in its neighbourhood. In practice this is found to be a very unsuitable terminal for many reasons: it is impossible to locate the point if the cranium is either missing or badly damaged as is frequently the case; if the skull be available but some, or all, of the teeth are missing it may not be possible to place the mandible in the natural position, though some approximation to that can be made by supporting the mandible with plasticine; damaged or decayed condyles will introduce a difficulty of the same kind and, finally, even if the bone be complete, it is often impossible to place the mandible in any unique position which can be supposed its natural one. For these reasons the pogonion B, though in itself a point of great importance, appears to us to be a very unsuitable one to take measurements from and

The positions of all the points defined above are marked on the bone in pencil before any measurements are taken from them.

Another mandibular plane has to be defined. The standard sagittal plane of the mandible is the plane perpendicular to the standard basal plane, passing through the intradental and perpendicular to the bigonial axis; it should also contain the infradental point, the two pogonia and the gnathion.

(*b*) *The Mandible Board.*

All the measurements, with the exception of those made with small callipers, Flower's callipers, or the steel tape, are taken when the bone is resting, in one position or another, on the mandible board; an instrument similar in design to Broca's "goniomètre mandibulaire*." A fixed horizontal board (*A*) has hinged to it at one end a movable board (*B*)- to be called the "rameal wing" of the instrument—so that the latter can be inclined at any angle from 0° to 180° with the horizontal. The angle between *A* and *B* can be read off directly from a semicircular scale attached to a vertical board (*C*) so that the centre of the scale is in a line with the centre of the hinge. Both the rectangular boards *A* and *B* have mm. scales along one edge fixed so that the centre of the hinge is their common zero. With the aid of this instrument several mandibular angles can be found directly. The bone rests on the horizontal board *A* and is held on the second left molar and heights of points above that plane, which is that of the standard basal plane, are determined by adjusting the point of a scriber to their level and then reading off the height of the latter above the basal plane on the scale attached to the board *B*, when that is in a vertical position. A solid set-square is provided for the purpose of measuring the projections of lengths. This consists of two lengths of wood, of which the section is square, fastened together at one of their extremities, but not let into each other, so that they are at right angles. If one of these branches is resting on the mandible board parallel to the hinge, the other will be parallel to the length of the board, i.e. the edge to which the scale is fastened and with its upper surface on a line with it. Then the solid set-square is slid along until it comes into contact with the point on the mandible from which the measurement has to be made; the other extremity of the length is the zero line, i.e. the central line of the hinge. The projected length can be read off from the horizontal scale. The heights of the coronoidal and condylar processes can be found in a similar way by using the board with the rameal wing vertical.

(*c*) *Measurements and Methods of Measurement.*

The following is a list of the direct measurements of the mandible which I have determined for the Tibetan series. All measurements, unless otherwise stated,

accordingly, the more conventionally defined pogonion A—the point always referred to below unless the other is specifically named—has been used for that purpose. It makes only a slight difference to the position of the gnathion whether it is defined with reference to the Frankfurt horizontal or standard basal plane, but the two pogonia may be several mms. apart.

* See *Instructions Craniologiques*, p. 95.

are given for the left side only. The lengths w_1, w_2*, h_1, and zz were provided by Fawcett for the Naqada Egyptians and have been determined for several other races by subsequent workers in the Biometric Laboratory.

(i) *Lengths measured with callipers or tape.*

w_1, condylar width. Greatest width of mandible at condyles from outside of one condyle to outside of second; with small callipers. w_2, greatest width of mandible at angles from outside of one angle to outside of the other; with small callipers. h_1, sagittal height of mandible; measured from gnathion to intradental†. zz, least distance between inner rims of *foramina mentalia*. $c_l c_r$, coronial breadth from coronion to coronion. rb, least breadth of ramus parallel to basis. As a guide to measurement a pencil line parallel to the basis is drawn on the ramus in the region of its least breadth while it is orientated on the standard basal plane. rb', least breadth of ramus in any direction. G_2', breadth of mandible between inner alveolar walls at middle of second molars This is the counterpart of the measurement G_2 of palate breadth. $c_y c_r$, condylion to coronion. $g_o g_o$, gonion to gonion. $g_n g_o (l)$, gnathion to left gonion. $g_n g_o (r)$, gnathion to right gonion $c_y l$, greatest length of condyle. $c_y b$, greatest breadth of condyle perpendicular to $c_y l$. $m_2 p_1$, distance between outer alveolar margin at the middle of the second molar to middle of first premolar. $p_a d_t$, pogonion to intradental. $p_a g_n$, pogonion to gnathion $p_a d_f$, pogonion to infradental. $g_n d_f$, gnathion to infradental. $p_a p_b$, pogonion A to pogonion B. $g_o p_a g_o$, bigonial arc, measured with the steel tape from one gonion to the other through the pogonion; a very uncertain measurement.

(ii) *Heights and Lengths; using Mandible Board.*

Unless otherwise stated, it is always to be understood that the mandible is in the standard basal position. Heights above the plane of the basis are found either by the aid of a scrib-awl or with the solid set-square, and projections are found with the solid set-square (see p. 256).

ih, least height of incisura, with scrib-awl. ih', greatest depth of incisura from a line joining condylion to coronion. The mandible is inverted so that it is supported on the left side by the condylion and coronion and by one of those points on the right ramus. Then the measurement is the greatest height of the incisura above the supporting surface; read by aid of scrib-awl. $c_r h$, height of coronion. Projection on to vertical rameal wing of board, using solid set-square. $c_y h$, condylar height. Vertical projection as for $c_r h$. $d_t h$, height of intradental; with scrib-awl. $m_2 h$, height of outer alveolar margin at middle of second molar; with scrib-awl. $p_1 h$, height of outer alveolar margin at middle of first premolar; with scrib-awl. $c_p l$, length of corpus. The rameal wing of the mandible board is brought up into the position in which it touches two points of the left ramus and one of the right. The measurement $c_p l$ is the projection from the zero line of the board to the most

* See *Biometrika*, Vol. I. p. 417. Fawcett gave the letters W_1 and W_2 to these two measurements, but Thomson altered them to w_1 and w_2 as W seemed more appropriate for weight of skull. I have used small letters to denote all mandibular measurements.

† This and all the following lengths are measured with small callipers unless otherwise stated.

advanced point of the chin—the pogonion A. rl, length of ramus. Projection of ramus on to rameal wing of board with position as for finding $c_p l$. ml, mandibular length. Projection from posterior points of condyles to pogonion A, with rameal wing of board vertical.

(iii) *Angular Measurements using Mandible Board and Craniophor.*

$M\angle$, mandibular angle, i.e. angle between the standard basal and rameal planes; position on board as for finding the length $c_p l$. $R\angle$, angle of condylar-coronoidal line with ramus tangent. The mandible is inverted so that it rests on the left coronion and condylion and one point of the right ramus. Then adjustment is made so that the rameal wing of the board is in the position of the left ramus tangent while making contact with one point on the right ramus. $G\angle$, gnathio-gonional angle, i.e. the angle subtended at the gnathion by the two gonia; determined from the measured lengths $g_o g_o$, $g_n g_o(l)$, and $g_n g_o(r)$ with the aid of Pearson's Trigonometer. $C\angle$, angle between the standard basal plane and the line in the sagittal plane joining pogonion A to intradental; found with the goniometer resting on a horizontal plane parallel to the standard basal plane. $C'\angle$, angle between standard basal plane and the line joining pogonion A to infradental; similar to $C\angle$. $C\angle$ and $C'\angle$ are the "mental angles" of the mandible. $L\angle$, angle between line joining pogonion A to infradental and Frankfurt horizontal plane; measured with goniometer on skull and mandible adjusted on craniophor to Frankfurt horizontal with base of skull uppermost. The angle is measured as on the left *norma lateralis* (the skull being vertex uppermost) from the facial profile clockwise to the horizontal, so $L\angle$ will be obtuse if the infradental is more advanced than the pogonion. $L'\angle$, angle between line joining pogonion B to infradental and Frankfurt horizontal plane; measured with goniometer in same sense as $L\angle$. $L\angle$ and $L'\angle$ are the "mandibular profile angles." $F\angle$, angle between line joining pogonion B to nasion and Frankfurt horizontal plane. This is the total profile angle (*angulus lateralis totalis*). $S\angle$, the angle between the Frankfurt horizontal and standard basal planes; this is $(L\angle - C'\angle)$ and no great reliance can be placed on its accuracy since $L\angle$ is rather an uncertain measurement.

(iv) *Indicial measurements.*

From the foregoing direct measurements the following indicial ones are calculated: $100\, c_r h/ml$, height-length; $100\, c_r c_r/ml$, breadth-length; $100\, g_o g_o/c_p l$, corpus breadth-length; $100\, rb'/rl$, rameal breadth-length; $100\, c_y b/c_y l$, condylar breadth-length; $100\, g_o g_o/c_r c_r$, gonional-coronoidal breadths; $100\, c_y h/c_r h$, condylar-coronoidal heights; $100\, ih'/c_r c_y$, incisural depth-length; $100\, d_t h/c_r h$, intradental-coronoidal heights.

The mean measurements of the Tibetan mandibles are given in the following table.

As we have so far no comparative material it is not possible to estimate the true significance of the differences between the mean mandibular measurements of the two Tibetan series. The B type, which has the larger and more muscular and rugous skull, exceeds the A type for the majority of mandibular lengths, it is equal to it

TABLE XXIII. *Tibetan Mandibular Mean Measurements** (mms.).

		Lengths with Callipers															
		w_1	w_2	h_1	zz	c_rc_r	rb	rb'	G_2'	c_yc_r	g_og_o	$g_ng_o(l)$	$g_ng_o(r)$	c_yl	c_yb	m_2p_1	p_ad_t
Tibetan A	Mean No.	117·8 16	95·3 16	30·7 15	45·1 16	94·6 16	36·8 16	31·4 16	47·6 16	34·8 16	92·9 16	83·4 16	83·7 16	18·6 16	7·9 16	29·3 15	25·9 15
Tibetan B ...	Mean No.	122·0 12	99·4 12	35·9 12	47·0 12	93·9 12	37·3 12	34·2 12	48·6 12	38·1 12	96·5 11	88·7 11	88·0 12	19·9 12	8·2 12	29·2 12	31·2 12

		Lengths with Callipers				Tape	Lengths and Heights on Mandible Board										Indices	
		p_ag_n	p_ad_f	g_nd_f	p_ap_b	$g_op_ag_o$	ih	ih'	c_rh	c_yh	d_th	m_2h	p_1h	c_pl	rl	ml	c_rh/ml	c_rc_r/ml
Tibetan A .	Mean No.	7·7 16	23·0 15	28·2 15	3·6 16	187·4 16	40·4 16	16·3 16	60·7 16	49·6 16	34·8 15	23·3 16	28·4 16	74·0 16	57·8 16	105·1 16	57·8 16	89·8 16
Tibetan B .	Mean No.	7·8 12	28·1 12	33·8 12	4·0 7	201·0 12	43·6 12	16·6 12	65·2 12	54·9 12	40·1 12	27·6 12	33·9 12	78·9 12	61·0 12	109·6 12	59·7 12	86·0 12

		Indices							Angles								
		g_og_o/c_pl	rb'/rl	c_yb/c_yl	g_og_o/c_rc_r	c_yh/c_rh	ih'/c_rc_y	d_th/c_rh	$M\angle$	$R\angle$	$G\angle$	$C\angle$	$C'\angle$	$L\angle$	$L'\angle$	$F\angle$	$S\angle$
Tibetan A ...	Mean No.	125·8 16	54·5 16	42·7 16	98·4 16	81·7 16	46·9 16	57·6 15	126·1 16	71°·4 16	67°·7 16	64°·4 15	61°·4 15	96·8 15	95°·1 15	91°·7 16	35·4 15
Tibetan B ..	Mean No.	121·7 11	56·3 12	41·3 12	102·9 11	84·0 12	43·8 12	61·7 12	123°·0 12	73°·1 12	66°·2 11	66°·7 12	63·1 12	101·1 7	99°·8 7	95°·5 7	38°·0 –

* The values given in this table for the lengths g_og_o, $g_ng_o(r)$ and $g_op_ag_o$ are not theoretically correct, though they are close approximations to the true values, because they were not determined by the position of the right gonion in the way described above (p. 255). The right gonion should be the point on the right angle nearest to the zero axis of the board when the mandible is held on the second *left* molar and the rameal wing of the board is in contact with two points on the *left* ramus and one on the right ramus. Instead of this point I marked, and took measurement from, the point on the right angle which was nearest to the zero axis of the board when the mandible was held on the second *right* molar and the rameal wing was in contact with two points on the *right* ramus and one on the left ramus. This point must be very close to the true gonion.

for a few measurements and markedly less for none. Some of the mean characters of the two types differ by more than 5 mm. and these differences appear to be greater, in proportion, for heights than for horizontal lengths. The indices are very similar and perhaps their differences have no significance, but the angles appear to be more characteristic; these show that the B type has the more receding chin when the Frankfurt horizontal is the plane of reference ($L\angle$, $L'\angle$ and $F\angle$) and consequently a less prominent one than the A type when resting on the standard basal plane ($C\angle$ and $C'\angle$).

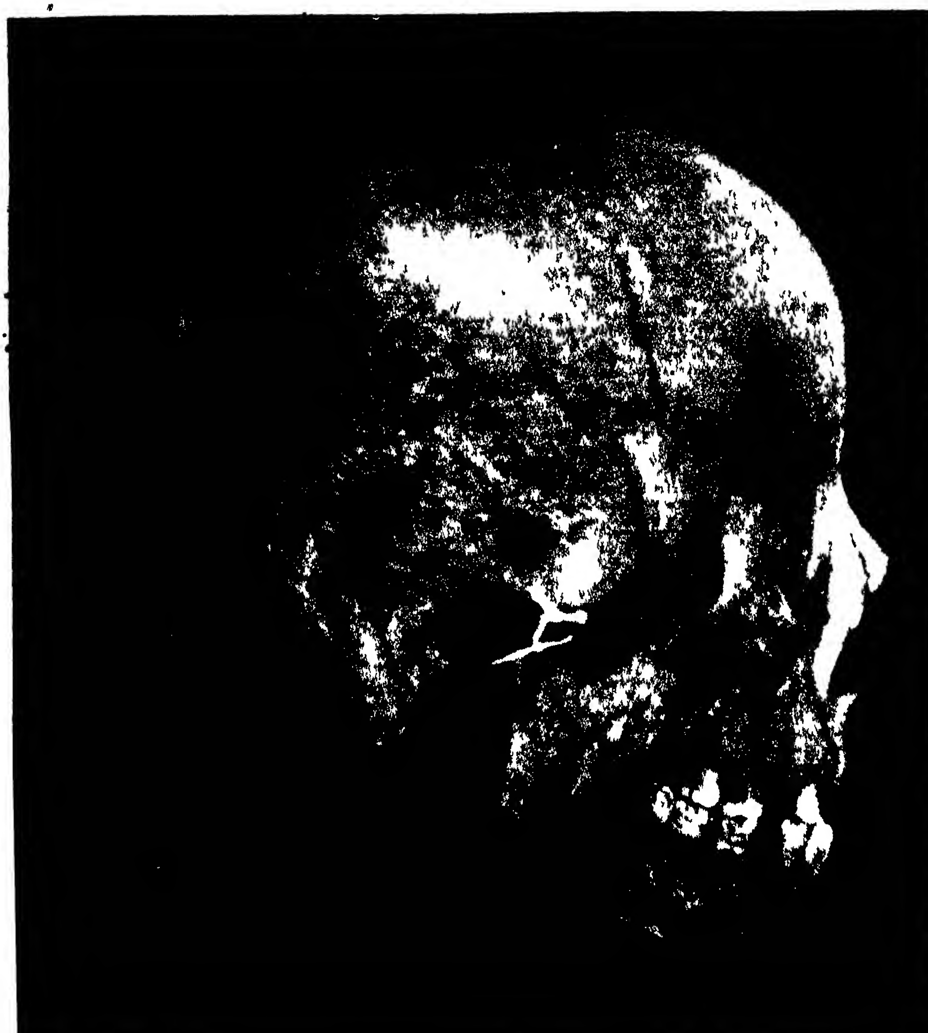
DESCRIPTION OF PLATES.

Plates I to X give the five aspects of two typical skulls of the Tibetan A and B types respectively. Neither the *norma lateralis* nor the *norma facialis* brings out clearly the greater facial flattening of the skulls of the A type which was one characteristic which distinguished them from the skulls of the B type.

Plate	I,	Tibetan	A, No. 29,	*Norma lateralis*
„	II,	„	B, No. 27,	„ „
„	III,	„	A, No. 29,	*Norma facialis*
„	IV,	„	B, No. 27,	„ „
„	V,	„	A, No 29,	*Norma basalis*
„	VI,	„	B, No 27,	„ „
„	VII,	„	A, No 29	*Norma verticalis*
„	VIII,	„	B, No 27,	„ „
„	IX,	„	A, No 29,	*Norma occipitalis*
„	X,	„	B, No 27,	„ „

The remaining plates (XIV (right) excepted) illustrate anomolies

Plate	XI	(left),	Tibetan	B, No 30,	*Os triangulare*
„	„	(right),	„	A, No 25,	Bipartite *os triangulare*
„	XII	(left),	„	B No 19,	Bipartite interparietal.
„	„	(right),	„	A, No 1,	Ossicles of lambdoid suture
„	XIII	(left),	„	A, No 8,	Asymmetrical precondyles
„	„	(right),	„	B, No 17,	Single precondyle
„	XIV	(left),	„	A, No 3,	Asymmetrical precondyles
„	„	(right),	„	B, No 14,	This photograph shows how common tangent to *lineae nuchae superiores* passes nearly through superior inion



Tibetan Skull, Type A, No. 29.

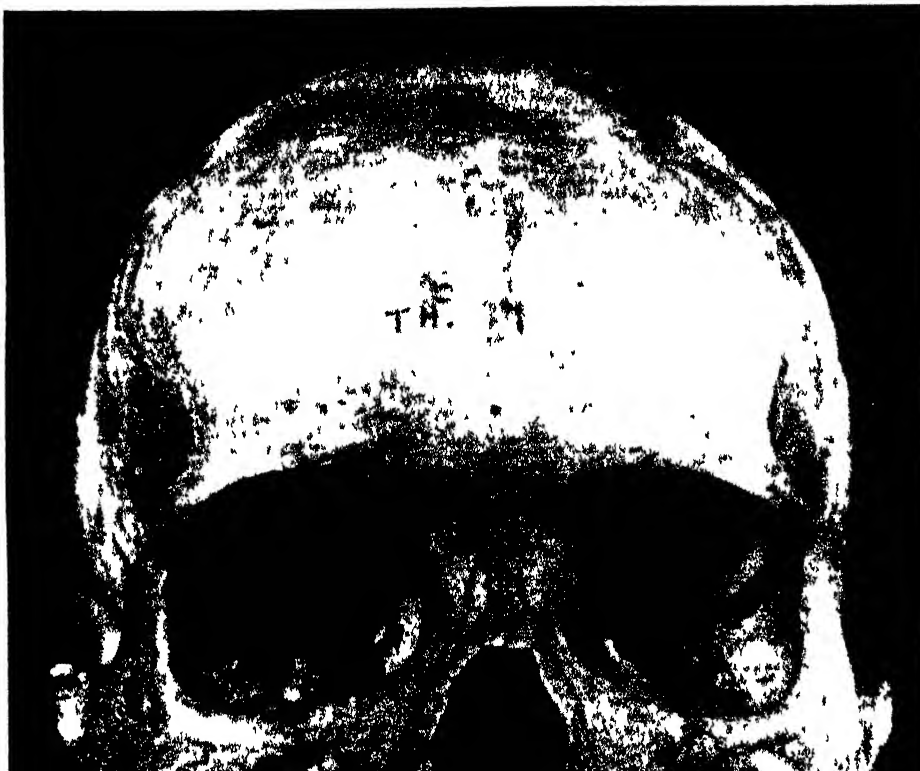
Norma lateralis

Tibetan Skull, Type B, No. 27.

Norma lateralis.

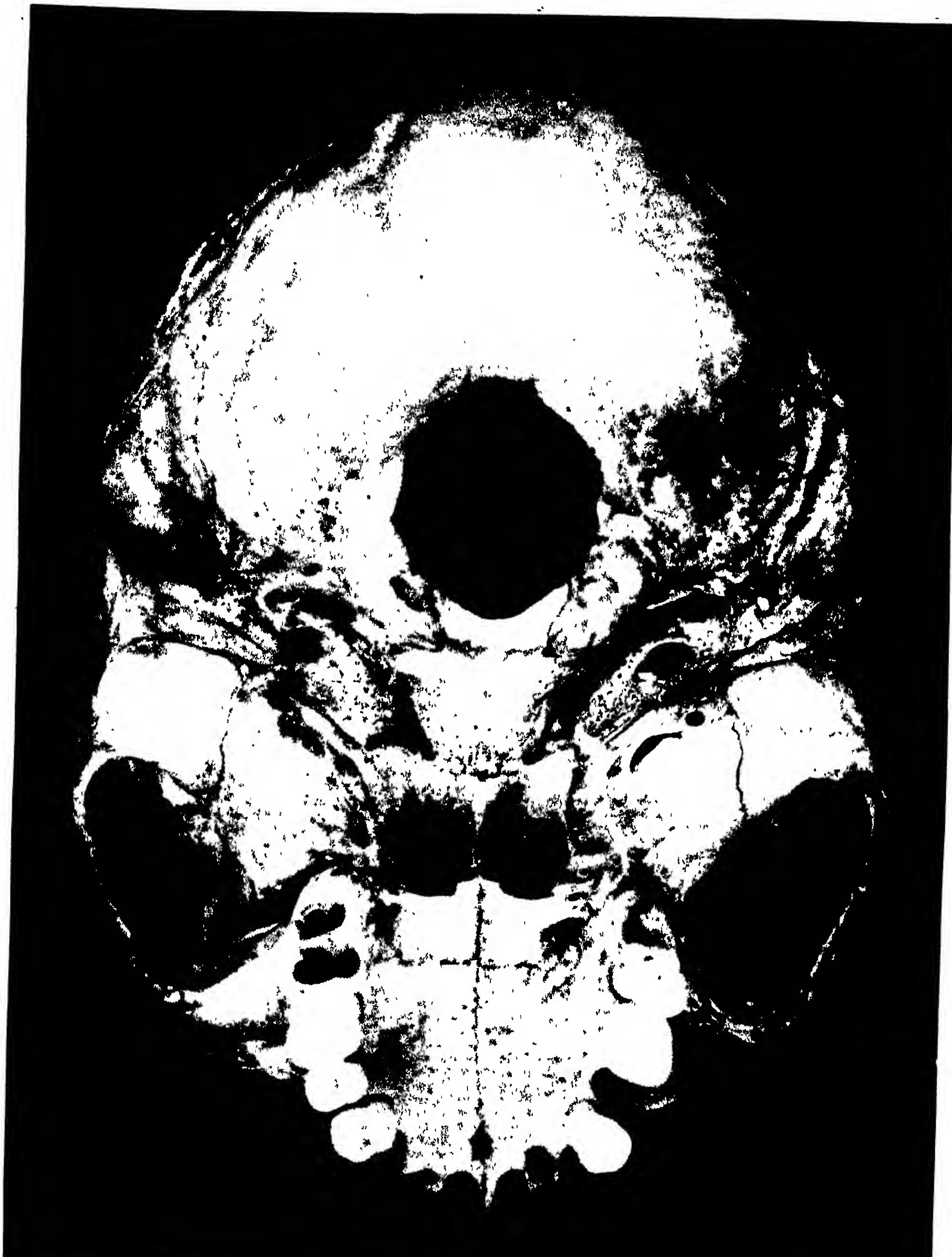
Tibetan Skull, Type A, No. 29.

Norma facialis.



Tibetan Skull, Type B, No. 27.

Norma facialis.



Tibetan Skull, Type A, No. 29.

Norma basalis.

Tibetan Skull, Type B, No 27.

Norma basalis

Morant Tibetan Skull

Tibetan Skull, Type A, No. 29.

Norma verticalis

Tibetan Skull, Type B, No 27

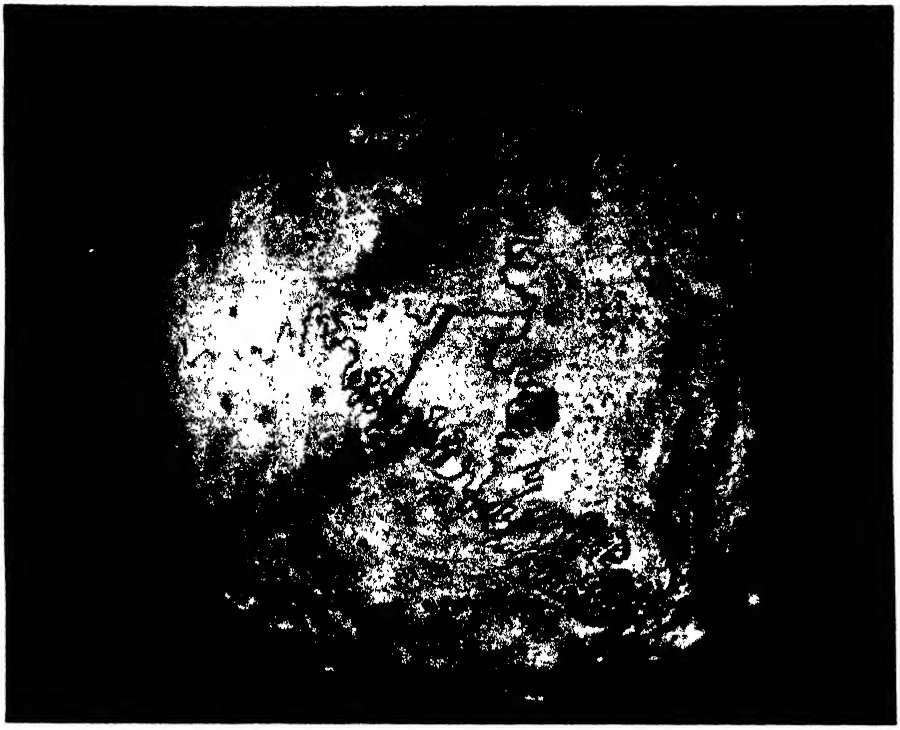
Norma verticalis

Tibetan Skull, Type A, No. 29.

Norma occipitalis.

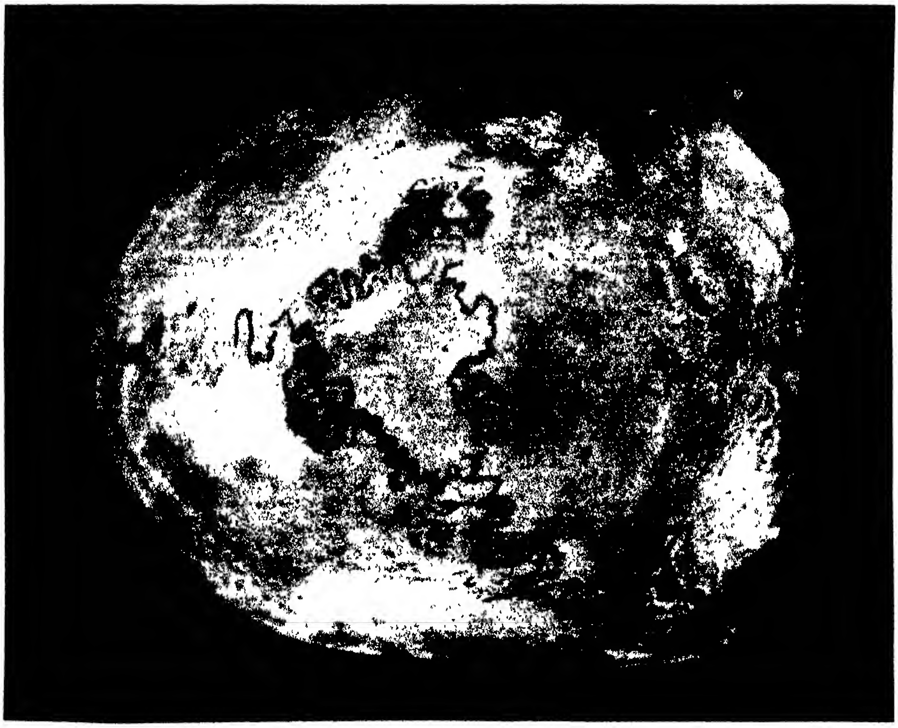
Tibetan Skull, Type B, No. 27.

Norma occipitalis.



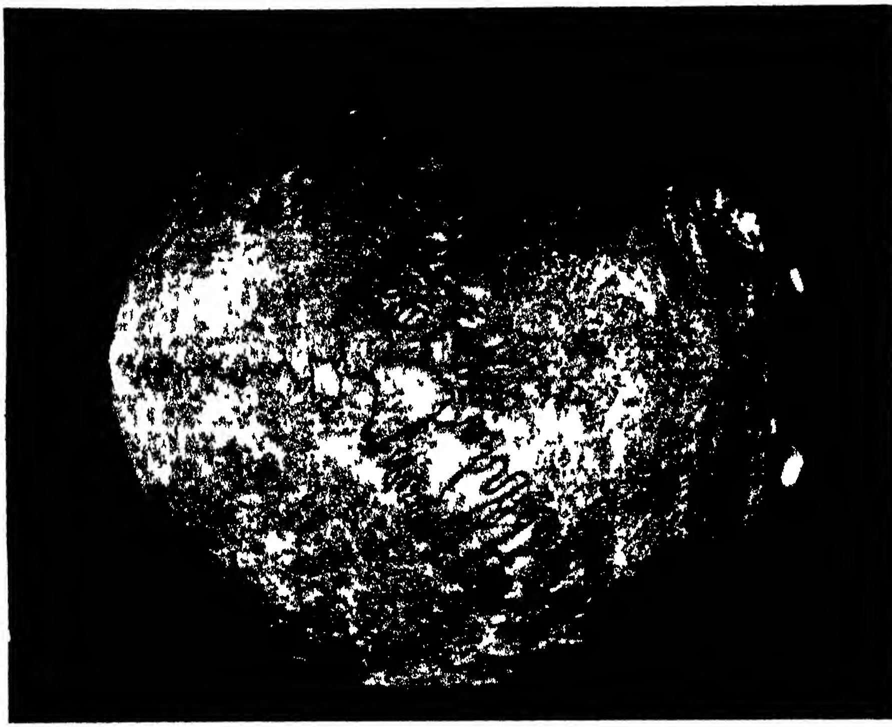
Special Skull, No. 25.

The os triangulare replacing the os pentagonale becomes bipartite.



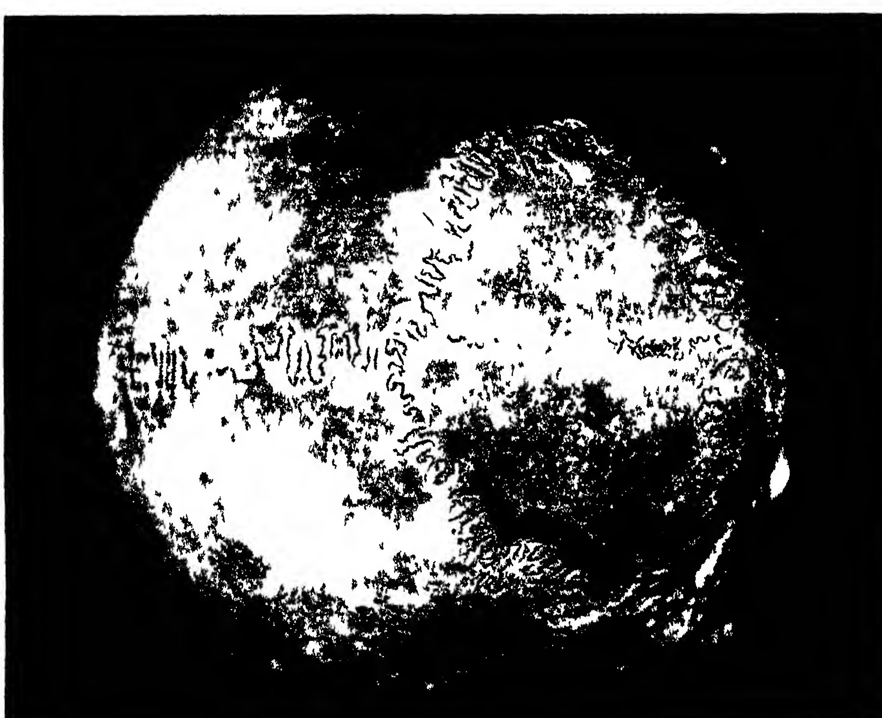
Special Skull, No. 30.

An os triangulare replaces the more usual os pentagonale.



Special Skull, No. 1.

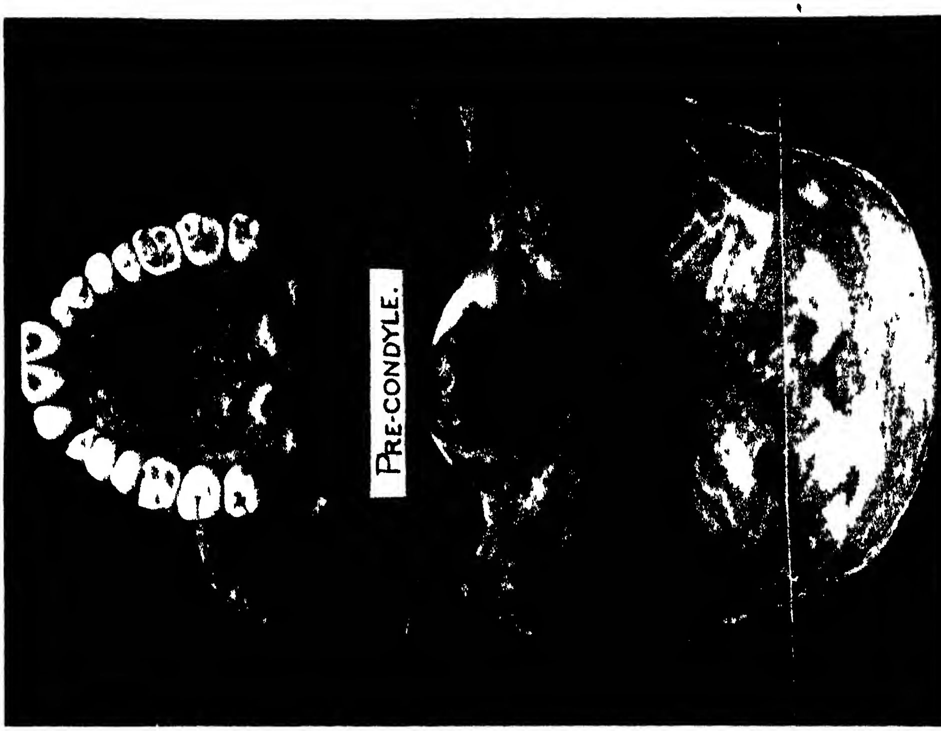
Numerous Ossicles of lambdoid Suture.



Special Skull, No 19.

Bipartite Interparietal

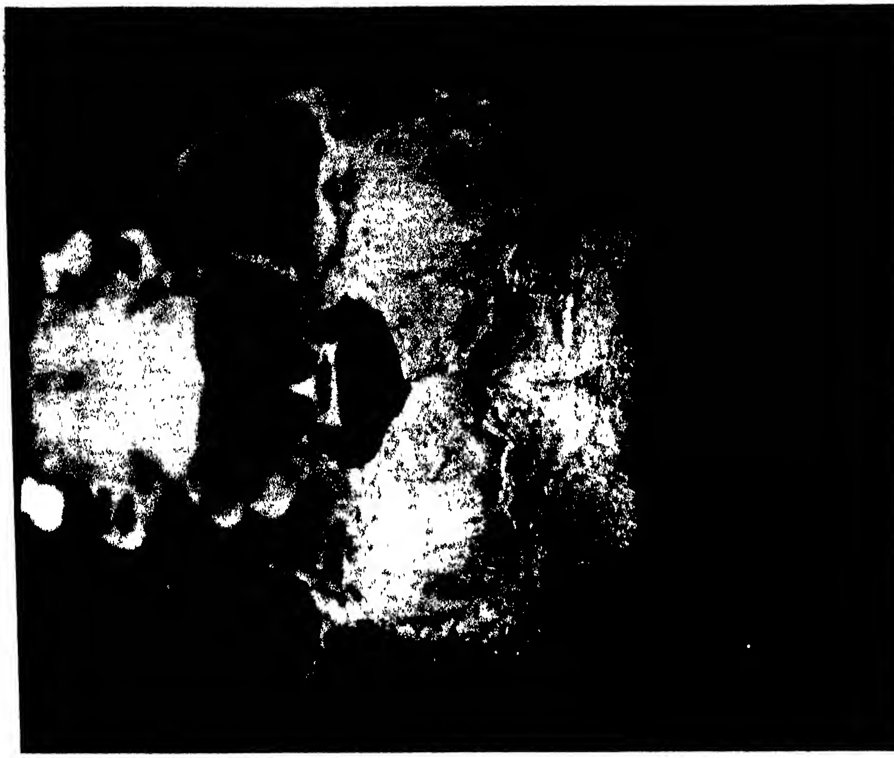
Mount *Ilitin Skull*



Special Sku o. 17

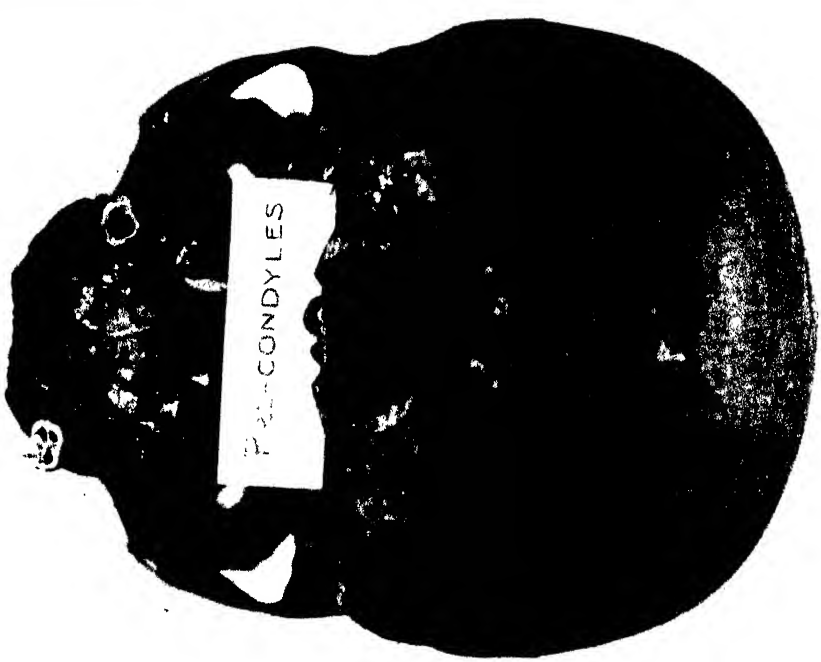
Special Skull, No. 8.
Asymmetrical Precondyles

Morant, *Tibetan Skulls*



Special Skull, No. 14.

S owing how common tangent to lineae nuchae superiores passes nearly through superior inion.



Special Skull, No. 3.

symmetrica Precondyles

THE PROBABLE ERROR OF A CLASS-INDEX CORRELATION.

By EGON S. PEARSON, B.A.

(1) The correlation between the quantitative value of a variate and its class-index when the variate is classed or "indexed" in broad categories is termed the "class-index correlation." As a rule this class-index is taken as the value of the mean of the variates falling within the category, but it may also be taken as their median, or as the mid-point of the sub-range corresponding to the broad category. In this paper the class-index adopted is the mean of the variates falling into the broad category. The importance of the class-index correlation arises from its use as a correction for correlations determined from broad-category classifications, whether these are treated by contingency or by correlation-ratio methods. It has accordingly become a problem of very real interest to determine the probable error of a class-index correlation. Is such a correlation subject to a large error, so that it cannot reasonably be used as a correction? The object of this paper is to answer this question.

In previous papers* the probable errors have been given of various constants required in the description of the frequency distribution in a sample taken from a larger population. If the particular character in the population under consideration is classified only by broad qualitative categories, it is generally most convenient to fit the data to a Normal Scale, and the following notation is that in common use.

M is the total population supposed "indefinitely large," from which a sample N, small compared to M, is taken. On the other hand we are not dealing with "small samples" and shall neglect second order terms with factors $1/N$‡.

$$y_s = \frac{N}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{x_s^2}{\sigma^2}}, \qquad n_s = \int_{x_s}^{\infty} \frac{N}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx,$$

where σ represents the standard deviation, n_s the frequency beyond x_s in a *sample*, and x is measured from the mean of the sample. As the data are divided into qualitative categories only, we do not know σ, y_s and x_s but only n_s†, and thence from Sheppard's Tables of the Normal Curve can find

$$h_s = \frac{x_s}{\sigma}; \quad H_s = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h_s^2}; \quad \text{and } h_{r,s} = \frac{x_{r,s}}{\sigma} = \frac{H_s - H_r}{(n_s - n_r)/N},$$

where $x_{r,s}$ is the abscissa of the centroid of the class or group lying between x_s and x_r. For the *total population* we shall have corresponding constants $\tilde{h}_s$, $\tilde{h}_{r,s}$, $\tilde{H}_s$ and $\tilde{n}_s$, where

$$h_s = \tilde{h}_s + \delta h_s, \ h_{r,s} = \tilde{h}_{r,s} + \delta h_{r,s}, \ H_s = \tilde{H}_s + \delta H_s, \ n_s = \tilde{n}_s N/M + \delta n_s.$$

* *Biometrika*, Vol. II. pp. 273—281; Vol. IX. pp. 1—10; Vol. XIII. pp. 118—132.

† n_s/N is of course the $\frac{1}{2}(1-\alpha)$ of Sheppard's Tables.

‡ This is what a member of the Biometric School understands when he replaces "statistical differentials" by "mathematical differentials."

Now it has been shown in the earlier papers referred to in the footnote on p. 261, that if statistical differentials may be treated as mathematical differentials, or in other words $1/\sqrt{N}$ is small compared to unity, the following fundamental relations hold for the variations, δn_s and δn_r, in a sample of N from an indefinitely larger population:

$$\text{Mean}(\delta n_s^2) = \sigma^2_{n_s} = \frac{N\tilde{n}_s}{M}\left(1 - \frac{\tilde{n}_s}{M}\right)$$

$$= n_s\left(1 - \frac{n_s}{N}\right) \text{ approximately} \quad \ldots\ldots\ldots\ldots\ldots\text{(i)},$$

$$\text{Mean}(\delta n_s \delta n_r) = \frac{N\tilde{n}_r}{M}\left(1 - \frac{\tilde{n}_s}{M}\right)$$

$$= n_r\left(1 - \frac{n_s}{N}\right) \text{ approximately} \quad \ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

where in the second equation the frequency n_s includes n_r.

Starting from these two equations it is proposed in the present paper to find the probable errors due to random sampling when the sampled population is approximately normal,

(*a*) of $h_{r,s} = \frac{a_{r,s}}{\sigma}$, or the distance, in terms of the standard deviation as unit, of the mean of a group from the mean of the whole sample;

(*b*) of $r_{x c_x}$, the Class-Index Correction* or correlation of a variate with its class-mark given by the relation

$$r^2_{x c_x} = \frac{\sigma^2_{c_x}}{\sigma^2} = S\left\{\frac{n_s - n_{s+1}}{N} h^2_{s+1, s}\right\} \quad \ldots\ldots\ldots\ldots\ldots\text{(iii)},$$

where σ_{c_x} is the standard deviation of the means of the groups or class-marks, and the summation is for all groups.

(2) As a preliminary step it is necessary to find the following mean values of the squares and products of the variations of the constants in samples of N:

$\text{Mean}(\delta H_s^2) = \sigma^2_{H_s}$; $\text{Mean}(\delta H_s \delta H_r)$, $\text{Mean}(\delta H_s \delta n_r)$ and $\text{Mean}(\delta H_s \delta n_s)$,

where n_s includes n_r.

Now
$$\tilde{n}_s = \int_{\tilde{h}_s}^{\infty} \frac{M}{\sqrt{2\pi}} e^{-\frac{1}{2}h^2}\, dh,$$

and it follows, if we may treat statistical differentials as mathematical, that

$$\delta n_s = -\frac{N}{M}\frac{M}{\sqrt{2\pi}} e^{-\frac{1}{2}\tilde{h}_s^2}\,\delta h_s = -N\tilde{H}_s \delta h_s,$$

δn_s being a variation in a sample of size N;

$$\delta H_s = -\frac{\tilde{h}_s}{\sqrt{2\pi}} e^{-\frac{1}{2}\tilde{h}_s^2}\,\delta h_s = -\tilde{h}_s \tilde{H} \delta h_s,$$

and therefore
$$\delta H_s = \frac{1}{N}\tilde{h}_s \delta n_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

* *Biometrika*, Vol. IX. pp. 116—139.

Making use of (i), (ii) and (iv), it is found at once that

$$\text{Mean}\,(\delta H_s^2) = \sigma^2_{H_s} = \frac{\tilde{h}_s^2}{N}\frac{\tilde{n}_s}{M}\left(1-\frac{\tilde{n}_s}{M}\right) \quad \text{.................(v)},$$

$$\text{Mean}\,(\delta H_s \delta H_r) = \frac{\tilde{h}_s\tilde{h}_r}{N}\frac{\tilde{n}_r}{M}\left(1-\frac{\tilde{n}_s}{M}\right) \quad \text{.....................(vi)},$$

$$\text{Mean}\,(\delta H_s \delta n_r) = \tilde{h}_s\frac{\tilde{n}_r}{M}\left(1-\frac{\tilde{n}_s}{M}\right) \quad \text{.....................(vii)},$$

$$\text{Mean}\,(\delta H_r \delta n_s) = \tilde{h}_r\frac{\tilde{n}_r}{M}\left(1-\frac{\tilde{n}_s}{M}\right) \quad \text{.....................(viii)},$$

$$\text{Mean}\,(\delta H_s \delta n_s) = \tilde{h}_s\frac{\tilde{n}_s}{M}\left(1-\frac{\tilde{n}_s}{M}\right) \quad \text{.......................(ix)},$$

where in all cases n_s includes n_r.

The values of the constants in the sampled population are generally unknown, and for practical purposes it is therefore necessary to put into the above equations the values found from the sample*, or h_s, h_r, $\frac{n_s}{N}$ and $\frac{n_r}{N}$.

(3) $$\tilde{h}_{r,s} = \frac{\tilde{H}_s - \tilde{H}_r}{(\tilde{n}_s - \tilde{n}_r)/M},$$

and hence

$$\delta h_{r,s} = \frac{M}{\tilde{n}_s - \tilde{n}_r}(\delta H_s - \delta H_r) - \frac{M(\tilde{H}_s - \tilde{H}_r)}{(\tilde{n}_s - \tilde{n}_r)^2}(\delta n_s - \delta n_r)\frac{M}{N}$$

δn_s and δn_r being variations in a sample of size N,

$$= \tilde{h}_{r,s}\frac{\delta H_s - \delta H_r}{\tilde{H}_s - \tilde{H}_r} - \frac{\tilde{h}^2_{r,s}}{N}\frac{\delta n_s - \delta n_r}{\tilde{H}_s - \tilde{H}_r}.$$

After forming the equation of differentials, we may substitute $h_{r,s}$ for $\tilde{h}_{r,s}$ and H_s for $\tilde{H}_s$, etc., in this equation; then squaring both sides and taking the mean for all possible variations in random samples, we have

$$\begin{aligned}\text{Mean}\,(\delta h^2_{r,s}) = \frac{h^2_{r,s}}{(H_s - H_r)^2}\Big\{&\text{Mean}\,(\delta H_s^2) + \text{Mean}\,(\delta H_r^2) - 2\times\text{Mean}\,(\delta H_s\delta H_r)\\ &+\frac{h^2_{r,s}}{N^2}(\text{Mean}\,(\delta n_s^2) + \text{Mean}\,(\delta n_r^2) - 2\times\text{Mean}\,(\delta n_s\delta n_r))\\ &-2\frac{h_{r,s}}{N}(\text{Mean}\,(\delta H_s\delta n_s) + \text{Mean}\,(\delta H_r\delta n_r) - \text{Mean}\,(\delta H_s\delta n_r) - \text{Mean}\,(\delta H_r\delta n_s))\Big\}.\end{aligned}$$

Making use of the relations (v)—(ix) modified by the substitution of the

* For example $n_s = \frac{N\tilde{n}_s}{M} \pm \lambda\sigma_{n_s}$ where λ will on the average be greater than 2 in 4·5 °/。 of samples, and greater than 3 in only about 0·3°/。. Hence if the substitution of $\frac{n_s}{N}$ for $\frac{\tilde{n}_s}{M}$ is to be justifiable, $\frac{\lambda\sigma_{n_s}}{N}$ must be small compared to $\frac{\tilde{n}_s}{M}$ or $\frac{n_s}{N}$. Using Equation (i) it is found that this condition implies that $\lambda^2\left(1-\frac{n_s}{N}\right)$ should be small compared with n_s. $1-\frac{n_s}{N}$ is always less than unity, so that we see that in arranging our material, it is important to avoid groups containing less than 10 to 20 individuals; smaller cell contents should be combined together, or else the methods of "small samples" must be resorted to.

sample instead of the population values of the constants, and collecting terms, we find that

$$\sigma^2_{h_{r,s}} = \text{Mean}\,(\delta h^2_{r,s})$$

$$= \frac{N^2}{(n_s - n_r)^2}\left\{\frac{n_s}{N}\left(1 - \frac{n_s}{N}\right)\left(\frac{h_s^2}{N} + \frac{h^2_{r,s}}{N} - 2\frac{h_{r,s}h_s}{N}\right) + \frac{n_r}{N}\left(1 - \frac{n_r}{N}\right)\left(\frac{h_r^2}{N} + \frac{h^2_{r,s}}{N} - 2\frac{h_{r,s}h_r}{N}\right)\right.$$

$$\left. + 2\frac{n_r}{N}\left(1 - \frac{n_s}{N}\right)\left(-\frac{h_s h_r}{N} - \frac{h^2_{r,s}}{N} + \frac{h_{r,s}h_s}{N} + \frac{h_{r,s}h_r}{N}\right)\right\},$$

$$\sigma_{h_{r,s}} = \frac{\sqrt{N}}{n_s - n_r}\left\{\frac{n_s}{N}\left(1 - \frac{n_s}{N}\right)(h_s - h_{r,s})^2 + \frac{n_r}{N}\left(1 - \frac{n_r}{N}\right)(h_r - h_{r,s})^2\right.$$

$$\left. - 2\frac{n_r}{N}\left(1 - \frac{n_s}{N}\right)(h_s - h_{r,s})(h_r - h_{r,s})\right\}^{\frac{1}{2}} \quad \text{.........}\ \text{..(x).}$$

If the sample be large, no great error will be involved in assuming that the variations, $\delta h_{r,s}$, follow a normal distribution, so that we may take $\pm\,{\cdot}67449\,\sigma_{h_{r,s}}$ as the probable error of $h_{r,s}$.

(4) Let us suppose that our material is divided into p categories. Then by definition $\tilde{h}_0 = -\infty$, $\tilde{h}_p = +\infty$, $\tilde{H}_0 = 0 = \tilde{H}_p$, $\tilde{n}_0 = M$, $\tilde{n}_p = 0$. Further since

$$\tilde{h}_{s+1,\,s} = \frac{\tilde{H}_s - \tilde{H}_{s+1}}{(\tilde{n}_s - \tilde{n}_{s+1})/M},$$

we shall have for the means of the end categories

$$\tilde{h}_{10} = \frac{-\tilde{H}_1}{1 - \dfrac{\tilde{n}_1}{M}}, \qquad \tilde{h}_{p,\,p-1} = \frac{\tilde{H}_{p-1}}{\tilde{n}_{p-1}/M}.$$

Now
$$\tilde{r}^2_{x\,c_x} = \overset{p-1}{\underset{s=0}{S}}\left\{\frac{\tilde{n}_s - \tilde{n}_{s+1}}{M}\,\tilde{h}^2_{s+1,\,s}\right\} \quad \text{... . ...(iii)}\ bis$$

$$= M\overset{p-1}{\underset{s=0}{S}}\left\{\frac{(\tilde{H}_s - \tilde{H}_{s+1})^2}{\tilde{n}_s - \tilde{n}_{s+1}}\right\},$$

where it must be remembered that terms in $\tilde{H}_0$ and $\tilde{H}_p$ do not occur. Now if we treat statistical differentials as mathematical differentials, we have for the equation connecting the variations in $r_{x.c_x}$ with the variations in the $\tilde{H}$'s and $\tilde{n}$'s,

$$2\tilde{r}_{x\,c_x}\,\delta r_{x\,c_x} = 2M\overset{p-1}{\underset{s=0}{S}}\left\{\frac{(\tilde{H}_s - \tilde{H}_{s+1})(\delta H_s - \delta H_{s+1})}{\tilde{n}_s - \tilde{n}_{s+1}}\right\}$$

$$- M\overset{p-1}{\underset{s=0}{S}}\left\{\frac{(\tilde{H}_s - \tilde{H}_{s+1})^2\,\dfrac{M}{N}(\delta n_s - \delta n_{s+1})}{(\tilde{n}_s - \tilde{n}_{s+1})^2}\right\},$$

and on replacing $\tilde{H}_s$ by H_s, $\dfrac{\tilde{n}_s}{M}$ by $\dfrac{n_s}{N}$, etc., as we may do after differentiation, we find that

$$r_{x.c_x}\,\delta r_{x.c_x} = \overset{p-1}{\underset{s=1}{S}}\{h_{s+1,\,s}(\delta H_s - \delta H_{s+1})\} - \tfrac{1}{2}\overset{p-1}{\underset{s=0}{S}}\left\{\frac{h^2_{s+1,\,s}}{N}(\delta n_s - \delta n_{s+1})\right\}$$

$$= \overset{p-1}{\underset{s=1}{S}}\{\delta H_s(h_{s+1,\,s} - h_{s,\,s-1})\} - \tfrac{1}{2}\overset{p-1}{\underset{s=1}{S}}\left\{\frac{\delta n_s}{N}(h^2_{s+1,\,s} - h^2_{s,\,s-1})\right\},$$

since there are no terms in δH_0 and δH_p.

Squaring both sides of this equation, we find

$$\begin{aligned} r^2_{x c_x}(\delta r_{x c_x})^2 = \mathop{S}_{s=1}^{p-1} \Big\{ & \delta H_s^2 (h_{s+1,s} - h_{s,s-1})^2 + \frac{1}{4N^2}\delta n_s^2 (h^2_{s+1,s} - h^2_{s,s-1})^2 \\ & - \frac{1}{N}\delta H_s \delta n_s (h_{s+1,s}-h_{s,s-1})(h^2_{s+1,s}-h^2_{s,s-1})\Big\} \\ & + 2 \mathop{S'}_{j,k}\{\delta H_j \delta H_k (h_{j+1,j}-h_{j,j-1})(h_{k+1,k}-h_{k,k-1})\} \\ & + \frac{2}{4N^2} \mathop{S'}_{j,k}\{\delta n_j \delta n_k (h^2_{j+1,j}-h^2_{j,j-1})(h^2_{k+1,k}-h^2_{k,k-1})\} \\ & - \frac{1}{N} \mathop{S'}_{j,k}\{\delta H_j \delta n_k (h_{j+1,j}-h_{j,j-1})(h^2_{k+1,k}-h^2_{k,k-1})\} \\ & - \frac{1}{N} \mathop{S'}_{j,k}\{\delta H_k \delta n_j (h_{k+1,k}-h_{k,k-1})(h^2_{j+1,j}-h^2_{j,j-1})\}, \end{aligned}$$

where $\mathop{S'}_{j,k}$ indicates the summation for all pairs of integers (j not equal to k) between 1 and $p-1$. Now if we sum both sides of the last equation for all possible variations of $r_{x c_x}$, of the H's and of the n's, and take the mean value, we have on making use of relations (iv) to (ix), and then collecting terms,

$$\begin{aligned} r^2_{x c_x}\,\text{Mean}\,[(\delta r_{x c_x})^2] &= r^2_{x c_x}\sigma^2_{r_{x c_x}} \\ &= \mathop{S}_{s=1}^{p-1}\left\{\frac{n_s}{N}\left(1-\frac{n_s}{N}\right)\left[(h_{s+1,s}-h_{s,s-1})^2\frac{h_s^2}{N} + (h^2_{s+1,s}-h^2_{s,s-1})^2\frac{1}{4N}\right.\right. \\ &\qquad\left.\left. - (h_{s+1,s}-h_{s,s-1})(h^2_{s+1,s}-h^2_{s,s-1})\frac{h_s}{N}\right]\right\} \\ &\quad + 2\mathop{S'}_{j,k}\left\{\frac{n_k}{N}\left(1-\frac{n_j}{N}\right)\left[(h_{j+1,j}-h_{j,j-1})(h_{k+1,k}-h_{k,k-1})\frac{h_j h_k}{N}\right.\right. \\ &\quad + (h^2_{j+1,j}-h^2_{j,j-1})(h^2_{k+1,k}-h^2_{k,k-1})\frac{1}{4N} \\ &\quad - (h_{j+1,j}-h_{j,j-1})(h^2_{k+1,k}-h^2_{k,k-1})\frac{h_j}{2N} \\ &\quad \left.\left. - (h_{k+1,k}-h_{k,k-1})(h^2_{j+1,j}-h^2_{j,j-1})\frac{h_k}{2N}\right]\right\}, \end{aligned}$$

where n_j includes n_k. Hence

$$\begin{aligned} r^2_{x c_x}\sigma^2_{r_{x c_x}} &= \mathop{S}_{s=1}^{p-1}\left\{\frac{n_s}{N^2}\left(1-\frac{n_s}{N}\right)(h_{s+1,s}-h_{s,s-1})^2\left[h_s - \tfrac{1}{2}(h_{s+1,s}+h_{s,s-1})\right]^2\right\} \\ &\quad + \mathop{S'}_{j,k}\left\{\frac{2n_k}{N^2}\left(1-\frac{n_j}{N}\right)(h_{j+1,j}-h_{j,j-1})(h_{k+1,k}-h_{k,k-1})\left[h_j - \tfrac{1}{2}(h_{j+1,j}+h_{j,j-1})\right]\right. \\ &\qquad \left.\times\left[h_k - \tfrac{1}{2}(h_{k+1,k}+h_{k,k-1})\right]\right\} \quad \ldots\ \text{(xi)}, \end{aligned}$$

where the *first summation* is for all values of s from 1 to $p-1$, e.g. if the data are divided into 7 categories giving dichotomies at $-\infty, h_1, h_2 \ldots h_6, +\infty$ there will be six terms in the summation.

The *second summation* is for all possible pairs of integers (j not equal to k) between 1 and $p-1$; there will thus be $\frac{(p-1)(p-2)}{2}$ terms in the summation, e.g. if $p=7$, there will be 15 terms.

In order to calculate the standard deviation and therefore (on the assumption of the normality of the distribution of $\delta r_{x\,c_x}$) the probable error of a class-index correction, the following expressions are required for each dichotomy, h_s:

(1) $\frac{n_s}{N}=\alpha_s$, (2) $1-\frac{n_s}{N}=\beta_s$,

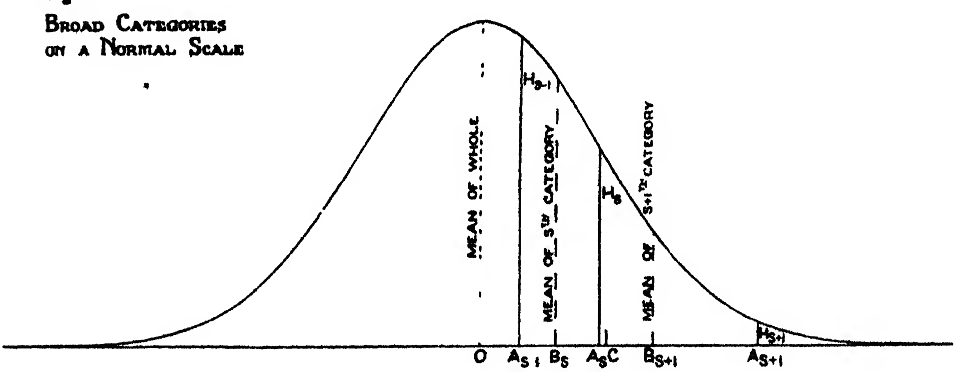
(3) $h_{s+1,\,s}-h_{s,\,s-1}=\gamma_s$, (4) $h_s-\frac{1}{2}(h_{s+1,\,s}+h_{s,\,s-1})=\delta_s$,

whence we have from (xi)

$$\sigma_{r_{c_x}}=\frac{1}{\sqrt{N}}\frac{1}{r_{c_x}}\sqrt{\overset{p-1}{\underset{s=1}{S}}(\alpha_s\beta_s\gamma_s^2\delta_s^2)+2\underset{j,k}{S'}(\alpha_k\beta_j\gamma_j\gamma_k\delta_j\delta_k)}\;\text{.............(xii).}$$

In Figure I, $OA_{s-1}=h_{s-1}$, $OA_s=h_s$, $OA_{s+1}=h_{s+1}$,
$OB_s=h_{s,\,s-1}$, $OB_{s+1}=h_{s+1,\,s}$,
C is the mid-point of B_sB_{s+1}.

Fig I
Broad Categories on a Normal Scale



Then $\gamma_s=B_sB_{s+1}$, $\delta_s=CA_s$. Both γ_s and δ_s will decrease as the number of groups is increased: δ_s is likely to be a very small quantity unless we are dealing with a few unequal groups. These points will be illustrated in the numerical examples given below.

(5) The problem is not completed, until we have considered the manner in which the errors in $r_{x.c_x}$ and $r_{y\,c_y}$ affect the corrected coefficients of correlation or of mean square contingency, or the corrected correlation ratio. We know that

$$r_{xy}=\frac{r_{c_x\,c_y}}{r_{x.c_x}r_{y\,c_y}}\;\text{..... ....(xiii),}$$

where $r_{c_x.c_y}$ is the coefficient of correlation between the categories or class-marks of

the variates, and r_{xy} is the correlation between their true quantitative values. Or again using the coefficient mean square contingency

$$r_{xy} = \frac{C_2}{r_{x.c_x} r_{y.c_y}} \quad \text{.................................(xiv).}$$

Lastly if we are calculating a correlation ratio, and $H_{y_{c_x}}$ is the value found when the y variate is finely classified so that the mean of an array can be determined, while the x variate is divided into broad classes only, then the corrected η is given by

$$\eta_{y_x} = \frac{H_{y_{c_x}}}{r_{x.c_x}} \quad \text{....................................(xv).}$$

If we take logarithmic differentials* of (xiv) and square both sides of the equation, we have

$$\frac{\delta r^2_{xy}}{\tilde{r}^2_{xy}} = \frac{\delta C_2^2}{\tilde{C}_2^2} + \frac{\delta r^2_{x.c_x}}{\tilde{r}^2_{x.c_x}} + \frac{\delta r^2_{y.c_y}}{\tilde{r}^2_{y.c_y}} - \frac{2\delta C_2 \delta r_{x.c_x}}{\tilde{C}_2 \tilde{r}_{x.c_x}} - \frac{2\delta C_2 \delta r_{y.c_y}}{\tilde{C}_2 \tilde{r}_{y.c_y}} + \frac{2\delta r_{x.c_x} \delta r_{y.c_y}}{\tilde{r}_{x.c_x} \tilde{r}_{y.c_y}},$$

where $\tilde{r}_{xy}$, $\tilde{C}_2$, $\tilde{r}_{x.c_x}$ and $\tilde{r}_{y.c_y}$ are the values of the constants in the sampled population, and the δ's denote the variation from these values of the constants calculated from the material of a sample. If we sum for all possible variations, and take the mean value of both sides of the equation, we have

$$\sigma^2_{r_{xy}} = \tilde{r}^2_{xy} \left\{ \frac{\sigma^2_{c_2}}{\tilde{C}_2^2} + \frac{\sigma^2_{r_{x.c_x}}}{\tilde{r}^2_{x.c_x}} + \frac{\sigma^2_{r_{y.c_y}}}{\tilde{r}^2_{y.c_y}} - \frac{2\,\text{Mean}\,(\delta C_2\, \delta r_{x.c_x})}{\tilde{C}_2 \tilde{r}_{x.c_x}} \right.$$
$$\left. - \frac{2\,\text{Mean}\,(\delta C_2\, \delta r_{y.c_y})}{\tilde{C}_2 \tilde{r}_{y.c_y}} + \frac{2\,\text{Mean}\,(\delta r_{x.c_x} \delta r_{y.c_y})}{\tilde{r}_{x.c_x} \tilde{r}_{y.c_y}} \right\} \quad \text{............(xvi).}$$

The calculation of the three mean value expressions is likely to be complex, and it is not proposed to make the attempt in the present paper. The correlations between the values of $\tilde{C}_2$, $\tilde{r}_{x.c_x}$ and $\tilde{r}_{y.c_y}$ must however lie between $+1$ and -1, so that it follows that

$$\sigma_{r_{xy}} \leqslant \tilde{r}_{xy} \left\{ \frac{\sigma_{c_2}}{\tilde{C}_2} + \frac{\sigma_{r_{x.c_x}}}{\tilde{r}_{x.c_x}} + \frac{\sigma_{r_{y.c_y}}}{\tilde{r}_{y.c_y}} \right\} \quad \text{.....................(xvii).}$$

In the same way it follows from (xv) that

$$\sigma_{\eta_{y_x}} \leqslant \tilde{\eta}_{y_x} \left\{ \frac{\sigma_{H_{y_{c_x}}}}{\tilde{H}_{y_{c_x}}} + \frac{\sigma_{r_{x.c_x}}}{\tilde{r}_{x.c_x}} \right\} \quad \text{.........................(xviii).}$$

As the values of the constants in the sampled population will generally be unknown, we shall as usual in (xvii) and (xviii) put for $\tilde{C}_2$, $\tilde{r}_{x.c_x}$, $\tilde{r}_{y.c_y}$, etc. the values of these constants found in the sample. The results of the examples worked out below suggest that the standard deviations of the class-index corrections $r_{x.c_x}$ and

* As before we assume that statistical differentials may be treated as mathematical differentials, or that $\frac{1}{\sqrt{N}}$ is small compared with unity.

$r_{y\,c_y}$ are in general likely to be so small compared to the standard deviations of $H_{y_{c_x}}$ and C_2, that rough approximations for the upper limits of the probable errors sufficient for practical purposes will be:

(*a*) for r_{xy} found by the corrected mean square contingency method,

$$\pm \cdot 67449 \frac{r_{xy}}{C_2} \sigma_{C_2} = \pm \cdot 67449 \frac{\sigma_{C_2}}{r_{x.c_x}\, r_{y.c_y}},$$

(*b*) for η_{yx},

$$\pm \cdot 67449 \frac{\eta_{yx}}{H_{y_{c_x}}} \sigma_{H_{y_{c_x}}} = \pm \cdot 67449 \frac{\sigma_{H_{y_{c_x}}}}{r_{x\,c_x}}.$$

(6) *Illustrative Examples.*

Example 1.

At the top of the table below is given a distribution, divided into five broad classes, of the health among 1918 schoolboys*.

TABLE I.

Health in Schoolboys (5 *categories*).

Very Strong	Strong	Normally Healthy	Rather Delicate	Delicate	Total
70·5	605·5	907	313	22	1918

s	1	2	3	4
$\alpha_s = \frac{n_s}{N}$	·96324	·64755	·17466	·01147
$\beta_s = 1 - \frac{n_s}{N}$	03676	·35245	·82534	·98853
h_s	$-1·7896$	$-·3787$	$+·9359$	$+2·2744$

r, s	1, 0	2, 1	3, 2	4, 3	5, 4
$h_{r,s}$	$-2·1882$	$-·9214$	$+·2408$	$+1·3937$	$+2·6181$

s	1	2	3	4
γ_s	$+1·2668$	$+1·1622$	$+1·1529$	$+1·2244$
δ_s	$-·2348$	$-·0384$	$+1187$	$+·2685$

* Taken from the table in *Biometrika*, Vol. III. p. 166.

If these data are fitted to a normal scale, we find the values given in the table for the proportional frequencies $\frac{n_s}{N}$ and $1-\frac{n_s}{N}$, and for the positions of the dichotomic lines, h_s, and of the centroids of the groups, $h_{r,s}$. Putting these values into Equation (x) and remembering that $1-\frac{n_0}{N}$ and $\frac{n_5}{N}$ are zero, it is found that the probable errors of the distances of the centroids or means of the groups from the mean of the sample (referred to the standard deviation as unit) are for h_{10}, $\pm$ ·0314; for h_{21}, $\pm$ ·0166; for h_{32}, $\pm$·0149; for h_{43}, $\pm$ ·0204 and for h_{54}, $\pm$ ·0492. The largest probable errors occur in the small groups at the tails of the distribution.

The relation
$$r^2_{x.c_x} = \overset{4}{\underset{s=0}{S}} \left\{\frac{n_s - n_{s+1}}{N} h^2_{s+1,s}\right\}$$
gives
$$r^2_{x.c_x} = \cdot867055, \qquad r_{x.c_x} = \cdot9312.$$

To find the probable error of $r_{x.c_x}$ we calculate first the values of the auxiliary quantities γ_s and δ_s (defined on p. 266); these are given in the last two rows of Table I; α_s and β_s or $\frac{n_s}{N}$ and $1-\frac{n_s}{N}$ have been calculated already. Substituting these expressions in Equation (xii) it is found first that

$$\overset{4}{\underset{s=1}{S}} (\alpha_s\beta_s\gamma_s^2\delta_s^2) = + \cdot007513, \quad \underset{j,k}{S'} (\alpha_k\beta_j\gamma_j\gamma_k\delta_j\delta_k) = + \cdot000005,$$

and finally,
$$\sigma_{r_{x.c_x}} = \cdot00213.$$

Hence on the assumption that the distribution of variations in samples of $r_{x.c_x}$ may be treated as approximately normal, the probable error of $r_{x.c_x}$ is $\pm$ ·0014, or

$$r_{x.c_x} = \cdot9312 \pm \cdot0014.$$

Example 2.

In order to see how a reduction in the number of groups affects the probable errors, I have combined the Very Strong and Strong groups, and the Rather Delicate and Delicate groups. The altered values of the constants are given in Table II. The probable errors of the $h_{r,s}$'s are now

for h_{10}, $\pm$ ·0141 : for h_{21}, $\pm$ ·0149; for h_{32}, $\pm$ ·0180.

and the class-index correction is $r_{x.c_x} = \cdot8934$,

$$\overset{2}{\underset{s=1}{S}} (\alpha_s\beta_s\gamma_s^2\delta_s^2) = + \cdot001644, \quad \underset{j,k}{S'} (\alpha_k\beta_j\gamma_j\gamma_k\delta_j\delta_k) = \alpha_2\beta_1\gamma_1\gamma_2\delta_1\delta_2 = + \cdot000427,$$

whence $\sigma_{r_{x.c_x}} = \cdot00116$, giving a probable error of $\pm$ ·0008.

If these results are compared with those of Example 1, it will be seen that by getting rid of the small tail groups, the probable errors of the means have been reduced and at the same time, while the class-index correction itself has a lower value, its probable error is considerably smaller in the second system of grouping. The low value for $\sigma_{r_{x.c_x}}$ is remarkable; its value has been checked by another method described in section (7) below. Now suppose that we had some other character of the 1918 schoolboys—weight, for example—and that we had found a

TABLE II.

Health in Schoolboys (3 *categories*)

Very Strong and Strong	Normally Healthy	Rather Delicate and Delicate	Total
676	907	335	1918

$a_s = \frac{n_s}{N}$	·64755	·17466
$\beta_s = 1 - \frac{n_s}{N}$	·35245	·82534
h_s	− ·3787	+ ·9359

r, s	1, 0	2, 1	3, 2
$h_{r,s}$	−1·0536	+ ·2408	+ 1·4741

	1	2
	+1·2944	+1·2333
δ_s	+ ·0277	+ ·0785

correlation ratio, $H_{y_{c_x}}$, of ·40, between this character and the five health class-marks; the probable error of $H_{y_{c_x}}$ would be given approximately as

$$\pm\,·67449\,\frac{1-H^2}{\sqrt{N}} = \pm\,·0129,$$

while the corrected $\eta_{y_x} = \frac{4000}{·9312} = ·4296$. Then (xviii) gives the upper limit of the probable error of η_{y_x} as

$$\pm\,\frac{1}{r_{x\,c_x}}\,(\text{P.E. of } H_{y_{c_x}} + \eta_{y_x} \times \text{P.E. of } r_{x\,c_x})$$

$$= \pm\,\frac{1}{·9312}\,(·0129 + ·4296 \times ·0014) = \pm\,·0145$$

Thus the probable error is for rough purposes given by

$$\pm\,·67449\,\frac{\sigma_{H_{y_{c_x}}}}{\sqrt{N}} = \pm\,·0139$$

Example 3

In this example I have taken data of non-Gaussian frequency, divided into groups of very unequal distribution These are the figures for barometric heights at Southampton and Laudale distributed in a 3×3-fold table, as below*

TABLE III *Frequency of Barometric Heights (in inches)*

Southampton

Laudale	31 05—30 15	30 15—29 25	29 25—28 35	Totals
30 85 – 29 85	808 25	733 25	0	1541 5
29 85—28 85	83 75	1223·25	45 5	1352 5
28 85– 27 85	0	14	14	28 0
Totals	892 0	1970 5	59 5	2922

The Southampton distribution will correspond to x, and the Laudale to y Then we find for the Southampton marginal totals

TABLE IV

Southampton

$\alpha_s = \frac{n_s}{}$	69473		02036
$\beta_s = 1 \; \frac{n_s}{N}$	·30527		97964
	− 5093		+ 2 0463
γ_s	+ 1 5946		+ 1 9673
	− 1587		+ 6160
	1, 0	2 1	3, 2
	− 1 1479	+ 4467	+ 2 4140

Giving $r_{x\,c_x} = 8096$ and $\sigma_{r_{x\,c_x}} = 00452$,

and the probable error $\pm\,·0030$

* The table is taken from *Biometrika*, Vol IX p 137 from the paper on the Influence of "Broad Categories" on Correlation Slips which there seem to have occurred in the calculation of $r_{x\,c_x}$ and $r_{y\,c_y}$ have now been corrected The values of $k_{1,s}$ and $k_{r\,s}$ do not quite agree to the last figure with those given in the paper quoted, but the present values were obtained using the "Tables of Deviates and of Ordinates of the Normal Curve for each Permille of Frequency," *Tables for Statisticians and Biometricians*, Cambridge University Press, Table I, and *Biometrika*, Vol XIII p 428 respectively, while the original values were probably obtained by working to more figures The difference is not significant, as it makes no difference to the value of $r_{x\,c_x}$ or $r_{y\,c_y}$ calculated to four figures

For Laudale, we have Table V:

TABLE V.

Laudale.

$\alpha_s' = \frac{n_s}{N}$	·47245	·00958
$\beta_s' = 1 - \frac{n_s}{N}$	·52755	·99042
k_s	+ ·0691	+ 2·3424
γ_s	+ 1·5585	+ 1·8753
	+ ·0441	+ ·6006

r, s	1, 0	2, 1	3, 2
$k_{r,s}$	− ·7543	+ ·8042	+ 2·6795

Giving $\qquad r_{y\,c_y} = ·8175, \quad \sigma_{r_{y.c_y}} = ·00268,$

and the probable error $\qquad \pm ·0018.$

A glance at the marginal totals would suggest that the y-distribution with its small tail group of 28 would give a less reliable class-index correction than the x-distribution. That $\sigma_{r_{y.c_y}}$ is actually smaller than $\sigma_{r_{x.c_x}}$ is mainly due to the fact, that $\delta_1'(+·0441)$ is less than one-third of $\delta_1(-·1587)$, the mean of k_{10} and k_{21} nearly coinciding in position with k_1. This is illustrated diagrammatically in Figure II.

Now using the method of mean square contingency I find that $\phi^2 = ·3662\ 9808$ and $C_2 = ·51778$, hence

$$r_{xy} = \frac{C_2}{r_{x.c_x}\, r_{y.c_y}} = ·7823.$$

In order to find the probable error of C_2, I have calculated σ_{ϕ^2} by the method of Pearson and Young, *Biometrika*, Vol. XI. pp. 215—230*, and obtain

$$\sigma_{\phi^2} = ·064745,$$

whence $\qquad \sigma_\phi = \frac{1}{2\phi}\sigma_{\phi^2} = ·053488,$

and $\qquad \sigma_{c_2} = \sigma_\phi/(1+\phi^2)^{\frac{3}{2}} = ·033492,$

giving finally the probable error of C_2 as $\pm ·02259$.

* I have used the approximative form (D) of p. 229 of that paper, where the sign of $2c$ in the second bracket should be negative, as corrected in *Biometrika*, Vol. XII. p. 259.

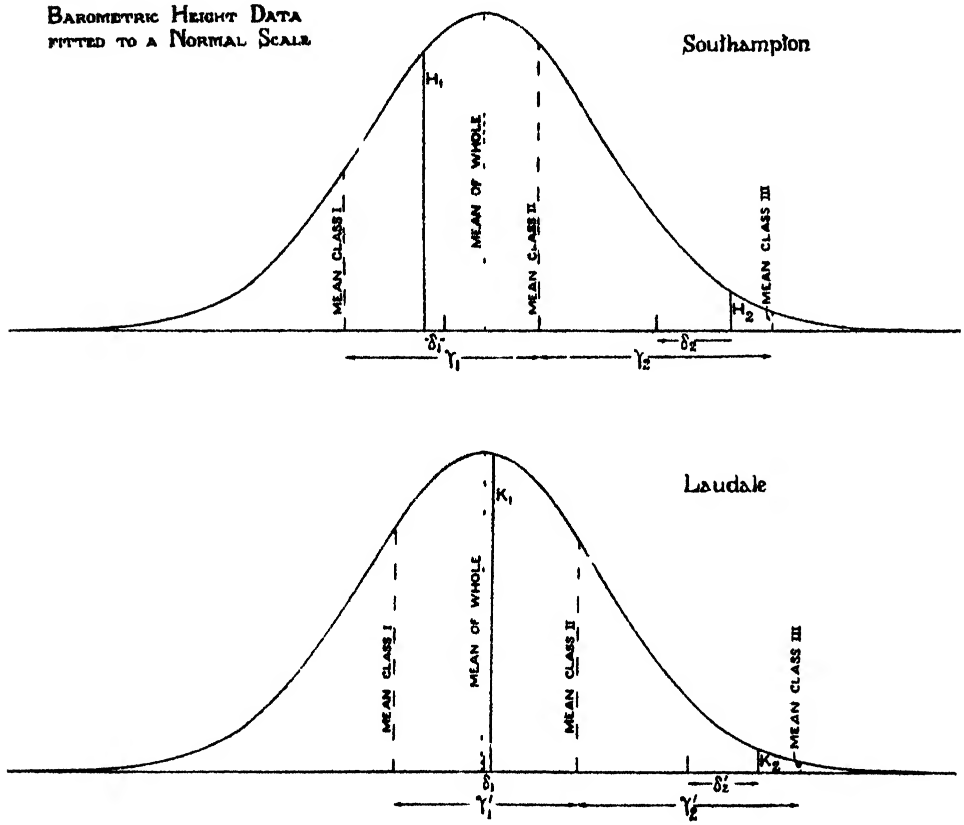
If now these values of C_2, $r_{x.c_x}$ and $r_{y.c_y}$ and of their probable errors are put into the relation (xvii), we have for the upper limit of the probable error of the corrected r_{xy}*,

$$\pm \cdot 7823 \left(\frac{\cdot 0226}{\cdot 5178} + \frac{\cdot 0030}{\cdot 8096} + \frac{\cdot 0018}{\cdot 8175}\right)$$

$$= \pm (\cdot 0341 + \cdot 0029 + \cdot 0017)$$

$$= \pm \cdot 0387.$$

Fig II
BAROMETRIC HEIGHT DATA FITTED TO A NORMAL SCALE



Here again a rough approximation to the probable error is given by the first term, or

$$\pm \cdot 67449 \frac{\sigma_{c_*}}{r_{x.c_x} r_{y.c_y}}.$$

Example 4.

In the previous examples the total number N in the sample has been fairly large. To examine the value of the probable error of $r_{x.c_x}$ in a smaller sample, and

* Making the correction to the value of r_{xy} given in the previous paper referred to above (*Biometrika*, Vol. IX. pp. 137, 138), we have to compare the value found by corrected mean square contingency, (·7828) instead of (·7570), with that found by the product moment method (·7802 ± ·0049).

its relation to the probable error of C_2, I have taken the contingency table originally given by Miss Elderton in her Memoir on *The Measure of Resemblance of First Cousins**, for which C_2 and its probable error have been calculated by Young and Pearson†. This is a 6×6-fold table, but as it is symmetrical about a diagonal, $r_{x.c_x}$ and $r_{y.c_y}$ are the same.

TABLE VI.

Contingency between Hair Colours of Female Cousins.

First Female Cousin.

Second Female Cousin

Tints	Very Dark	Dark Brown	Brown	Light Brown	Fair	Red	Totals
Very Dark .	9	11	3·5	11	1·5	0	36
Dark Brown	11	13	12	10	9	1	56
Brown . .	3·5	12	8	9·75	3·25	3	39·5
Light Brown	11	10	9·75	16·5	9·25	1	57·5
Fair	1·5	9	3·25	9·25	1	0	24
Red	0	1	3	1	0	0	5
Totals	36	56	39·5	57·5	24	5	218

The values of the constants h_s, $h_{r,s}$ and α_s, β_s, γ_s, δ_s are given in Table VII From these it is found that

$$r_{x\,c_x} = \cdot 9626, \quad \overset{5}{\underset{s=1}{S}} (\alpha_s\beta_s\gamma_s^2\delta_s^2) = \cdot 0017036, \quad \underset{j,k}{S'} (\alpha_k\beta_j\gamma_j\gamma_k\delta_j\delta_k) = -\cdot 0002368,$$

TABLE VII.

s	1	2	3	4	5
$\alpha_s = \frac{n_s}{N}$	·83486	·57798	·39679	·13303	·02294
$\beta_s = 1 - \frac{n_s}{N}$	·16514	42202	·60321	86697	·97706
h_s	−·9735	−·1967	+·2616	+1·1122	+1·9965
γ_s	+·9476	+·5882	+·6150	+·8116	+·9117
δ_s	+·0567	+·0656	−·0777	+·0596	+·0823

r, s	1, 0	2, 1	3, 2	4, 3	5, 4	6, 5
$h_{r,s}$	−1·5040	−·5564	+·0318	+·6468	+1·4584	+2·3701

* *Eugenics Laboratory Memoirs*, IV. Cambridge University Press, 1907. The group of 5 with red hair is undesirably small.

† *Biometrika*, Vol. XI. pp. 225—227 and correction for error in *Biometrika*, Vol. XII. p 260.

and hence $\sigma_{r_x c_x} = \cdot 00247$, and the probable error of $r_{x.c_x}$ is $\pm \cdot 0017$, or is only slightly greater than in Example 1, where N was 1918 instead of the present 218.

Now $\phi^2 = 14895$, $C_2 = \cdot 86005$, $r_{xy} = \dfrac{C_2}{r_{x\,c_x} r_{y.c_y}} = \cdot 3886$, $\sigma_{\phi^2} = \cdot 0719$, $\sigma_{c_2} = \cdot 0756$, and the probable error of $C_2 = \pm \cdot 0510$.

Using relation (xvii), we find the upper limit of the probable error of the corrected r_{xy} to be

$$\pm \cdot 3886 \left(\frac{\cdot 0510}{\cdot 3600} + 2 \times \frac{\cdot 0017}{\cdot 9626}\right)$$
$$= \pm (\cdot 0550 + \cdot 0014) = \pm \cdot 0564,$$

where the second term due to the probable errors of the class-index correction is again insignificant compared to the first, due to the probable error of C_2.

(7) In section (4), $\delta r_{x\,c_x}$, or a variation in the class-index correction found from a sample out of the population, was expressed in terms of the variations in H_s and n_s, it can also be expressed in terms of the variations in $h_{s+1,s}$ and n_s. Thus differentiating (iii),

$$2r_{x\,c_x}\delta r_{x\,c_x} = \overset{s\,p-1}{\underset{s=0}{S}} \left\{\frac{\delta n_s - \delta n_{s+1}}{N} h^2_{s+1,s} + 2h_{s+1,s}\,\delta h_{s+1,s}\frac{n_s - n_{s+1}}{N}\right\} \quad \ldots\ldots \text{(xix)}.$$

If we square both sides of (xix), sum for all possible variations in samples, and take the mean value, we shall be able to express $\sigma^2_{r_x\,c_x}$ in terms of the mean values of the squares and products of the differentials of n_s and $h_{s+1,s}$, $h_{r+1,r}$, etc., or of

$$\text{Mean}\,(\delta n_s^2) = \sigma^2_{n_s}, \quad \text{Mean}\,(\delta n_s \delta n_r), \quad \text{Mean}\,(\delta h^2_{s+1,s}) = \sigma^2_{h_{s+1,s}},$$
$$\text{Mean}\,(\delta n_s\,\delta h_{s+1,s}), \quad \text{Mean}\,(\delta n_s\,\delta h_{r+1,r}), \text{ etc}$$

Of these the first three expressions are known from Equations (i), (ii) and (x), and the others can be calculated in terms of n_s, h_s, $h_{s+1,s}$, etc. As a check the standard deviation of $r_{x\,c_x}$ was worked out by this method for the simple case of the three groupings of Example 2 above. It was found to be $\cdot 00116$, agreeing exactly with the value given on p 269. This appears to give a satisfactory confirmation of the correctness of both Equations (x) and (xi) as well as of the accuracy of the arithmetic in this particular example

(8) There is another method of calculating the probable error of $r_{x\,c_x}$ if the data are recorded on a *quantitative* scale. In this case $r_{x\,c_x} = \dfrac{\sigma_{c_x}}{\sigma_x}$, where σ_{c_x}, the standard deviation of the means of the groups, and σ_x, the standard deviation of the whole sample, are both known numerically The problem is broadly the same as that of finding the probable error of a correlation ratio, η, for both $r_{x\,c_x}$ and η are the ratios of the standard deviation of the means of independent groups to the standard deviation of the total distribution. Now the standard deviation of the values of η due to random sampling has been worked out in a previous paper*, and

* *Drapers' Company Research Memoirs.* No. XIV "On the General Theory of Skew Correlation and Non-linear Regression," 1905.

with slight modification the relations there obtained will give the standard deviation of $r_{x.c_x}$.

We have first that

$$\frac{\delta r_{x.c_x}}{r_{x.c_x}} = \frac{\delta \sigma_{c_x}}{\sigma_{c_x}} - \frac{\delta \sigma_x}{\sigma_x},$$

and on squaring, summing for all random samples and dividing by the number of such samples, we find:

$$\frac{\sigma^2_{r_{x.c_x}}}{r^2_{x.c_x}} = \frac{\sigma^2_{\sigma_{c_x}}}{\sigma^2_{c_x}} + \frac{\sigma^2_{\sigma_x}}{\sigma^2_x} - \frac{2 \text{ Mean } (\delta \sigma_{c_x} \delta \sigma_x)}{\sigma_{c_x} \sigma_x} \quad \text{............ ...(xx).}$$

Now we know that*

$$\sigma^2_{\sigma_x} = \frac{\sigma_x^2}{4N} \frac{\mu_4 - \mu_2^2}{\mu_2^2} = \frac{\sigma_x^2}{4N} (\beta_2 - 1) \quad \text{..................(xxi),}$$

where μ_4 and μ_2 are the 4th and 2nd moment coefficients of the sample.

Making the necessary modification to Equation (xxvii) of p. 16 of the Memoir referred to in the footnote on p. 275, we find that

$$\frac{\sigma^2_{\sigma_{c_x}}}{\sigma^2_{c_x}} = \frac{1}{4N} \left\{ \frac{\lambda_4 - \lambda_2^2}{\lambda_2^2} + 4 \underset{s}{S} \left(\frac{m_s (\bar{x}_s - \bar{x})^2 {}_s\pi_2}{N \lambda_2^2} \right) \right\} \quad \text{...... .....(xxii),}$$

where $\bar{x}$ is the mean of the sample,

$\bar{x}_s$ is the mean of the sth class or category into which the material is grouped,

m_s is the size of the sth category,

${}_s\pi_q = \frac{1}{m_s} \underset{t}{S} \{n_{s_t} (x_{s_t} - \bar{x}_s)^q\}$ or is the qth moment coefficient of the observations in the sth category about the mean of that category,

$\lambda_q = \frac{1}{N} \underset{s}{S} \{m_s (\bar{x}_s - \bar{x})^q\}$ or is the qth moment coefficient of the weighted means of categories about the mean of the whole sample,

$\underset{s}{S}$ indicates summation for all categories,

$\underset{t}{S}$ indicates summation for all variates within the sth category.

Similarly after modification, Equation (xxxii) of the Memoir gives

$$\frac{\text{Mean } (\delta \sigma_{c_x} \delta \sigma_x)}{\sigma_{c_x} \sigma_x} = \frac{1}{4N} \left[\frac{\lambda_4}{\sigma_x^2 \lambda_2} - 1 + 5 \underset{s}{S} \left\{ \frac{m_s (\bar{x}_s - \bar{x})^2 {}_s\pi_2}{N \sigma_x^2 \lambda_2} \right\} + 2 \underset{s}{S} \left\{ \frac{m_s (\bar{x}_s - \bar{x}) {}_s\pi_3}{N \sigma_x^2 \lambda_2} \right\} \right] \quad \text{...........(xxiii).}$$

Substituting from (xxi), (xxii) and (xxiii) into (xx), we find that

$$\sigma^2_{r_{x.c_x}} = \frac{r^2_{x.c_x}}{4N} \left[\beta_2 + \frac{\lambda_4}{\lambda_2^2} \left(1 - 2 \frac{\lambda_2}{\sigma_x^2} \right) + 2 \left(2 - 5 \frac{\lambda_2}{\sigma_x^2} \right) \underset{s}{S} \left\{ \frac{m_s (\bar{x}_s - \bar{x})^2 {}_s\pi_2}{N \lambda_2^2} \right\} - 4 \underset{s}{S} \left\{ \frac{m_s (\bar{x}_s - \bar{x}) {}_s\pi_3}{N \sigma_x^2 \lambda_2} \right\} \right],$$

* *Biometrika*, Vol. II. p. 276.

	Southampton																	
Height	31·0	30·9	30·8	30·7	30·6	30·5	30·4	30·3	30·2	30·1	30·0	29·9	29·8	29·7	29·6	29·5	29·4	29·3
Frequency	1	4	4	30·5	52·5	107·5	140·5	237	315	395·5	382·5	339·5	288	201	150·5	98·5	65	50
Group Frequencies of Example 3	892·0								1970·5									

	Southampton								Laudale									
Height	29·2	29·1	29·0	28·9	28·8	28·7	28·6	28·5	30·8	30·7	30·6	30·5	30·4	30·3	30·2	30·1	30·0	29·9
Frequency	23·5	15·5	7·5	7	3	—	1	2	2	14	36	64	141	200	263	260·5	277·5	283·5
Group Frequencies of Example 3	59·5								1541·5									

	Laudale																	
Height	29·8	29·7	29·6	29·5	29·4	29·3	29·2	29·1	29·0	28·9	28·8	28·7	28·6	28·5	28·4	28·3	28·2	28·1	28·0	27·9
Frequency	277·5	245	212	192	135	97·5	67·5	63	38·5	24·5	11	7·5	4·5	—	2·5	0·5	—	—	1	1
Group Frequencies of Example 3	1352·5										28·0									

Total for both Southampton and Laudale = 2922.

or, remembering that

$$r^2_{x\,c_x} = \frac{\sigma^2_{c_x}}{\sigma_x^2} = \frac{\lambda_2}{\sigma_x^2},$$

and putting

$$\frac{\lambda_4}{\lambda_2^2} = B_2,$$

we have finally

$$\sigma^2_{r_x\,c_x} = \frac{r^4_{x\,c_x}}{4N}\left[\beta_2 + B_2(1 - 2r^2_{x\,c_x}) + 2(2 - 5r^2_{x\,c_x})\,\underset{s}{S}\left\{\frac{m_s(\bar{x}_s - \bar{x})^2\,{}_s\pi_2}{N\sigma^4_{c_x}}\right\} - 4\underset{s}{S}\left\{\frac{m_s(\bar{x}_s - \bar{x})\,{}_s\pi_3}{N\sigma_x^2\,\sigma^2_{c_x}}\right\}\right] \quad \text{.......(xxiv).}$$

In this equation we cannot, as in the case of the standard deviation of η, make any assumption as to the values of B_2, ${}_s\pi_2$ or ${}_s\pi_3$. When dealing with a few categories only, ${}_s\pi_3$ is likely to be quite significant particularly in the unsymmetrical tail groups.

(9) As an illustration of Equation (xxiv) we may take the barometric-height data of Example 3, above. All that is required for the present purpose are the marginal frequencies for Southampton and Laudale*; these are given in Table VIII, and in Table IX are given the constants required in (xxiv) calculated from these frequencies. Substituting in (xxiv) it is found that,

for Southampton $\sigma_{r_x\,c_x} = \cdot 00608$,

for Laudale $\sigma_{r_y\,c_y} = \cdot 00386$.

TABLE IX.

	Southampton			Laudale		
	$\bar{x} = 29''\cdot 9839$; $\sigma_x^2 = \cdot 105629$; $\beta_2 = 3\cdot 612029$			$\bar{x} = 29''\cdot 8488$; $\sigma_x^2 = \cdot 154629$; $\beta_2 = 3\cdot 194947$		
	Group 1	Group 2	Group 3	Group 1	Group 2	Group 3
$\bar{x}_s$	30″·3416	29″·8499	29″·0597	30″·1503	29″·5304	28″·6357
${}_s\pi_2$	·022250	·045256	·029129	·038140	·055822	·052653
${}_s\pi_3$	+ ·003611	− ·007100	− ·007808	+ ·004128	− ·010449	− ·023435
	$\sigma^2_{c_x} = \cdot 068561$; $r^2_{x\cdot c_x} = \cdot 649070$; $B_2 = 4\ 2699$			$\sigma^2_{c_y} = \cdot 108982$; $r^2_{x\cdot c_x} = \cdot 704797$; $B_2 = 2\cdot 5148$		

If these values are compared with those found above from (xi), by the method of fitting the broad group totals to a normal scale, viz.

$$\sigma_{r_x\,c_x} = \cdot 00452,$$

$$\sigma_{r_y\cdot c_y} = \cdot 00268,$$

* The complete data are given in Table IX, p. 453 of the paper by Pearson and Lee in *Phil. Trans.* Vol. 190, A, 1898.

it will be seen that in both cases the standard deviation found from (xxiv) is about 1·4 times as great as that found from (xi).

It appears therefore that by fitting our material to a normal scale and not dealing with the crude data, we have ensured a slightly greater degree of consistency in the values of the class-index correction calculated from different samples of N. If we compare the values of $r_{x.c_x}$ found by the two methods we find

	Southampton	Laudale
Qualitative method	·8096	·8175
Quantitative method	·8056	·8395

There is a significant difference between the two values of the class-index correction for the Laudale distribution, but if the very small size of the tail group (i.e. 28·0) is remembered this discrepancy is perhaps not surprising.

It is also possible to use Equation (xxiv) when dealing with material which is divided into qualitative categories only. For in this case, fitting the groups as before to a normal scale, we have

$$\beta_2 = 3\ ;\quad B_2 = \frac{1}{r^4_{x.c_x}} \underset{s}{S} \left\{ \frac{(n_s - n_{s+1})}{N} h^4_{s+1,s} \right\};$$

$$\underset{s}{S} \left\{ \frac{m_s (x_s - x)^2 {}_s\pi_2}{N \sigma^4_{c_x}} \right\} = \frac{1}{r^4_{x.c_x}} \underset{s}{S} \left\{ h^2_{s+1,s} (n_s - n_{s+1})\, {}_s\pi_2' \right\},$$

$$\underset{s}{S} \left\{ \frac{m_s (x_s - \bar{x})\, {}_s\pi_3}{N \sigma_x^2 \sigma^2_{c_x}} \right\} = \frac{1}{r^4_{x.c_x}} \underset{s}{S} \left\{ h_{s+1,s} (n_s - n_{s+1})\, {}_s\pi_3' \right\},$$

where ${}_s\pi_q'$ is the qth moment coefficient about its mean ordinate of the section of the normal curve

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h^2},$$

lying between the dichotomic ordinates at h_s and h_{s+1}.

Taking the Southampton distribution, and making use of the values of $h_{s+1,s}$, h_s, $r_{x.c_x}$, etc., found in the working of Example 3, I calculated ${}_s\pi_2'$ and ${}_s\pi_3'$ for each of the three categories by a rough quadrature. Equation (xxiv) then gave

$$\sigma_{r_{x.c_x}} = ·0049.$$

We may thus compare the values of the standard deviation of $r_{x.c_x}$ for the Southampton distribution found by the three methods:

(1) from (xi) using broad categories only and fitting to a normal scale ·0045,
(2) „ (xxiv) „ „ „ „ ·0049,
(3) „ (xxiv) using the crude quantitative data ... ... ·0061.

As should be the case the results of methods (1) and (2) agree closely. Without further numerical testing it cannot be asserted that for all other arrangements of grouping the class-index correction found by the broad category method will have a smaller probable error than that calculated using the true quantitative scale, but since the arrangement of grouping could hardly be more unfavourable than in the

present example it seems probable that the results of these two lines of approach will not in general differ seriously. Method (1) is undoubtedly the shortest, and in the majority of problems in which the class-index correction is required, the quantitative scaling of the variate required in method (3) will not be available.

(10) *Conclusion.* Expressions have been found for the probable error due to random sampling of the distance (in terms of the standard deviation as unit) between the mean of a class group or broad category and the mean of the whole sample, in material fitted to a normal scale, (x); and for the probable error of a class-index correction, (xi), (xii), and (xxiv). The first problem is straightforward and calls for no comment. The second has been approached from several points of view, and illustrated by examples covering a fairly wide range, which suggest that the probable errors of $r_{x.c_x}$ and $r_{y.c_y}$ are likely to be so small compared to the probable error of the coefficient to be corrected—whether $r_{c_x.c_y}$, $H_{y c_x}$ or C_x—that they may be neglected in the rough appreciation of the probable error of the corrected constant which is all that is usually required for practical purposes. The greatest values are likely to occur when there are a few unevenly divided groups, but Example 3 with its contingency table (Table III) of exceptionally unfavourable grouping, shows that even in such cases the probable error of the class-index corrections will remain very small provided that the size of the sample is reasonably large.

Thus this paper indicates that a class-index correlation based on broad categories is not subject to a large probable error, but only to an error of the same, or even a less order, than a product moment coefficient of correlation.

ON THE VARIATE DIFFERENCE METHOD.

Being a Paper read before the Society of Biometricians and Statisticians.

By KARL PEARSON, F.R.S. AND ETHEL M. ELDERTON, *Galton Fellow.*

(1) Every one is familiar with data which show a secular trend with time, say the deaths from tuberculosis as measured by the corrected deathrate from 1865 down to the present or again the falling birthrate per married woman between 15 and 50.

Now the data for such matters do not when plotted take the form of smooth curves corresponding to continuous mathematical functions; they exhibit general trends with the time, but they rise and fall with apparent, but not indeed necessary irregularity, above and below a sort of average curve representing the secular trend. The words "sort of average curve" are vague and intended to be so, because much of the matter in dispute turns on the manner in which the curve representing the secular trend is to be determined. The curve which represents this secular trend may be a long period periodic term or it may not Generally we have absolutely no reason to suppose it is so, but we may imagine it so, if we have a monomania for the representation of time-functions by periodic analysis*. We have, however, no reason for supposing that increased consumption of apples or bananas, or increasing expenditure on the navy, per head of the population are periodic in their nature; they may rise or fall according to circumstance; but we are no more justified in assuming that they follow a time-function $A \sin(nt + \alpha)$ than $A \sin(nf(t) + \alpha)$, indeed the latter, as it is more general, is more likely to describe the result. A still more general attitude is simply to assume that the trend is given by $f(t)$ itself for this involves no suggestion of periods for $t = f^{-1}\left(\frac{2\pi i}{n}\right)$ of the previous formula.

Of course in any case in which it seemed reasonable to use a harmonic curve, one would naturally do so, but in the cases we have chiefly in mind this is certainly not so, and it is far more reasonable to suppose that the secular trend can be given by a single high order parabola, or by the series of such parabolae involved in a good smoothing process, than by any products of periodic analysis.

Now we know that if we correlate the falling phthisis deathrate with the falling birthrate we shall have a correlation of the order 0·9. But no one is likely to believe there is an *organic* relationship between the two of this order,—any more than one

* Any function, e.g. a triangle, can be fairly closely represented by an adequate number of periodic terms for a finite base But it does not follow that the representation has any organic significance or would be of the slightest value if used for extrapolation, i.e. prediction.

believes that the correlation between the cancer deathrate and the increasing expenditure on apples per head of the population, the value of which is 0·89, is a true organic relationship, i.e. is due to one or more common factors in the two variates. Such high correlations as arise from common growth or decline with time, when interpreted as causal or semicausal relationships, are in our opinion perfectly idle, indeed are only too apt to be mischievous, and we shall reach nothing, or less than nothing—knighthoods,—by the investigation of them.

But when we take the *apparently* random deviations from the secular trend, it does seem a perfectly legitimate problem to ask: Is there any relationship between them?

If the deviations of two variates from their secular trends be X_t and Y_t, we want to discover their correlation r_{xy}. All are agreed, we think, as to the desirability of finding this correlation,—including even Mr Yule, although he apparently confesses that he cannot find any source for such correlation except in common periodic terms*. Now the real problem before us is this: Having by means of a high order parabola or an adequate smooth got rid of the secular trend, will the variate difference method give us r_{xy} or what does it give us?

There are certain considerations which may be referred to first. As a rule our data are given for annual periods, a year is our usual time unit. Now the most important periodic factor is the annual seasonal change. This with the year as unit will not be of any influence. We know very little yet as to periods in meteorological or climatological phenomena. There are those who talk of an eleven-year period in the former, but it is far from certain, and if it existed, it would for our purposes be of no importance, as its effects would be reduced to insignificance in taking a very few differences. Climatic periodicity seems to be a matter of tens of thousands of years, rather than tens of years, if we may judge by geological considerations; and such periodicity for anything we can say to the contrary may be included in our secular trend. Mr Yule apparently holds that periodicities of 1·2 to 6 times the unit of grouping would be fatal to the variate difference method. It certainly seems to us that any one, who suggests in annual tabling periods of 1·2 to 6 years, is called upon to show how such periods arise, what natural phenomena they are peculiar to, and why such natural causes influence the variates in question. If we take the example already referred to, let us say an isosceles triangle on a time base of a year, then it is clear that a three months periodic term would be an important component in its representation by periodic analysis; but it does not follow that such a period would have the least organic relation to the origin of the triangle. Because we can *represent* a fluctuating variable by a series of periodic terms discovered by periodogram analysis, it does not by any means follow of necessity that the constituent factors of the variate had themselves such periods. The periodogram may *suggest* periods, but that is of small value until the natural

* *Journal of the R. Statistical Society*, Vol. xxxiv. p. 502. Of course any occurrence which affects both variates happens in time, but it is not therefore a function of time, still less a periodic function of time, i.e. it cannot be quantitatively predicted from a knowledge of the time of its occurrence.

factors have been discovered which have these periods, and it can be shown also that these natural factors influence the variates in question.

Again we would add that in our opinion the deviations X and Y from the secular trends of the two variates, such as occur in vital and economic statistics, are dependent on factors which are obviously non-periodic in character. They are summed up in sanitation, legislation, new routes and methods of transport, over- or under-production, new methods of agriculture, wars, famines, transfer of population and thousands of other factors which make up civilised human life. We might define them as "historical factors," history takes place in time, but its events are not mathematical functions of the time, still less periodic functions, whatever folk-experience may whisper about history repeating itself.

It is only legitimate to call the effects of these historical factors random fluctuations, if that term is used in a special sense as Mr Yule appears to use it, i.e. for everything which is not due to a periodic variation. Such "random fluctuations" are by no means as Mr Yule would seem to suggest due only to errors of observation or the deviations of random sampling; they are due to non-periodic causes which may affect both the A and B variates or may not. The question is to what extent have A and B common causes behind their fluctuations, apart from growth with time. We think this is a perfectly legitimate question to ask, and that in asking it we are not open to the insinuation, contained in the use of the term "random variations," of asking whether pure chance fluctuations are or are not correlated*.

Let us suppose for a moment that the distribution of X or Y or of both is purely random, then whether they are in excess or defect of the secular trend would be indifferent, and might be ascertained by a tossing experiment. Now if we take a series of tosses there will be more changes from head to tail or tail to head than anything else, i.e. runs of one are most frequent, then come runs of two and so on, long runs are very rare. But to the casual observer of a graph, runs of one suggest

* An example may possibly render the matter clearer. A financier starts a company to grow cotton in a newly opened African district, and the company in chartering ships to fetch the cotton home finds it of value to export hardware for sale in the district. The company may or may not be successful or may change the nature of its exports and imports. But its contributions to cotton imports and hardware exports are not periodic. Many such transactions—and is not all trade ultimately of this character?—would produce correlated fluctuations in special imports and exports which are not due to periodic factors. Or again take another example. Let us take an English and a Zulu baby, and measure their weights monthly from babyhood to manhood. We should obtain fluctuating values which might be plotted, and the ordinates would give a very high correlation. But if we smoothed these growth data and correlated the fluctuations from the smoothed curves we should anticipate very small correlation. The reason for this would be obvious, the environmental conditions are very different. But if we take the weight curves of two English children and consider the corresponding fluctuations in the same way, we should expect correlation, emphasised if the children were of the same social class and still more if of the same family. There would be a legislative time for vaccination, there would be a traditional or racial time for weaning, there would be local differences in climate and in summer and winter, there would be local epidemics, traditional times for departure for school, for university, for professional life; there would be racial times for puberty, marriage, prime, decay. All these things would leave their impressions on the weight curve. But none of these factors are periodic in the growth curve, yet their relationship as indicated by the correlation of the fluctuations would be of very great interest as measuring the difference between what are generally human and what are especially racial factors.

two-year (or other unit) periods, runs of two, especially if they follow each other, four-year periods, and it is not difficult to be impressed with the presence of short periods by a graph, which is of really random distribution. But when we come more carefully to examine such graphs we find that the runs do not always give a chance distribution. If X_t be in considerable excess or considerable defect, X_{t+1} is more likely to be somewhat in excess or defect also. In other graphs, however, we find a compensating influence in successive years; instead of an excess of permanencies there is an excess of changes. In the former cases we anticipate a positive, in the latter cases a negative correlation between X_t and X_{t+1}. This result is inconsistent with a fundamental hypothesis of the Variate Difference Method as originally stated by "Student" and in this matter it seems to us that there is need to extend the theory.

Now the old method we had adopted in the Biometric Laboratory before the publication of "Student's" memoir was to fit high order parabolae to both variates, and then take the differences between the ordinates of these parabolae and the observed data for X and Y. These were then correlated, and the question of whether or not X_t and X_{t+1} were or were not correlated was immaterial. The difficulty of this process was two-fold: (i) The same parabola was used throughout the whole system; this involved not only a very lengthy pièce of work by least squares, but it was obvious that in many cases a change of parabola would be advantageous. Accordingly it is better to replace this high order parabola by a series of parabolae such as are provided by Sheppard's* or Rhodes'† system of smoothing. (ii) It was not *a priori* possible to select aptly the order requisite for the parabola, nor having selected it to settle, except from the general appearance of the graph, whether it was adequate or we must go through the great labour of fitting entirely anew a parabola of a still higher order‡.

On the contrary it was fairly easy in the Variate Difference Method to test the adequacy of the order of differences taken and so determine whether we were approaching an elimination of the time-factor. If not, it was not so serious a matter to take still another difference. We take it that Mr Yule whatever method he would select for smoothing, would agree that the fundamental point is to correlate X and Y after such a smoothing process. This does not, however, seem to be the opinion of another critic, Dr Warren M. Persons§ of Harvard. He does admit that curve fitting may be preferable to taking "moving averages"—which procedure appears to the present writers a most fallacious manner of smoothing—but he demands that the smoothing curve shall "increase or decrease regularly according to some principle"§. This axiom, whatever may be its exact significance, allows him to accept a sloping straight line, or to consider two sloping straight lines as good as a second order parabola. Nevertheless he admits a second order parabola, or the "compound

* *Fifth International Congress of Mathematicians*, Cambridge, 1912 (II).

† *Tracts for Computers*, No. VI. Smoothing. Cambridge University Press.

‡ Tchebycheff's method to some extent surmounts this difficulty, but the computations are even more laborious than those of Least Squares.

§ *Quarterly Publications of the American Statistical Association*, Vol. XVI. June, 1917.

interest law" $x = a . C^t$. He seems, however, to have some objection to an nth order parabola form of smoothing; why, we cannot say, unless it is because it takes more labour to fit, or because for some reason or other he supposes it does not fall under his axiom. The secular trend of the gêneral deathrate could certainly not be fitted by two straight lines, a second order parabola or the "compound interest law." In such a case it appears much better to take a high order parabola, but not so high as to introduce the sinuosities peculiar to the Sprague and King methods of osculatory graduation. It is very clear that we should get better results with 3rd to 5th order parabolae than Dr Persons does* with straight lines in the cases of Sauerbeck's Index Number of Wholesale Prices or the London Bank-Clearings.

What both Dr Persons and Mr Yule state, i.e. that we must be careful to see that our data give enough decimal places to cover our higher differences is, of course, correct. It is not a criticism of the method, but of application of the method beyond its proper range in the case of inadequate data. In matters like death- and birth-rates of large populations adequate decimal places can generally be obtained. This, however, may not be possible with some of the index numbers provided by various authorities.

On the other hand Dr Persons' statement, made after citing "Student," Dr Anderson, Miss Cave and one of ourselves, appears to us quite incorrect, namely: that "the writers on the Variate Difference correlation method all assume that 'the true r_{XY}' is for pairs concurrent in time" (*loc. cit.* p. 3).

Now Dr Persons' paper was issued in June 1917 and in May 1915 more than two years earlier the present authors published a paper in the number of *Biometrika* following that of Dr Anderson and the Cave-Pearson paper in which one of the essential features was the correlation of X_t with Y_{t+1}, Y_{t+2}, Y_{t+3} and Y_{t+4}. This paper was actually written at the same time as the Cave-Pearson paper and was not published till the following number on account of space. We were therefore quite conscious of the possibility of "lag." Indeed the possibility of lag had been considered many years before 1914 in correlating barometric heights on either side of the Atlantic by differences. What we overlooked was that the co-existence of these correlations might invalidate "Student's" assumptions.

This brings us to the main problem, the question of the assumptions made originally by "Student." They were—taking X_t and Y_t as the variates after removal of the secular trend—as follows:

$$X_t \text{ and } X_{t\pm\tau} \text{ are not correlated,}$$
$$Y_t \text{ and } Y_{t\pm\tau'} \text{ are not correlated,}$$
$$X_t \text{ and } Y_{t\pm\tau'} \text{ are not correlated.}$$

The latter assumption will allow for the lag $t'-t$, if we merely replace it by saying X_t and $Y_{t'\pm\tau'}$ are not correlated, i.e. X_t and $Y_{t'}$ are supposed to be correlated. In fact all we are doing is shifting the Y curve backwards or forwards on the X curve.

* Figs. 9 and 10, *loc. cit.*

We pay therefore no attention to this phase of Dr Persons' criticism, because the possible existence of "lag" was realised long before Dr Persons wrote. Putting this aside there seem to us two main criticisms of the Variate Difference correlation method: Dr Persons' that we cannot assume that X is solely correlated with a single Y, and that the series of X's is not intercorrelated, nor the series of Y's. We think this is a valid criticism which has to be met, either by showing that there are many cases in which the causes which produce the X's and Y's do not last over more than one interval, or else by enlarging our method and supposing that correlations between the X's and Y's of the above character really exist. We think this can be done. But we will postpone its consideration until we have dealt with the second main criticism, that of Mr Yule. This criticism is the following one, namely: If there exists a short periodic term it will tend to dominate the whole investigation. In other words, if we are dealing with years as intervals, a two-year or a three-year period according to Mr Yule will of necessity swamp the non-periodic fluctuations.

Let us first examine the type of proof Mr Yule gives, and then elaborate the whole problem mathematically. A short cut can be made to the desired result by the use of central differences.

(2) Mr Yule starts from a periodic term, $x = A \sin(nt+\alpha)$, and assumes that this is what we have to deal with in investigations of this kind. Now we think this is very frequently incorrect, and that an error of a serious kind affects not only Mr Yule's reasoning, but a good deal of that of other workers on periodic analysis. The variate itself may well be of the form $A \sin(nt+\alpha)$, but in most cases we are not given values of the variate itself, but of its sum or integral for the unit of time, and it is this integral that we have to deal with in our range returns. Thus:

$$x_p = \int_{(p-1)h}^{ph} A \sin(nt+\alpha)\,dt = A\frac{\tau}{\pi}\sin\frac{\pi h}{\tau}\sin\left\{\alpha + \frac{2\pi(p-\frac{1}{2})h}{\tau}\right\} \quad \ldots (\text{i}),$$

where h is the subrange, $\tau = 2\pi/n$, the period.

Now $(p-\frac{1}{2})h$ is the time at the centre of the pth range h, and we can write with Mr Yule

$$x_p = A' \sin\left\{\alpha + 2\pi(p-\tfrac{1}{2})\frac{h}{\tau}\right\},$$

but to do so is to overlook the fact that this amplitude A' will as a rule contain a function of the period τ itself.

Next if we take central differences

$$\begin{aligned}\delta^2 x_p &= x_{p+1} + x_{p-1} - 2x_p \\ &= -A\frac{\tau}{\pi}\sin\frac{\pi h}{\tau}\,2^2\sin^2\frac{\pi h}{\tau}\sin\left(\alpha + \frac{2\pi(p-\frac{1}{2})h}{\tau}\right),\end{aligned}$$

and

$$\begin{aligned}\delta^{2q} x_p &= (-1)^q A\frac{\tau}{\pi}\sin\frac{\pi h}{\tau}\left(2^2\sin^2\frac{\pi h}{\tau}\right)^q\sin\left(\alpha + \frac{2\pi(p-\frac{1}{2})h}{\tau}\right) \\ &= (-1)^q\left(2^2\sin^2\frac{\pi h}{\tau}\right)^q x_p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{ii}).\end{aligned}$$

Now let us consider the mean value of $\delta^{2q} x_p$ using square brackets here and throughout this paper to denote a mean value, i.e. $[z]$ = mean value of z.

Accordingly:

$$[\delta^{2q} x_p] = (-1)^q \left(2^2 \sin^2 \frac{\pi h}{\tau}\right)^q [x_p]$$

$$= (-1)^q \left(2^2 \sin^2 \frac{\pi h}{\tau}\right)^q A' \left[\sin\left\{\alpha + \frac{2\pi(p-\frac{1}{2})h}{\tau}\right\}\right]$$

$$= (-1)^q \left(2^2 \sin^2 \frac{\pi h}{\tau}\right)^q \frac{A' \sin\left(\alpha + \frac{\pi\lambda h}{\tau}\right) \sin\frac{\pi\lambda h}{\tau}}{\lambda \sin\frac{\pi h}{\tau}} \quad \ldots\text{(iii)},$$

where λ is the number of intervals under consideration.

If λh be small as compared with τ, this expression will not tend to vanish with λ, but give a limit $\sin\alpha$. On the other hand if τ be small as compared with λh, the total range of observation, the term is of the order $1/\lambda$, and will tend to be negligible, i.e. for short period terms. Assuming then that we are dealing with "short period" terms we can now proceed to find the standard deviation of the qth central difference

$$[(\delta^{2q} x_p)^2] = \left(2^2 \sin^2 \frac{\pi h}{\tau}\right)^{2q} A'^2 \left[\sin^2\left(\alpha + \frac{2\pi(p-\frac{1}{2})h}{\tau}\right)\right] \ \ldots\ldots\text{(iv)},$$

and we require to find the mean sum of squares of sines, or

$$\left[\frac{1}{2}\left\{1 - \cos\left(2\alpha + \frac{4\pi(p-\frac{1}{2})h}{\tau}\right)\right\}\right] = \frac{1}{2}\left\{1 - \frac{\cos\left(2\alpha + \frac{2\pi\lambda h}{\tau}\right)\sin\frac{2\pi\lambda h}{\tau}}{\lambda \sin\frac{2\pi h}{\tau}}\right\}.$$

Here the transcendental term will disappear for terms of short period and we have:

$$[(\delta^{2q} x_p)^2] = \frac{1}{2}\left(2^2 \sin^2 \frac{\pi h}{q}\right)^{2q} A^2 \frac{\tau^2}{\pi^2} \sin^2 \frac{\pi h}{\tau} \quad \ldots \ \ldots\ldots\ldots\text{(v)}$$

Lastly, let us find the correlation between two periodic terms, say x_p and $y_p = B\sin(n't+\beta)$. Here y_p refers only to the pth interval of the y variate, which may or may not be concurrent with the pth interval for x. Let $n' = 2\pi/\tau'$. Then as before for "short periods"

$$[(\delta^{2q} y_p)^2] = \frac{1}{2}\left(2^2 \sin^2 \frac{\pi h}{\tau'}\right)^{2q} B^2 \frac{\tau'^2}{\pi^2} \sin^2 \frac{\pi h}{\tau'} \quad \ldots\ldots\ldots\ldots\text{(vi)},$$

$$[\delta^{2q} x_p\, \delta^{2q} y_p] = \left(2^2 \sin^2 \frac{\pi h}{\tau}\right)^q \left(2^2 \sin^2 \frac{\pi h}{\tau'}\right)^q AB \frac{\tau\tau'}{\pi^2} \sin\frac{\pi h}{\tau} \sin\frac{\pi h}{\tau'}$$

$$\times \left[\sin\left(\alpha + \frac{2\pi(p-\frac{1}{2})h}{\tau}\right)\sin\left(\beta + \frac{2\pi(p-\frac{1}{2})h}{\tau'}\right)\right].$$

First suppose τ and τ' to be different. Then we have to find:

$$\left[\sin\left(\alpha + \frac{2\pi(p-\frac{1}{2})h}{\tau}\right)\sin\left(\beta + \frac{2\pi(p-\frac{1}{2})h}{\tau'}\right)\right],$$

which may be expressed as:

$$=\frac{1}{2}\left\{\left[\cos(\alpha-\beta)+2\pi h\,(p-\tfrac{1}{2})\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)\right]-\left[\cos\left(\alpha+\beta+2\pi h\,(p-\tfrac{1}{2})\left(\frac{1}{\tau}+\frac{1}{\tau'}\right)\right)\right]\right\}$$

$$-\frac{1}{2\lambda}\left\{\frac{\cos\left\{\alpha-\beta+\pi\lambda h\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)\right\}\sin\pi h\lambda\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)}{\sin\pi h\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)}\right.$$

$$\left.\frac{\cos\left\{\alpha+\beta+\pi\lambda h\left(\frac{1}{\tau}+\frac{1}{\tau'}\right)\right\}\sin\pi h\lambda\left(\frac{1}{\tau}+\frac{1}{\tau'}\right)}{\sin\pi h\left(\frac{1}{\tau}+\frac{1}{\tau'}\right)}\right.$$

If the periods be short the $\frac{1}{\lambda}$ term makes this very small. The exception is when $\tau=\tau'$, then

$$\frac{\sin\pi h\lambda\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)}{\lambda\sin\pi h\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)}$$

takes the value unity and we have

$$[\delta^{2q}x_p\,\delta^{2q}y_p]=\left(2^2\sin^2\frac{\pi h}{\tau}\right)^q\left(2^2\sin^2\frac{\pi h}{\tau'}\right)^q AB\frac{\tau\tau'}{\pi^2}\sin\frac{\pi h}{\tau}\sin\frac{\pi h}{\tau'}\times\tfrac{1}{2}\cos(\alpha-\beta)\quad\text{(vii)},$$

the second term being negligible compared to this.

Accordingly the correlation between two short period terms of equal period is simply

$$\cos(\alpha-\beta)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(viii)},$$

or perfect if there be no difference in phase, zero if $\frac{1}{4}$ period difference in phase, and perfect but of negative sign if $\frac{1}{2}$ period difference in phase. The result is independent of the difference used if we work with central differences. If therefore the only term were a single short period term the correlation of successive central differences ought to give us a constant correlation. If we find this correlation negligible we are driven to the conclusion that there is no such single short period term in existence common to both variates or that there must be $\frac{1}{4}$ period difference in phase.

Turn now to the theory of non-periodic fluctuations. We know that if X and Y be the non-periodic fluctuations on the theory of "Student"

$$\left.\begin{aligned}[(\delta^{2q}X_p)^2]&=\frac{(4q)!}{(2q)!\,(2q)!}\sigma_X^2\\ [(\delta^{2q}Y_p)^2]&=\frac{(4q)!}{(2q)!\,(2q)!}\sigma_Y^2\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\text{(ix)},$$

$$[\delta^{2q}X_p\delta^{2q}Y_p]=\frac{(4q)!}{(2q)!\,(2q)!}\sigma_X\sigma_Y r_{XY}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(x)},$$

and accordingly

$$r_{XY}=\text{correlation of }\delta^{2q}x_p\text{ and }\delta^{2q}y_p,$$

without consideration of removing secular trend. This is true if only *one* X is correlated with *one* Y—it is not needful to suppose these values of X and Y to be concurrent as Dr Persons asserts those who used the method have supposed. On the other hand if the series of X's and the series of Y's and again the series of X, Y's have correlation, we must introduce factors, which we will call for a moment $\phi(\rho')$, $\phi(\rho'')$ and $\phi(\rho)$, into these results

$$\left.\begin{aligned}[(\delta^{2q}X_p)^2] &= \frac{(4q)!}{(2q)!(2q)!}\phi(\rho')\sigma_x^2\\ [(\delta^{2q}Y_p)^2] &= \frac{(4q)!}{(2q)!(2q)!}\phi(\rho'')\sigma_y^2\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\text{(xi)},$$

$$[(\delta^{2q}X_p\delta^{2q}Y_p)] = \frac{(4q)!}{(2q)!(2q)!}\phi(\rho)\sigma_x\sigma_y r_{xy} \quad \ldots\ldots\ldots\ldots\text{(xii)},$$

and we shall have

$$r_{xy} = \frac{\sqrt{\phi(\rho')\phi(\rho'')}}{\phi(\rho)} \times \text{correlation of } \delta^{2q}x_p \text{ and } \delta^{2q}y_p \ldots\ldots\ldots\text{(xiii)}.$$

We will discuss these factors shortly.

(3) Supposing, however, "Student's" theory to hold, let us investigate what happens if we have a combination of secular trend, periodic terms and non-periodic fluctuations:

$$\left.\begin{aligned}x_p &= a_0 + a_1 t + \ldots + a_{n-1}t^{n-1} + S_\tau\left\{A_\tau'\sin\left(\alpha_\tau + \frac{2\pi}{\tau}(p-\tfrac{1}{2})h\right)\right\} + X_p,\\ y_p &= b_0 + b_1 t + \ldots + b_{m-1}t^{m-1} + S_\tau\left\{B_\tau'\sin\left(\beta_\tau + \frac{2\pi}{\tau}(p-\tfrac{1}{2})h\right)\right\} + Y_p,\end{aligned}\right\} \ldots\text{(xiv)}.$$

Or, if $2q$ be greater than n and m,

$$[\delta^{2q}x_p\delta^{2q}y_p] = S_\tau\left\{\tfrac{1}{2}A_\tau'B_\tau'\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q}\cos(\alpha_\tau-\beta_\tau)\right\} + \frac{(4q)!}{(2q)!(2q)!}\sigma_x\sigma_y r_{x_p y_p} \ldots\text{(xv)},$$

$$\left.\begin{aligned}[(\delta^{2q}x_p)^2] &= S_\tau\left\{\tfrac{1}{2}A_\tau'^2\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q}\right\} + \frac{(4q)!}{(2q)!(2q)!}\sigma_x^2\\ [(\delta^{2q}y_p)^2] &= S_\tau\left\{\tfrac{1}{2}B_\tau'^2\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q}\right\} + \frac{(4q)!}{(2q)!(2q)!}\sigma_y^2\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\text{(xvi)}.$$

Now we must here make a two-fold confession: first that one of us actually queried in proof Dr Anderson's statement in *Biometrika*, Vol. x p 279, about the elimination of periodic terms, and took the earliest opportunity when he did not modify it to express disagreement with what we thought to be his views (*Biometrika*, Vol. x. p. 503 footnote). In that place we also say that: "Should a more extended experience show that there is a real, if slight *positive* correlation between deathrates at three years' interval, while there is considerable *negative* correlation at one and two years' interval we should be compelled to discuss whether there is not something periodic in the nature of the heavy and light deathrates of infancy and childhood" In view of that statement it is almost absurd for Mr Yule to suggest that we overlooked the possibility of a short period term, yet he actually appears to be slightly annoyed because we inserted this footnote! But that footnote continues: "We have been unable to trace any sign of such periodicity either in the deathrate

or the graphs drawn, but we do not believe that a very short periodicity would be eliminated by the variate difference method using any moderate number of differences." This appears to have been also Dr Anderson's view throughout, only both Mr Yule and we misread his statements. We overlooked, perhaps with some justification, his qualifying phrase: "Ja mehr noch, man kann beweisen, daß überhaupt aller mehr oder minder 'glatten Reihen,' alle bei denen eine genügende positive Korrelation zwischen den Nachbargliedern bemerkbar ist, für die Praxis beim endlichen Differenzieren verschwinden" (*Biometrika*, Vol. x. p. 279). The short period terms demand a *negative* correlation. As a matter of fact as we shall indicate later, no short periodic terms of any importance exist in the material used by us in our first paper; our original opinion is confirmed by more ample examination. But we must here confess our second point: we were actually indifferent to the short period terms if such existed, we felt comparatively certain of their insignificance in our material, and further, if they actually were present, they seemed to us just as likely to be a product of the organic relationship we were seeking, as a complete mask of it, which we suppose to be Mr Yule's view. It is possible, if such terms exist, that they form a reasonable element in the correlation of X_p and Y_p after removal of the secular trend. What we were concerned with was to get out of the morass which Sir Arthur Newsholme, Mortara and other statisticians had got us into, by correlating time growths of various factors and interpreting the resulting intense correlations as causal factors. That end seemed to us achieved by the variate difference method and therefore we praised it highly as breaking new ground.

Mr Yule criticises us and says (i) he does not believe in the existence of fluctuating variations and (ii) if such existed they would be swamped by differencing and the shortest period terms would alone survive. He postulates the existence of the latter. His processs of demonstrating appears to us remarkable. He notes the relative size of the multiplying factors on differencing, i.e.

$$\left(2^2 \sin^2 \frac{\pi h}{\tau}\right)^{2q} \quad \text{and} \quad \frac{(4q)!}{(2q)!\,(2q)!},$$

and shows that the former will be the greater for short period terms. Now the former will be the greater when

$$\sin \frac{\pi h}{\tau} > \frac{1}{2}\left(\frac{(4q)!}{(2q)!\,(2q)!}\right)^{\frac{1}{4q}}.$$

For second differences this gives				τ	$< 3{\cdot}496h$,
,,	fourth	,,	,,	,,	$< 3{\cdot}090h$,
,,	sixth	,,	,,	,,	$< 2{\cdot}901h$,
,,	eighth	,,	,,	,,	$< 2{\cdot}787h$,
,,	tenth	,,	,,	,,	$< 2{\cdot}708h$.

It is unnecessary to go further because it is unlikely that we could go beyond 10th differences owing to the imperfection of our data, or the great labour of computation.

Now we think there is an error in Mr Yule's reasoning. For the variate difference investigation X and Y are the whole fluctuations above the secular trend. Mr Yule compares σ_X with A_τ', that is to say he compares the standard deviation of one portion of his fluctuation with the amplitude of another portion! But at least he ought to have compared $\frac{1}{\sqrt{2}} A_\tau'$ with σ_Y, and even this is not legitimate.

The fluctuation is

$$S_\tau A_\tau' \sin\left\{a_\tau + \frac{2\pi}{\tau}(p - \tfrac{1}{2})h\right\} + X_p,$$

the S.D.² of the whole is

$$S_\tau(\tfrac{1}{2}A_\tau'^2) + \sigma_X^2,$$

and accordingly σ_X^2 should be compared with $S_\tau(\frac{1}{2}A_\tau'^2)$ and not with $\frac{1}{2}A_\tau'^2$, still less with $(A_\tau')^2$ as Mr Yule compares it.

Does, however, $\frac{1}{2}A_\tau'^2$ necessarily form the largest term in the summation? We see no reason for supposing it in the least likely.

$$\tfrac{1}{2}A_\tau'^2 = \tfrac{1}{2}A_\tau^2 \frac{\tau^2}{\pi^2} \sin^2 \frac{\pi h}{\tau},$$

and this is not a maximum when τ is least but when

$$\sin \frac{\pi h}{\tau} \Big/ \frac{\pi h}{\tau}$$

is a maximum, i.e. when $\sin z/z$ is a maximum, or when $z = \tan z$

In fact if we evaluate the function $\frac{\tau^2}{h^2}\sin^2\left(\frac{\pi h}{\tau}\right)$, we have the series of values in the following table.* They show that *for equal amplitudes* the short period terms contribute *least* to the S.D.² of the fluctuations, and that we cannot straight off assert as Mr Yule has done that the low period terms of less than 3 years will dominate the result reached

$\tau = 1$	$\frac{\tau^2}{h^2}\sin^2\left(\frac{\pi h}{\tau}\right)$	τ	$\frac{\tau^2}{h^2}\sin^2\left(\frac{\pi h}{\tau}\right)$
h	0·000	$10h$	9·549
$2h$	4·000	$12h$	9·646
$3h$	6·750	$18h$	9·786
$4h$	8·000	$25h$	9·818
$5h$	8·637	$30h$	9·834
$6h$	9·000	$40h$	9·849
$7h$	9·225	$50h$	9·855
$8h$	9·373	$100h$	9·866
$9h$	9·475	$150h$	9·868

Now it seems to us that we have to take for any comparison at all some ratio between σ_X and the total S.D. due to periodic terms, or we may take

$$\sigma_X^2 = m^2 S_\tau\left(\tfrac{1}{2}A_\tau^2 \frac{\tau^2}{\pi^2}\sin^2\frac{\pi h}{\tau}\right),$$

and if we assume any one value to be more important here than another, it will not be the term of *lowest* period. In any case it is not needful to consider that the term of lowest period contributes most to the S.D. of the fluctuations.

Now return to the S.D.^{2} of the $2q$th difference. We have to consider the parts of the expression

$$S_\tau\left\{\tfrac{1}{2}A_\tau^2\left(\frac{\tau^2}{\pi^2}\sin^2\frac{\pi h}{\tau}\right)\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q}\right\}+\sigma_\chi^2\frac{(4q)!}{(2q)!(2q)!}.$$

Accordingly if we assume $m^2=1$ or the random terms and the periodic of the same order, we have to investigate the relative order of

$$\frac{S_\tau\left\{\tfrac{1}{2}A_\tau^2\left(\frac{\tau^2}{\pi^2}\sin^2\frac{\pi h}{\tau}\right)\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q}\right\}}{S_\tau\left\{\tfrac{1}{2}A_\tau^2\left(\frac{\tau^2}{\pi^2}\sin^2\frac{\pi h}{\tau}\right)\right\}}\text{ and }\frac{(4q)!}{(2q)!(2q)!}\quad\text{......(xvii)},$$

and in considering only

$$\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q}\text{ and }\frac{(4q)!}{(2q)!(2q)!}$$

to obtain a dominant term Mr Yule has omitted to consider the multiplier

$$\tfrac{1}{2}A_\tau^2\left(\frac{\tau^2}{\pi^2}\sin^2\frac{\pi h}{\tau}\right)\Big/S_\tau\left\{\tfrac{1}{2}A_\tau^2\left(\frac{\tau^2}{\pi^2}\sin^2\frac{\pi h}{\tau}\right)\right\}.$$

We cannot reduce this to unity because we have just seen that the lowest τ term is not necessarily the dominant term in the denominator.

We have to evaluate $\left(\frac{\tau}{h}\right)^2\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q+1}$ for various values of τ/h and q to discover where we are. We will do this for second, fourth, sixth, eighth and tenth differences, and for $\tau=2h$, $3h$, $4h$, $6h$, $8h$ and $10h$, placing also on record the value of $(4q)!/\{(2q)!(2q)!\}$. We have the following table:

Table of $\left(\frac{\tau}{h}\right)^2\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q+1}$.

$\tau/h=$	2	3	4	6	8	10	$(4q)!/\{(2q)!(2q)!\}$
$q=1$	256	243	128	36	12·87	5·57	6
2	4096	2187	512	36	4·41	0·81	70
3	65536	19683	2048	36	1·51	0·12	924
4	1048576	177147	8192	36	0·52	0·02	12870
5	16777216	1594323	32768	36	0·17	0·00	184756
$q=0$	16	27	32	36	37·49	38·20	—

Now it is clear that if we suppose only *one* short periodic term to occur, it will have no influence at all, if the period be four times or more the subrange of tabling. In other words unless the non-periodic fluctuation was very small as compared to the periodic all trace of the periodic term would have disappeared by the 8th or 10th differencing. Anderson would therefore have been justified in his statement

as we at first read it had he confined it to periodic terms of four or more times the subrange.

When we turn to terms of three times the subrange if there was only one periodic term, its order would at the 8th be in the ratio of 10 to 20, and at the 10th difference in the ratio of 10 to 31. Such a term therefore would not swamp the non-periodic fluctuation; it would in fact be considerably smaller than it. If it were combined with two or three other terms of longer period but about the same or larger amplitude, it would be swamped by the non-periodic fluctuation.

Now turning to the term where $\tau/h = 2$, we notice that if only a single such term existed, its influence at the tenth difference would be as $6:1$, at the eighth difference as about $5:1$ and at the sixth difference as about $4:1$ relative to the non-periodic fluctuation. If it was associated with two or three other terms of longer periodicity, its influence would be somewhat *less* than that of the non-periodic term. What is clear is that even here it would be exaggeration to talk of swamping by the term of short period. The contributions of the two terms are much of the same order, and which will dominate in the S.D. of the fluctuations will depend on how far those fluctuations are due to a series of periodic terms, or to the non-periodic part of the fluctuations.

Further the actual correlation in $\delta^{2q}x$ and $\delta^{2q}y$ depends not only on the S.D.'s of the fluctuations, but on the correlations, on whether the periodic terms in x and y are, or are not, in the same phase.

It is therefore not really a case of one or other factor being swamped. The process removes the secular trend and correlates the residuals. In these residuals the factors which contribute chiefly to the correlation are the non-periodic and very short periodic terms if the latter exist. If they exist it would indicate that the residuals have a common factor, and that seems to me the very point we set out to inquire into. If one has a two element period and the other has not, that will lessen the correlation between the non-periodic fluctuations, but that is surely what we should anticipate.

On the whole it does not seem to us that Mr Yule's criticisms really lead one far, especially when he concludes them by stating that the time problem is that of discussing the relations "between oscillations of different durations, such oscillations being in all probability not strictly periodic, but up and down movements of greater or less rapidity" (p. 524). If they are not periodic, why confuse the issue by endeavouring to show that *true* periodic terms would swamp the residuals which he holds are not *true* periodic terms?

If terms of very small period exist, then those who desire to do so can find and eliminate them.

(4) The criticism that is raised by Persons that we cannot assume $S(X_p X_{p+s}) = 0$ seems to us more valid, although it is certainly not demonstrated if we test its validity by finding the X's from a couple of combined straight lines.

We think, however, that it is possible to meet this criticism. In the first place it can be met by simply taking out the secular trend by adequate smoothing and

correlating the residuals in as many ways as seem desirable. But theoretically it is also possible to learn something about these correlations by the variate-difference method, i.e. by the aid of the functions $\phi(\rho)$, $\phi(\rho')$, $\phi(\rho'')$ already referred to. This we now proceed to discuss.

Let us suppose the correlation of X_p, Y_{p+s} to be ρ_s and of $X_p\ Y_{p-s}$ to be ρ_{-s}. Let us call ρ_s' the correlation of X_p and X_{p+s}, while ρ'_{-s} is that of X_p and X_{p-s}. Lastly ρ_s'' is the correlation of Y_p and Y_{p+s}, ρ''_{-s} of Y_p and Y_{p-s}. Then we readily find

$$\frac{[\Delta^n X_p \Delta^n Y_p]}{\sigma_X \sigma_Y} = \frac{(2n)!}{n!\,n!}\left\{(-1)^n \frac{n!\,n!}{(2n)!}\rho_{-n} + \dots + \frac{n(n-1)}{(n+1)(n+2)}\rho_{-2} - \frac{n}{n+1}\rho_{-1}\right.$$
$$\left. + \rho_0 - \frac{n}{n+1}\rho_1 + \frac{n(n-1)}{(n+1)(n+2)}\rho_2 - \dots (-1)^n \frac{n!\,n!}{(2n)!}\rho_n\right\}$$
$$= \frac{(2n)!}{n!\,n!}\phi(n, \rho_0, \rho) \text{ say} \quad \dots\dots\dots\dots(\text{xviii}).$$

Similarly $$\frac{[\Delta^{n+1} X_p \Delta^{n+1} Y_p]}{\sigma_X \sigma_Y} = \frac{(2n+2)!}{(n+1)!\,(n+1)!}\phi(n+1, \rho_0, \rho) \quad \dots\dots\dots(\text{xix}),$$

and accordingly* $$\frac{[\Delta^{n+1} X_p \Delta^{n+1} Y_p]}{[\Delta^n X_p \Delta^n Y_p]} = \left(4 - \frac{2}{n+1}\right)\frac{\phi(n+1, \rho_0, \rho)}{\phi(n, \rho_0, \rho)} \quad \dots\dots\dots(\text{xx}).$$

Again

$$\frac{[(\Delta^n X_p)^2]}{\sigma_X^2} = \frac{(2n)!}{n!\,n!}\left\{(-1)^n \frac{n!\,n!}{(2n)!}\rho'_{-n} + \dots + \frac{n(n-1)}{(n+1)(n+2)}\rho'_{-2}\right.$$
$$\left. - \frac{n}{n+1}\rho'_{-1} + 1 - \frac{n}{n+1}\rho_1' + \frac{n(n-1)}{(n+1)(n+2)}\rho_2' - \dots (-1)^n \frac{n!\,n!}{(2n)!}\rho_n'\right\}$$
$$= \frac{(2n)!}{n!\,n!}\phi(n, 1, \rho') \text{ say} \quad \dots\dots\dots\dots(\text{xxi}),$$

and accordingly $$\frac{[(\Delta^{n+1} X_p)^2]}{[(\Delta^n X_p)^2]} = \left(4 - \frac{2}{n+1}\right)\frac{\phi(n+1, 1, \rho')}{\phi(n, 1, \rho')} \quad \dots\dots\dots\dots(\text{xxii}).$$

Similarly $$\frac{[(\Delta^{n+1} Y_p)^2]}{[(\Delta^n Y_p)^2]} = \left(4 - \frac{2}{n+1}\right)\frac{\phi(n+1, 1, \rho'')}{\phi(n, 1, \rho'')} \quad \dots\dots\dots\dots(\text{xxiii}).$$

Now it is clear that our three equations (xx), (xxi) and (xxii) separate off the ρ_s, the ρ_s' and ρ_s'', and accordingly we can confine our attention to any one type in considering the solution. Further, by putting $\tilde{\rho}_s = \rho_s/\rho_0$ we can write

$$\phi(n+1, \rho_0, \rho) = \rho_0 \phi(n+1, \tilde{\rho})$$

and drop the second variable (ρ_0 or 1) out.

After finding $\tilde{\rho}$, ρ' and ρ'' we shall have finally to determine ρ_0 from

$$\frac{[\Delta^n X_p \Delta^n Y_p]}{\sqrt{[(\Delta^n X_p)^2][(\Delta^n Y_p)^2]}} = \rho_0 \frac{\phi(n, \tilde{\rho})}{\sqrt{\phi(n, \rho')\,\phi(n, \rho'')}} \quad \dots\dots\dots(\text{xxiv}).$$

We may therefore leave ρ_0 out of account and consider we have to find the three series $\tilde{\rho}$, ρ', ρ''.

* We owe results (xviii) and (xxi) to Mr E. S. Pearson, who had used them in a thesis presented in 1921 and since published in *Biometrika*, Vol. XIV. pp. 37—39.

If we call the three observed quantities*

$$\tilde{Y}_{n,0}\left(4-\frac{2}{n+1}\right),\quad Y'_{n,0}\left(4-\frac{2}{n+1}\right),\quad Y''_{n,0}\left(4-\frac{2}{n+1}\right),$$

we have the three type equations

$$\tilde{Y}_{n,0}=\frac{\phi(n+1,\tilde{\rho})}{\phi(n,\tilde{\rho})},\quad Y'_{n,0}=\frac{\phi(n+1,\rho')}{\phi(n,\rho')},\quad Y''_{n,0}=\frac{\phi(n+1,\rho'')}{\phi(n,\rho'')},$$

to find the series $\tilde{\rho}$, ρ', ρ'' respectively. Now at first sight it might seem that all we have to do is to write down equations for enough values of n till we have adequate relations to find, say, the $\tilde{\rho}$'s, starting n after we have gone to a difference adequate to remove the secular trend. This would not, however, be possible, for each fresh equation introduces two more ρ's. Thus in $\tilde{Y}_n$ we have $\tilde{\rho}_{-(n+1)}\ldots\tilde{\rho}_{-1}\,\tilde{\rho}_1\ldots\tilde{\rho}_{n+1}$, but in $\tilde{Y}_{n+1}$ we introduce $\tilde{\rho}_{-(n+2)}$ and $\tilde{\rho}_{n+2}$ in addition. Thus every new equation gives us two further unknowns. At the same time these unknowns will probably be very small and multiplied by very small coefficients of the order $n!\,n!/(2n!)$, i.e. the inverses of the terms in the last column of the table on p. 292. Remembering that the ρ's will most probably be rapidly convergent also, we see that if we suppose the secular term disposed of by the fourth difference, we might well neglect the terms involving $\tilde{\rho}_4$ and $\tilde{\rho}_{-4}$. This would leave us with six unknowns to be determined, $\tilde{\rho}_{-3}$, $\tilde{\rho}_{-2}$, $\tilde{\rho}_{-1}$, $\tilde{\rho}_1$, $\tilde{\rho}_2$, $\tilde{\rho}_3$, or we should need six equations, or the value of $\tilde{Y}_n$ from $n=4$ to 9 involving up to 10th differences would suffice Theoretically the problem is solved, if we can be content with the correlations of seven adjacent fluctuations. These indeed if we found them would usually be quite adequate to determine the problem of "lag," and throw light by their rate of decay on Mr Yule's view as to "not strictly periodic oscillations." If "Student's" original hypotheses are correct, we should find all the ρ's but one sensibly zero, and this one would indicate the lag.

Now this method depends for safety on our accuracy in the high differences, and this can by no means always be attained. Accordingly we ask whether additional correlations cannot be found which will avoid the risk of too high differences. We have clearly still available the correlations of $\Delta^n X_p$ and $\Delta^n Y_{p+s}$. For example:

$$\begin{aligned}\frac{[\Delta^n X_p\,\Delta^n Y_{p+1}]}{\sigma_X\sigma_Y}=\frac{(2n)!}{n!\,n!}\Bigg\{&(-1)^n\frac{n!\,n!}{(2n)!}\rho_{-(n-1)}+\cdots\\&+\frac{n(n-1)}{(n+1)(n+2)}\rho_{-1}-\frac{n}{n+1}\rho_0+\rho_1-\frac{n}{n+1}\rho_2+\frac{n(n-1)}{(n+1)(n+2)}\rho_3\cdots\\&+(-1)^n\frac{n!\,n!}{(2n)!}\rho_{n+1}\Bigg\}\quad\ldots\ldots\ldots\ldots\text{(xxv)},\end{aligned}$$

$$\begin{aligned}\frac{[\Delta^n X_p\,\Delta^n Y_{p-1}]}{\sigma_X\sigma_Y}=\frac{(2n)!}{n!\,n!}\Bigg\{&(-1)^n\frac{n!\,n!}{(2n)!}\rho_{-(n+1)}+\cdots\\&+\frac{n(n-1)}{(n+1)(n+2)}\rho_{-3}-\frac{n}{n+1}\rho_{-2}+\rho_{-1}-\frac{n}{n+1}\rho_0+\frac{n(n-1)}{(n+1)(n+2)}\rho_1\cdots\\&+(-1)^n\frac{n!\,n!}{(2n)!}\rho_{n-1}\Bigg\}\quad\ldots\ldots\ldots\ldots\text{(xxvi)}.\end{aligned}$$

* The subscript zero is introduced into Y to mark that we are correlating concurrent $\Delta^n X_p$ and $\Delta^n Y_p$, not $\Delta^n X_p$ and $\Delta^n Y_{p+s}$. See p. 285.

These non-symmetrical functions will not converge as rapidly as the symmetrical, but even if we had to take an additional pair of ρ's (for example $\tilde{\rho}_{-4}$ and $\tilde{\rho}_4$) into account, we could get eight equations for finding them as well as an additional control equation by using only results up to the seventh differences, supposing the secular trend to disappear at the fourth.

As before, if $\tilde{\rho}_s = \rho_s/\rho_0$ and supposing E to be the usual finite difference operator acting on ρ_s but not on ρ_0, we have:

$$\frac{[\Delta^{n+1} X_p \Delta^{n+1} Y_{p+1}]}{[\Delta^n X_p \Delta^n Y_{p+1}]} = \left(4 - \frac{2}{n+1}\right) \frac{E\phi(n+1, \tilde{\rho})}{E\phi(n, \tilde{\rho})},$$

or if the known left-hand side be $\tilde{Y}_{n,1}\left(4 - \frac{2}{n+1}\right)$,

$$\tilde{Y}_{n,1} = \frac{E\phi(n+1, \tilde{\rho})}{E\phi(n, \tilde{\rho})} \quad \text{(xxvii)}.$$

Similarly
$$\tilde{Y}_{n,-1} = \frac{E^{-1}\phi(n+1, \tilde{\rho})}{E^{-1}\phi(n, \tilde{\rho})} \quad \text{(xxviii)}.$$

Similar equations:

$$Y'_{n,1} = \frac{E\phi(n+1, \rho')}{E\phi(n, \rho')}, \quad Y'_{n,-1} = \frac{E^{-1}\phi(n+1, \rho')}{E^{-1}\phi(n, \rho')},$$

$$Y''_{n,1} = \frac{E\phi(n+1, \rho'')}{E\phi(n, \rho'')}, \quad Y''_{n,-1} = \frac{E^{-1}\phi(n+1, \rho'')}{E^{-1}\phi(n, \rho')} \quad \text{(xxix)},$$

will help to give the ρ_s' and ρ_s'' respectively.

The above results will probably reduce our working, according to the nature of the secular trend, to something like 6th or 7th differences. Only practical experience can show how far the equations proposed are sensitive enough to give useful results. We propose shortly to test them by actually subtracting the secular trend and correlating the residuals, and also by finding the correlations by the present indirect method.

It will be observed of course that any ρ-system marks the rate of decadence of the correlated series of variates. For example ρ'_{-s} is the correlation of X_{p-s} with X_p and ρ'_{+s} of X_p with X_{p+s}. It would be no unreasonable assumption to suppose $\rho'_{-s} = \rho'_{+s}$ and this would reduce our requisite equations by nearly one half. The assumption that an oscillation can be represented by a harmonic term involves such a relationship. By retaining, however, the distinction it may be possible to allow for something of the nature of an asymmetrical oscillation.

One case is, however, of special interest: if $\rho_s' = \tilde{\rho}_s = \rho_s''$ for all values of s, then the correlation of $\Delta^n X_p$ and $\Delta^n Y_p$ is equal to the correlation of X and Y. That is we have "Student's" hypothesis replaced by a much more general one, namely the hypothesis that the correlation between the variates dies out at the same rate. This marks, we think, a considerable extension of the legitimate range of application of the simple Variate Difference Method. Let us examine the exact meaning of the needful relations. We have $\rho_s' = \tilde{\rho}_s$ or $\rho_s'\rho_0 - \rho_s = 0$, that is:

correlation of X_p, Y_{p+s} = correlation of X_p, Y_p × correlation of X_p, X_{p+s},

but the correlation of X_p and Y_p is that of X_{p+s} and Y_{p+s}. Hence correlation of X_p and Y_{p+s} for constant X_{p+s} is zero. Similarly the correlation of Y_p and X_{p+s} for constant X_{p+s} is zero. In other words there is no relation between any non-corresponding (we do not say non-concurrent because we may allow for "lag") X and Y, except indirectly through the X which corresponds to the Y. If we make the X which corresponds to any Y constant, then that Y will have zero correlation with all other X's. Y is indirectly affected by non-corresponding X's owing to its correlation with its own X's*.

We can reach, however, a still more general hypothesis. Let us suppose the three sets of correlations to degrade not at the same rate but at different rates, i.e. let

$$\rho_s = \epsilon^s \rho_0, \quad \rho_s' = \epsilon'^s, \quad \rho_s'' = \epsilon''^s.$$

We will call $\phi(n, \epsilon)$ the series

$$(-1)^n \frac{n!\,n!}{(2n)!}\epsilon^n + \ldots + \frac{n(n-1)}{(n+1)(n+2)}\epsilon^2 - \frac{n}{n+1}\epsilon + 1$$

Then we have

$$\frac{[\Delta^{n+1} x_p\, \Delta^{n+1} y_p]}{[\Delta^n x_p\, \Delta^n y_p]} = \left(4 - \frac{2}{n+1}\right)\frac{2\phi(n+1, \epsilon) - 1}{2\phi(n, \epsilon) - 1} \quad \ldots\ldots\ldots\text{(xxx)},$$

$$\frac{[(\Delta^{n+1} x_p)^2]}{[(\Delta^n x_p)^2]} = \left(4 - \frac{2}{n+1}\right)\frac{2\phi(n+1, \epsilon') - 1}{2\phi(n, \epsilon') - 1} \quad \ldots\ldots\text{(xxxi)},$$

$$\frac{[(\Delta^{n+1} y_p)^2]}{[(\Delta^n y_p)^2]} = \left(4 - \frac{2}{n+1}\right)\frac{2\phi(n+1, \epsilon'') - 1}{2\phi(n, \epsilon'') - 1} \quad \ldots\ldots\text{(xxxii)},$$

and since the left-hand sides are known we can find from them ϵ, ϵ' and ϵ''. Mr Henderson has kindly formed for us a table of the function on the right-hand side for $n = 1$ to 10 and ϵ from -1 to $+1$ by tenths†. It is possible by interpolation from this table to find the ϵ's readily. When they are found we have for the correlation of $\Delta^n x_p\, \Delta^n y_p$,

$$r_{\Delta^n x_p, \Delta^n y_p} = \frac{\{2\phi(n, \epsilon) - 1\}\, r_{XY}}{\sqrt{2\phi(n, \epsilon') - 1}\,\sqrt{2\phi(n, \epsilon'') - 1}},$$

or

$$r_{XY} = r_{\Delta^n x_p, \Delta^n y_p} \frac{\sqrt{2\phi(n, \epsilon') - 1}\,\sqrt{2\phi(n, \epsilon'') - 1}}{2\phi(n, \epsilon) - 1} \quad \ldots\ldots\text{(xxxiii)}.$$

We can test the goodness of the hypothesis by seeing if the values of ϵ, ϵ' and ϵ'' agree as determined from successive values of n.

If ϵ be positive then $2\phi(n, \epsilon) - 1$ always decreases with increase of n or increase of ϵ. Now we should suppose as a rule that the degradation of the correlation of the X's among themselves, or of the Y's among themselves, would be less rapid than that of X and Y. Accordingly it follows that in such cases

$$r_{XY} < r_{\Delta^n x_p, \Delta^n y_p},$$

* This is the Mendelian view of gametic constitution. There is a correlation between grandparents and grandchild, but if you fix the constitution of the parents this correlation vanishes.

† See Appendix to this paper, p. 310.

or the value obtained by "Student" is a superior limit to the correlation of the fluctuations from the secular trend. If on the other hand ϵ' and ϵ'' are negative, as would almost certainly be the case if there were a large two interval period, then r_{XY} will be larger than the value assigned by "Student," on the same hypothesis of degradation. In any case it appears to us that to calculate the ϵ's by aid of Mr Henderson's table will throw light on the inter-relation of the fluctuations from the secular trend, even if it be only a step preliminary to a fuller determination of the ρ's

We think the theory of the variate difference method thus generalised will meet such criticisms as those of Persons, which are valid. It may be harder to meet those of Mr Yule who after attempting to bludgeon the whole business with a two interval harmonic period, then turns round and states that such harmonic periods do not occur but up and down movements of greater or less rapidity, these movements from the secular trend, for some unprovided reason, not being identical with the non-periodic fluctuations which the believers in the variate difference method were attempting to study, nor with the "mehr oder minder glatten Reihen" which Anderson asserts disappear by differencing.

(5) Let us now turn to the case of a single periodic term in our variates and see how it will influence our equations.

The term in X_p being

$$A\,\frac{\tau}{\pi}\sin\frac{\pi h}{\tau}\sin\left\{\alpha + 2\pi\left(p-\tfrac{1}{2}\right)\frac{h}{\tau}\right\},$$

that in X_{p+s} will be

$$A\,\frac{\tau}{\pi}\sin\frac{\pi h}{\tau}\sin\left\{\alpha + 2\pi\left(p+s-\tfrac{1}{2}\right)\frac{h}{\tau}\right\}$$

$$= A\,\frac{\tau}{\pi}\sin\frac{\pi h}{\tau}\sin\left\{\alpha + \frac{2\pi sh}{\tau} + 2\pi\left(p-\tfrac{1}{2}\right)\frac{h}{\tau}\right\},$$

and accordingly the correlation between X_p and X_{p+s} will be

$$\rho'_s = \cos\frac{2\pi sh}{\tau} = \rho'_{-s} \quad\text{.........................(xxxiv)},$$

or there should be the same correlation between X_p and X_{p+s} or X_p and X_{p-s}.

Let us write $\gamma = 2\pi h/\tau$; then turning to (xxi)

$$\phi(n, 1, \rho') = 2\left\{1 - \frac{n}{n+1}\rho_1' + \frac{n(n-1)}{(n+1)(n+2)}\rho_2' - \dots + (-1)^n\frac{n!\,n!}{(2n)!}\rho_n'\right\} - 1$$

$$= 2\phi(n, \gamma) - 1 \quad\text{...(xxxv)},$$

where

$$\phi(n, \gamma) = 1 - \frac{n}{n+1}\cos\gamma + \frac{n(n-1)}{(n+1)(n+2)}\cos 2\gamma - \dots + (-1)^n\frac{n!\,n!}{(2n)!}\cos n\gamma \quad\text{.......................(xxxvi)}.$$

Hence $$2\phi(n, \gamma) - 1 = \frac{n!\,n!}{(2n)!}\left(2\sin\frac{\gamma}{2}\right)^{2n} \quad \text{.........(xxxvii)},$$

and $$\left(4 - \frac{2}{n+1}\right)\frac{2\phi(n+1, \gamma) - 1}{2\phi(n, \gamma) - 1} = 2(1 - \cos\gamma).$$

Accordingly $$\frac{[(\Delta^{n+1} X_p)^2]}{[(\Delta^n X_p)^2]} = 2(1 - \cos\gamma) \quad \text{.........(xxxviii)},$$

or $$\sin\frac{\pi h}{\tau} = \tfrac{1}{2}\left\{\frac{[(\Delta^{n+1} X_p)^2]}{[(\Delta^n X_p)^2]}\right\}^{\frac{1}{2}} \quad \text{.........(xxxix)}.$$

Thus if the ratio of successive S.D.'s of the differences approaches a limit, that limit will enable us to determine the period, if a single periodic term exists. Again returning to (xxi) we have

$$[(\Delta^n X_p)^2] = \sigma_X^2 \frac{(2n)!}{n!\,n!}\{2\phi(n, \gamma) - 1\} = \sigma_X^2\left(2\sin\frac{\gamma}{2}\right)^{2n}$$

or $$\sigma_X^2 = \frac{[(\Delta^n X_p)^2]}{\left(2\sin\frac{\gamma}{2}\right)^{2n}} \quad \text{.........(xl)},$$

which enables us after any form of smoothing has given us σ_X^2 to test how far a given periodic term will account for the fluctuations from the secular trend.

Again from $$\sin\frac{\pi h}{\tau'} = \tfrac{1}{2}\left\{\frac{[(\Delta^{n+1} Y_p)^2]}{[(\Delta^n Y_p)^2]}\right\}^{\frac{1}{2}} \quad \text{.........(xli)},$$

we can determine τ', and thus test whether there is really any common periodic term behind X and Y, as well as whether σ_Y^2 can be to any extent accounted for by such a periodic term.

Lastly suppose that there does exist a common periodic term in X and Y and that

$$X_p = A\frac{\tau}{\pi}\sin\frac{\pi h}{\tau}\sin\left\{\alpha + 2\pi(p - \tfrac{1}{2})\frac{h}{\tau}\right\},$$

$$Y_p = A\frac{\tau}{\pi}\sin\frac{\pi h}{\tau}\sin\left\{\beta + 2\pi(p - \tfrac{1}{2})\frac{h}{\tau}\right\}.$$

Then the correlation of X_p and Y_{p+s} will be

$$\rho_s = \cos\left(\alpha - \beta - \frac{2\pi sh}{\tau}\right)$$
$$= \cos(\alpha - \beta)\cos\frac{2\pi sh}{\tau} + \sin(\alpha - \beta)\sin\frac{2\pi sh}{\tau}.$$

Now substituting in equation (xviii), it will be clear that the parts of ρ_s and ρ_{-s} with the factor $\sin(\alpha - \beta)$ will cancel, and accordingly

$$\frac{[\Delta^n X_p \Delta^n Y_p]}{\sigma_X \sigma_Y} = \frac{(2n)!}{n!\,n!}\cos(\alpha - \beta)\{2\phi(n, \gamma) - 1\}$$
$$= \left(2\sin\frac{\gamma}{2}\right)^{2n}\cos(\alpha - \beta) \quad \text{.........(xlii)}.$$

It follows therefore that

$$\frac{[\Delta^{n+1} X_p \Delta^{n+1} Y_p]}{[\Delta^n X_p \Delta^n Y_p]} - \left(2 \sin \frac{\gamma}{2}\right)^2$$

$$= \frac{[(\Delta^{n+1} X_p)^2]}{[(\Delta^n X_p)^2]} = \frac{[(\Delta^{n+1} Y_p)^2]}{[(\Delta^n Y_p)^2]} \quad \text{.....................(xlii)} bis,$$

which will serve as a good test of whether the fluctuations can be attributed to a single periodic term, and lastly the difference of phase or "lag" of Y on X will be given by

$$r_{\Delta^n X \Delta^n Y} = \cos(\alpha - \beta) \quad \text{..........................(xliii)}.$$

Now we know that on "Student's" hypothesis, the ratios of the differences given in (xlii)*bis* tend to the value 4. They would only tend to 4 in the case of a single periodic term if the period were twice the fundamental unit of time, i.e. $\gamma = \pi$, for $\tau = 2h$. After the secular terms have been eliminated, the ratios in (xlii)*bis* and the correlation in (xliii) should always be the same in the case of a single periodic term. But in the case of correlated fluctuations, while the correlation in (xliii) would be constant, the ratio of the differences—of course on "Student's" hypothesis—would be $4 - \frac{2}{n+1}$ and thus only *tend* to a constant value $= 4$. On the other hand if ρ_s, ρ_s' and ρ_s'' be not a series of zero correlations (except of course for $s = 0$) then the ratios in (xlii)*bis* will tend more slowly to 4, because they depend on quantities like $\frac{n}{n+1}$ being equal to $\frac{n-1}{n}$ practically, as n becomes large.

(6) Having discussed these general points, it is desirable to see what light they throw on the results of our paper of 1914 and how far the criticisms raised by Dr Persons and Mr Yule affect our results.

The work we have undertaken is very laborious and may be considered under three headings:

(*a*) We have checked our arithmetic and calculated the mean products $[\Delta^n x \Delta^n y]$, $[(\Delta^n x)^2]$ and $[(\Delta^n y)^2]$, assuming n sufficiently high to assume that we have got rid of the secular terms. We have then analysed these results.

(*b*) We have smoothed the series by (i) Mr Rhodes' method and by (ii) Dr Sheppard's method, and taking the residuals as X_p, Y_p, have correlated X_p and Y_p, X_p and Y_{p+s}, X_p and X_{p+s}, Y_p and Y_{p+s}.

(*c*) We have applied periodogram analysis to X_p and Y_p to determine if there is really a short period term in the deathrates.

In carrying out this work we have to acknowledge the excellent computing assistance given to us by the members of the Senior Year Statistical Class at University College. They have done a large part of the smoothing, the correlation of the residuals and the periodogram work, computing in pairs and checking their results at each stage. Lengthy as the work has been, it would have consumed even more time of the authors had it not been for this most helpful laboratory-class aid.

We will now state the salient results of that analysis as they apply to our original conclusions and to our critics' statements.

In order to obtain a satisfactory smooth for the period 1859 to 1908, for which we worked in the 1915 paper, we added to our Tables I and II* the more recent results now available. These are given in the following additional tables. The smooth was made by aid of Rhodes' formulae†.

TABLES I α AND II α.

Deathrates in each Year of Life for groups born in the Year of 1st column.

Year	Males					Females					Year
	0—1	1—2	2—3	3—4	4—5	0—1	1—2	2—3	3—4	4—5	
1909	120·108	32·285	14·658	8·279	5·751	96·882	30·002	14·608	8·114	5·707	1909
1910	116·245	40·400	13·085	7·841	6·368	94·221	37·836	13·309	7·662	6·508	1910
1911	142·280	31·164	13·373	8·333	7·600	117·366	29·045	12·422	8·311	6·544	1911
1912	105·626	33·862	13·448	10·112	5·523	83·639	30·521	12·898	9·763	5·573	1912
1913	120·145	33·029	17·145	7·671	3·986	96·235	30·464	15·639	7·191	5·959	1913
1914	116·080	40·336	11·779	8·092	10·274	92·755	36·631	10·974	8·086	10·900	1914
1915	122·919	28·584	14·283	14·570	6·774	96·059	26·562	13·291	15·650	6·870	1915
1916	101·995	29·591	21·416	8·013	—	79·894	27·243	21·711	8·109	—	1916
1917	107·607	44·375	13·517	—	—	84·866	43·207	12·944	—	—	1917
1918	107·908	25·040	—	—	—	85·900	22·136	—	—	—	1918
1919	100·003	—	—	—	—	77·603	—	—	—	—	1919

Diagrams I and II (p. 307) are fair representations of the result of smoothing by Rhodes' formulae. The smooth, to judge from the graph, appears to contain no short period terms. The differences of the calculated smoothed values and the observations were now formed for the five columns of the males and are termed X_p, Y_p, Z_p, U_p and V_p respectively for the year p. A search was now made for the short periods with the results given in the following table:

TABLE III.

Amplitude of given Periods Males.

Period	X_p	Y_p	Z_p	U_p	V_p
2·0 years	0·8466	0·6397	0·3212	0·3505	0·3856
2·5 years	1·9095	0·6510	0·1520	0·1705	0·0808
3·0 years	2·9796	0·6485	0·5107	0·4095	0·4232
4·0 years	1·0488	1·2734	0·7089	0·4268	0·4061
5·0 years	3·3139	0·7524	0·7749	0·8328	0·8202
7·0 years	2·0356	1·1308	—		–

* *Biometrika*, Vol. x. pp. 491—492.

† *Tracts for Computers*, No. vi. p. 30.

To test whether these results are fairly steady, the results for X_p and Y_p in the case of female babies and children in the second year of life were worked out with the following results:

TABLE IV.

Amplitude of given Periods. Females

Period	X_p	Y_p
2·0 years	0·2476	0·6387
2·5 years	1·5484	0·4862
3·0 years	2·8312	1·0137
4·0 years	0·6301	1·1310
5·0 years	2·7242	0·9087
7·0 years	1·8783	0·8579

These results are in fair accordance with those of Table III, indicating periods of 3 and 5 years for X_p and 4 years for Y_p.

Now $\frac{1}{2}$(amplitude)^2 will be the contribution of any period to the total standard deviation squared of X_p, Y_p, Z_p, U_p or V_p respectively. These contributions are represented in Table V.

TABLE V.

Contributions to the Total Standard Deviation Squared of the Terms of each Period.

	Males					Females	
	X_p	Y_p	Z_p	U_p	V_p	X_p	Y_p
Total Standard Deviation Squared	40·6622	10·7394	3·8437	3·1898	1·8450	36·0721	10·4871
Periods							
2·0 years	·3584	·2046	·0516	·0614	·0743	·0307	·2040
2·5 years	1·8231	·2119	·0116	·0145	·0033	1·1988	·1182
3·0 years	4·4390	·2103	·1304	·0838	·0895	4·0078	·5138
4·0 years	·5500	·8108	·2513	·0911	·0825	·1985	·6396
5·0 years	5·4910	·2831	·3002	·3468	·3264	3·7106	·4129
7·0 years	2·0718	·6394	—	—	—	1·8018	·3680

In X_p for both male and female the three-year and the five-year periods are the least insignificant but they—if they existed—would not contribute more than $\frac{1}{9}$ and $\frac{1}{7}$ to the total standard deviation squared. Further both these are insignificant periodic terms in male and female Y_p. They cannot therefore possibly be the source of correlation between X_p and Y_p. We can test at once by aid of the table on p. 292 what the relative importance of the terms would be for the three-year period on differencing:

8th difference: $4{\cdot}4390 \times \frac{177147}{27}$ as against $40{\cdot}6622 \times 12870$,

i.e. as 29 to 523

as 1 to 18.

10th difference: $4{\cdot}4390 \times \frac{1594323}{27}$ as against $40{\cdot}6622 \times 184756$,

i.e. as 262 to 7512

as 1 to 29.

In any case there seems no possibility of a three-year period contributing anything of importance to the result or of its swamping the non-periodic but correlated variations The five-year period would produce even less effect.

It seems to us that Mr Yule should have investigated what real effect his two- or three-year periods would have produced on the result before publishing his criticism. He might further have investigated whether there were any *common* periods in X_p and Y_p, for without such his criticism also falls to the ground. As a matter of fact there are no common periods of any significance at all in X_p and Y_p for either males or females.

We now turn to the correlations as found from the residuals after smoothing, we have for males:

TABLE VI. *Values of the Correlations.*

ρ_1'	= correlation of	X_p and	X_{p+1}	$= -{\cdot}3792 \pm {\cdot}0817$
ρ_2'	= ,,	X_p ,,	X_{p+2}	$= +{\cdot}1071 \pm {\cdot}0943$
ρ_3'	= ,,	X_p ,,	X_{p+3}	$= -{\cdot}1851 \pm {\cdot}0921$
ρ_1''	= ,,	Y_p ,,	Y_{p+1}	$= -{\cdot}3352 \pm {\cdot}0847$
ρ_2''	= ,,	Y_p ,,	Y_{p+2}	$= +{\cdot}1936 \pm {\cdot}0919$
ρ_3''	= ,,	Y_p ,,	Y_{p+3}	$= -{\cdot}3290 \pm {\cdot}0851$
ρ_1'''	= ,,	Z_p ,,	Z_{p+1}	$= +{\cdot}0548 \pm {\cdot}0951$
ρ_2'''	= ,,	Z_p ,,	Z_{p+2}	$= -{\cdot}2211 \pm {\cdot}0907$
ρ_3'''	= ,,	Z_p ,,	Z_{p+3}	$= -{\cdot}2338 \pm {\cdot}0902$
ρ_1^{iv}	= ,,	U_p ,,	U_{p+1}	$= +{\cdot}2799 \pm {\cdot}0879$
ρ_2^{iv}	= ,,	U_p ,,	U_{p+2}	$= -{\cdot}1565 \pm {\cdot}0932$
ρ_3^{iv}	= ,,	U_p ,,	U_{p+3}	$= -{\cdot}3530 \pm {\cdot}0837$
ρ_1^{v}	= ,,	V_p ,,	V_{p+1}	$= +{\cdot}3168 \pm {\cdot}0858$
ρ_2^{v}	= ,,	V_p ,,	V_{p+2}	$= -{\cdot}3644 \pm {\cdot}0827$
ρ_3^{v}	= ,,	V_p ,,	V_{p+3}	$= -{\cdot}5176 \pm {\cdot}0698.$

Now these correlations show that the association of X_p and X_{p+1}, Y_p and Y_{p+1}, Y_p and Y_{p+3}, U_p and U_{p+1}, U_p and U_{p+3}, V_p and V_{p+1}, V_p and V_{p+2}, V_p and V_{p+3} cannot be taken as zero. Thus the criticism of Dr Persons is justified in this case. "Student's" simple hypothesis is not permissible for this material. Now while these correlations as far as X and Y are concerned show an alternating sign, which might be due to a periodic term, the series start with a negative term for death-rates in the first and second year of life, but the correlation becomes positive in the

third, fourth and fifth years of life, being followed by negative correlations; further the magnitudes of these correlations increase for the fourth and fifth years of life. In other words: if there be a two- to three-year period in the deathrates of the first two years of life, there is no correspondingly important period in the third, fourth and fifth years of life.

Supposing only a single substantial period to exist we can determine it from the above correlations. We have to find τ from

$$\rho_s = \cos \frac{2\pi sh}{\tau}.$$

This leads to the following results, the period being in years:

TABLE VII.

Values of Hypothetical Dominant Periods as found from each Deathrate Correlation.

	X	Y	Z	U	V
ρ_1	3·206	3·285	4·161	4·882	5·037
ρ_2	2·608	2·562	6·953	7·272	6·466
ρ_3	10·728	9·887	10·433	9·759	8·913

The results for the period as found from ρ_1, ρ_2, ρ_3 are so discordant that it is very difficult to believe that any one of the deathrates has a single dominant periodicity. The periods roughly suggested for Z, U, V are above the values which would in the least affect our difference results. If there were a two-year to three-year dominant period in X or Y, it is hard to suppose that ρ_3 in both cases could give so discordant a result, and there is no greater reason for rejecting the ρ_3 value than the ρ_1 or ρ_2. Supposing, however, we do reject it, then if we weight the periods as derived from ρ_1 and ρ_2 according to the probable errors of those correlations we find

$$\text{Period of } X = 2{\cdot}950 \text{ yrs.},$$
$$\text{,,} \quad \text{,,} \quad Y = 2{\cdot}953 \text{ yrs.},$$

results in agreement with each other and with the period of about three years which we found from direct periodogram analysis.

We see, however, no reason for excluding ρ_3 from such a computation; it has no relatively large probable error, but no result of any value can be obtained by its inclusion. We have already seen that a period of about three years in X and Y has no amplitude in the least accounting for the observed variations from the secular trend. Further without a lag it would lead to a high *positive* correlation between X_p and Y_p while that correlation is substantial and *negative*. If there be a lag, equation (xliii) shows us that it must be almost a third of a period, or one year, as we might naturally anticipate on the theory of natural selection. But Table IV and the non-accordance of ρ_3 suffice to indicate that the three-year period is of no importance in either X_p or Y_p. Thus Mr Yule's criticism appears to fail when we examine it directly by periodic analysis and indirectly by correlations.

But there are certain general results which may be drawn from these correlations:

(*a*) The geometrical dying out of the correlations is not an hypothesis suitable for application to the present data. Not only do the signs fail in Z, U, V of proper alternation but the magnitudes do not indicate any tendency to die out.

(*b*) In fact the correlations between the mortalities of individuals born in the same year of life at *three* years apart are all negative, and within the limits of their probable errors rise continuously with age. Thus if there be a heavy mortality in children born in year p in their 1st, 2nd, 3rd, 4th and 5th years of life, children born in year $p+3$ will have a light mortality in their 1st, 2nd, 3rd, 4th and 5th years of life. It is difficult to give a reason for such a relationship, especially its increasing intensity with the increasing age of the child

(*c*) The same relationship holds, in a less marked degree, for children of 3, 4 and 5 years of age born *two* years apart, but it is reversed in the case of children of 1 and 2 years of age, if we can in these cases give any weight to the correlations relatively to their probable errors.

(*d*) Lastly, if we consider children born in successive years (p and $p+1$), then their mortalities in their 1st and 2nd years of life are negatively correlated, and in their 3rd, 4th and 5th years positively correlated. As a matter of fact the correlations of mortalities at the same age of children born in successive years form an algebraic series continually rising $-\cdot 3792$, $-\cdot 3352$, $+\cdot 0548$, $+\cdot 2799$, $+\cdot 3168$. Of course none of the correlations (V_p and V_{p+3} excepted) between mortalities in the same year of life of children born in different years reach any considerable magnitude, but having regard to their probable errors, they cannot be straightway neglected.

Further they neither fit into geometrically dying out series, nor into anything of the nature of dominant harmonic periods.

A fine piece of work might be done in endeavouring to disentangle these correlations by differentiating the mortality from different causes. It would be very strenuous, but of great service. If it were done, we believe it would be best to try and obtain the crude figures for deaths and populations at risk, and not use any redistributed figures of populations in the 1st, 2nd, 3rd, 4th and 5th years of life.

We next proceed to consider the cross-correlations in order to determine whether we were correct in supposing X_p and Y_p negatively correlated. Still using the smooth by Rhodes' method we find:

TABLE VIII. *Correlations of First and Second Year Mortalities.*

Mortalities Correlated	Boys	Girls
$X_p\ Y_{p-2}$	$-\cdot 2077 \pm \cdot 0913$	$-\cdot 2730 \pm \cdot 0883$
$X_p\ Y_{p-1}$	$+\cdot 6521 \pm \cdot 0548$	$+\cdot 5935 \pm \cdot 0618$
$X_p\ Y_p$	$-\cdot 4584 \pm \cdot 0753$	$-\cdot 4904 \pm \cdot 0725$
$X_p\ Y_{p+1}$	$+\cdot 0284 \pm \cdot 0953$	$+\cdot 0033 \pm \cdot 0954$
$X_p\ Y_{p+2}$	$-\cdot 2619 \pm \cdot 0888$	$-\cdot 2838 \pm \cdot 0877$

Now those results are exceedingly suggestive and important, and at the same time some are very difficult to interpret. As to the values, those for girls are wholly in keeping with those for boys, the differences being everywhere within the probable errors of the differences. Let us remember that p denotes the year of birth, and the X denotes the excess mortality above the secular trend in the first year of life and Y the excess mortality above the secular trend in the second year of life. Now our results show that the numbers of those born in the same year who die in the first and the second year of their lives are *negatively* correlated, and that this correlation is substantial, somewhat under $-\cdot 5$. Thus the conclusion reached in the former paper by the present writers is confirmed by an entirely different form of investigation, but the values now obtained are 33 °/ₒ less than those obtained by the variate-difference method. This reduction undoubtedly flows from the neglect of correlations such as those of $X_p X_{p+s}$, $X_p Y_{p+s}$, $Y_p Y_{p+s}$ on "Student's" hypotheses. That neglect was not in our case justifiable, for it gave an exaggerated value to the influence of natural selection. That influence however still remains substantial.

Turning to the correlation of X_p and Y_{p-1}, i.e. the mortalities of children born in successive years, dying in the same year, we see that it is high and positive; it is probably due to diseases peculiar to certain years and attacking not only infants, but also children in the second year of life. Our direct value for the correlation of X_p and Y_p shows that the high positive correlation of X_p and Y_{p-1} is *not* the source of the negative correlation between X_p and Y_p as was suggested by one of our critics.

The correlation $X_p Y_{p+1}$ of deaths of infants born in year p with those of two-year-olds born in the following year is zero sensibly, as we should naturally anticipate. But we should also have anticipated that there would be no correlation $(X_p Y_{p+2})$ between the deaths of infants born in year p and that of two-year-olds born two years later and therefore dying three years later! The correlations are not very large, but they appear in both sexes, and it is difficult to consider them zero having regard to their probable error Such correlation might be accounted for by a periodic term common to X and Y with a suitable difference of phase, but to judge by Table VI there seems no common period of sufficient importance to justify the suggestion.

Lastly the first correlation of Table VIII is again one difficult of interpretation. The correlation of X_p and Y_{p-2} might possibly be considered zero for boys, but the value for girls suggests that it is really significant. Why should the dying of children in their second year in one year, be related to those dying in their first year in the following year? Had the correlation been *positive*, we might have considered epidemics lasting from one year into a second, but the correlation is *negative.*

The only explanation we have been able to think of will we fear be thought by many inadequate. It is that Y_{p-2}, X_p and Y_{p+2} are children born two years apart, and the *average* interval between children in the same family is somewhat over two years. Accordingly the groups mentioned above correspond to a considerable extent

to children of the *same* families. It is possible that when the previously born child has died greater care is taken of the next infant, and further that when a mother has a child alive between 1 and 2, her attention is not so wholly devoted to the new-born child and it may be less likely to survive. It may well be that the falling birthrate is really organically related to a falling infant deathrate, because it connotes fewer extremely young children to tend simultaneously*.

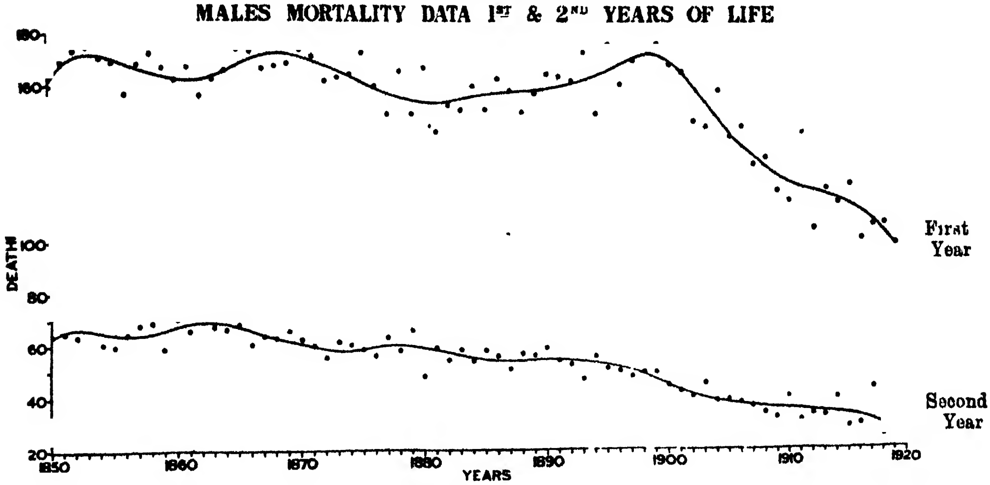


Diagram I.

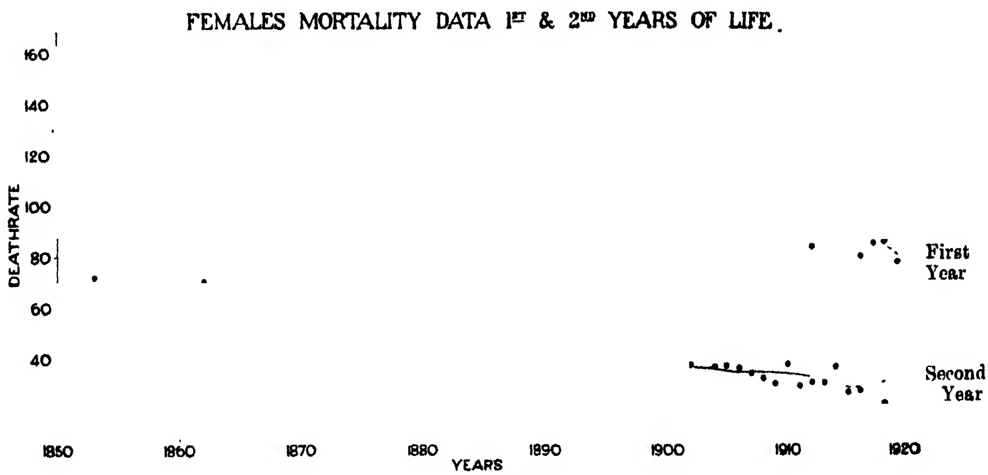


Diagram II.

We shall now show that our values for the correlation of X_p and Y_p are not peculiar to the particular method of smoothing adopted. The smooths of X_p and Y_p are shown in Diagrams I and II for boys and girls respectively. It will be seen

* We are well aware that the environmentalists attribute it largely to the spread of infant welcome and mother welfare centres, but we doubt if they have investigated the correlation between infant deathrate in towns with and without these centres.

on careful examination of these diagrams that the general rule is ***reversal of*** sign in X_p and Y_p. In Diagram I in 69 years there are 43 reversals of sign, in Diagram II there are 41 reversals, or roughly $\frac{3}{5}$ of the total. Thus graphical inspection shows a very considerable degree of negative correlation between deaths in the first and second years of life of the same group of individuals. But would another system of smoothing give the same general trend and an equal correlation? This was tested by an entirely different process of smoothing, namely that of Sheppard.

The result was*:

$$\begin{array}{lll} \text{Sheppard's process:} & r_{X_p Y_p} = -\cdot 4629 & \text{(Boys)} \\ \text{Rhodes' process:} & r_{X_p Y_p} = -\cdot 4584 & \text{(Boys)} \\ \quad " \qquad " & r_{X_p Y_p} = -\cdot 4904 & \text{(Girls)} \end{array}$$

We think it safe to say that there really does exist a substantial *negative* correlation between deaths of the same group in the first and second years of life. It is not as great as we found it in the previous paper using hypotheses, which, we admit, ought to have been tested; but it is quite adequate to indicate that natural selection is really at work.

The value of the $X_p Y_p$ correlations was exaggerated by adopting "Student's" hypotheses; the negative sign, however, was not, as suggested by Mr R. A. Fisher (*Journal of R. Statistical Society*, Vol. XXXIV. p. 534), due to the neglect of the correlations of children of different ages dying in the same year. Those correlations are very considerable and undoubtedly aid in modifying the value of $r_{X_p Y_p}$ as found by "Student's" method. Thus we have

$$\begin{array}{lll} r_{X_p Y_{p-1}}: & \text{for boys } \cdot 6521 & \text{and for girls } \cdot 5935 \\ r_{Y_p Z_{p-1}}: & \quad " \quad \cdot 6851 & \\ r_{Z_p U_{p-1}}: & \quad " \quad \cdot 9050 & \\ r_{U_p V_{p-1}}: & \quad " \quad \cdot 7481 & \end{array}$$

from which it appears that the liability of children to the same seasonal diseases is greatest in the 3rd and 4th years of life.

The increased negative value to the numerator of the expression (xxiv) due to the introduction of further negative terms (including of course $r_{X_p Y_{p-1}}$) is, however, corrected by increased *positive* terms in the expressions for $[(\Delta^n X_p)^2]$ and $[(\Delta^n Y_p)^2]$ of the denominator.

Assuming the difference order n to be 8, we calculated from the X_p, X_{p+s}, Y_p, Y_{p+s} and X_p, Y_{p+s} correlations, as far as we had worked them out, the values of

$$\phi(n, \rho_0, \rho), \text{ and } \phi(n, \rho'), \phi(n, \rho''),$$

and found, using seven terms in each of the latter and five in the former, that:

$$r_{\delta_8 x_p \delta_8 y_p} = -\cdot 68,$$

* Fitting single sixth order parabolas to X_p and Y_p was tried for 67 years, but they gave very inadequate smooths; it wants more than a single sixth order parabola to adequately describe the trends.

whereas "Student's" process gave the value $-\cdot 69$; the $-\cdot 01$ difference could very easily be accounted for by the small number of X_p and Y_{p+s} correlations that we were able to use.

This correspondence in value gives us some confidence in the proposal to deduce the $r_{X_p Y_{p+s}}$ and $r_{X_p X_{p+s}}$, $r_{Y_p Y_{p+s}}$ correlations—when we are certain that the correlations die out rapidly with s—from the extended Variate Difference Equations given in this paper. If this method is to give satisfactory results, however, we think it will be desirable to work with the same order of differences, and reach enough equations by correlating X_p with Y_{p+s}, X_p with X_{p+s} and Y_p with Y_{p+s}, i.e. not corresponding differences. We reserve, however, this matter for a fuller consideration later, probably on other material. It may turn out that the process is more lengthy than subtracting the trend by an adequate smooth, which at least enables us to correlate only the values we need.

We have published Mr Henderson's Table for geometrical decadence functions, because although it is of no service for these mortality data, we think it may well be so for other material. The law of geometrical decadence seems a reasonable one for many variates. We should like to try it on the growth data, say at weekly or monthly intervals for a series of the same years of age, in the case of brothers, to ascertain whether "spurts" and "checks" to growth are similar in two members of the same family. And we should welcome data of this kind that any reader of this paper may have in his power to lend us.

Our general conclusions so far must be

(i) that Mr Yule's criticisms as to periodic terms are not valid and that this could have been seen had the material been analysed at greater length;

(ii) that Dr Persons' criticism is valid, although his attempts to get rid of secular trends are from our standpoint wholly inadequate, and further his criticism was hypothetical. He suggested, but he did not show, that X_p and X_{p+s} or X_p and Y_{p+s} were in any of our cases really correlated.

That "Student's" hypotheses did not truly apply to our mortality data has, however, been shown in this paper. We do not think this signifies that the Variate Difference Method, either in its old, or in a more generalised form, will be of no service. It denotes only that it was not suited to the data to which we applied it. Because a non-justifiable method was adopted it does not follow that the conclusion we reached was erroneous. The present paper, we think, shows that mortality in the first year of life is negatively correlated with that in the second; the correlation is substantial but only about 70 °/$_\circ$ of what we deduced for it.

APPENDIX.

Table of Functions for Geometrical Decadence (J. HENDERSON).

$n=$	0	1	2	3	4	5	6	7	8	9	10
$\epsilon=-1{\cdot}0$	1·0	2·0	2·666667	3·200000	3·657144	4·063492	4·432900	4·773892	5·092152	5·391690	5·675460
	4·0	**4·0**	**4·000000**	**4·000000**	**4·000000**	**4·000000**	**4·000000**	**4·000000**	**4·000000**	**4·000000**	**4·000000**
−0·9	1·0	1·9	2·470000	2·908900	3·273375	3·588544	2·868073	4·120351	4·350953	4·563806	4·761794
	3·8	**3·9**	**3·925641**	**3·938538**	**3·946617**	**3·952281**	**3·956534**	**3·959875**	**3·962591**	**3·964852**	**3·966778**
−0·8	1·0	1·8	2·280000	2·635200	2·920732	3·160823	3·368545	3·551856	3·716004	3·864657	4·000490
	3·6	**3·8**	**3·852632**	**3·879236**	**3·895928**	**3·907632**	**3·916412**	**3·923305**	**3·928902**	**3·933560**	**3·937514**
−0·7	1·0	1·7	2·096667	2·378300	2·597260	2·776223	2·927253	3·057601	3·171987	3·273671	3·365004
	3·4	**3·7**	**3·781080**	**3·822230**	**3·848056**	**3·866138**	**3·879680**	**3·890289**	**3·898882**	**3·906017**	**3·912058**
−0·6	1·0	1·6	1·920000	2·137600	2·301074	2·430903	2·537778	2·628011	2·705649	2·773438	2·833334
	3·2	**3·6**	**3·711111**	**3·767664**	**3·803116**	**3·827873**	**3·846351**	**3·860784**	**3·872429**	**3·882066**	**3·890197**
−0·5	1·0	1·5	1·750000	1·912500	2·030357	2·121280	2·194298	2·254630	2·305556	2·349265	2·387288
	3·0	**3·5**	**3·642857**	**3·715686**	**3·761214**	**3·792880**	**3·816410**	**3·834702**	**3·849398**	**3·861503**	**3·871673**
−0·4	1·0	1·4	1·586667	1·702400	1·783360	1·844018	1·891551	1·930008	1·961877	1·988787	2·011857
	2·8	**3·4**	**3·576471**	**3·666447**	**3·722448**	**3·761182**	**3·789801**	**3·811921**	**3·829596**	**3·844080**	**3·856190**
−0·3	1·0	1·3	1·430000	1·506700	1·558403	1·596019	1·624793	1·647603	1·666177	1·681623	1·694687
	2·6	**3·3**	**3·512121**	**3·620104**	**3·686895**	**3·732772**	**3·766430**	**3·792275**	**3·812799**	**3·829521**	**3·843425**
−0·2	1·0	1·2	1·280000	1·324800	1·353874	1·374415	1·389761	1·401690	1·411245	1·419079	1·425624
	2·4	**3·2**	**3·450000**	**3·576811**	**3·654619**	**3·707607**	**3·746168**	**3·775563**	**3·798749**	**3·817526**	**3·833051**
−0·1	1·0	1·1	1·136667	1·156100	1·168231	1·176556	1·182634	1·187272	1·190930	1·193891	1·196337
	2·2	**3·1**	**3·390322**	**3·536726**	**3·625654**	**3·685609**	**3·728852**	**3·761554**	**3·787171**	**3·807785**	**3·824744**
0·0	1·0	1·0	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
	2·0	**3·0**	**3·333333**	**3·500000**	**3·600000**	**3·666667**	**3·714286**	**3·750000**	**3·777778**	**3·800000**	**3·818182**
0·1	1·0	0·9	·870000	·855900	·847774	·842508	·838824	·836104	·834015	·832361	·831019
	1·8	**2·9**	**3·279310**	**3·466771**	**3·577638**	**3·650634**	**3·702242**	**3·740631**	**3·770286**	**3·793873**	**3·813080**
0·2	1·0	0·8	·740667	·723200	·710217	·702029	·696411	·692323	·689218	·686780	·684817
	1·6	**2·8**	**3·228571**	**3·437167**	**3·558496**	**3·637324**	**3·692483**	**3·733182**	**3·764415**	**3·789139**	**3·809176**
0·3	1·0	0·7	·630000	·601300	·586060	·576695	·570381	·565845	·562432	·559772	·557642
	1·4	**2·7**	**3·181481**	**3·411292**	**3·542473**	**3·626522**	**3·684748**	**3·727381**	**3·759911**	**3·785541**	**3·806240**
0·4	1·0	0·6	·520000	·489600	·474103	·464808	·458638	·454253	·450979	·448443	·446421
	1·2	**2·6**	**3·138461**	**3·389217**	**3·529420**	**3·617995**	**3·678774**	**3·722972**	**3·756534**	**3·782866**	**3·804076**
0·5	1·0	0·5	·416667	·387500	·373214	·364832	·359341	·355473	·352603	·350391	·348633
	1·0	**2·5**	**3·100000**	**3·370965**	**3·519148**	**3·611481**	**3·674305**	**3·719723**	**3·754079**	**3·780934**	**3·802521**
0·6	1·0	0·4	·320000	·294400	·282331	·275384	·270881	·267731	·265405	·263618	·262202
	0·8	**2·4**	**3·066667**	**3·356517**	**3·511419**	**3·606711**	**3·671094**	**3·717421**	**3·752342**	**3·779589**	**3·801440**
0·7	1·0	0·3	·230000	·209700	·200460	·195224	·191856	·189512	·187788	·186466	·185420
	0·6	**2·3**	**3·039130**	**3·345780**	**3·505968**	**3·603410**	**3·668906**	**3·715886**	**3·751183**	**3·778684**	**3·800726**
0·8	1·0	0·2	·146667	·132800	·126074	·123243	·121047	·119524	·118406	1·117550	·116874
	0·4	**2·2**	**3·018181**	**3·338547**	**3·502493**	**3·601332**	**3·667553**	**3·714923**	**3·750467**	**3·778147**	**3·800290**
0·9	1·0	0·1	·070000	·063100	·060117	·058459	·057399	·056667	·056128	·055716	·055393
	0·2	**2·1**	**3·004762**	**3·334540**	**3·500553**	**3·600320**	**3·666918**	**3·714351**	**3·750048**	**3·777867**	**3·800053**
1·0	1·0	0·0	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000
	0·0	**[2·0]**	**[3·000000]**	**[3·333333]**	**[3·500000]**	**[3·600000]**	**[3·666667]**	**[3·714286]**	**[3·750000]**	**[3·777778]**	**[3·800000]**

The above table gives $2\phi(n, \epsilon)-1$ in ordinary type and $\left(4-\frac{2}{n+1}\right)\left\{\frac{2\phi(n+1, \epsilon)-1}{2\phi(n, \epsilon)-1}\right\}$ in heavy type for values of n from 0 to 10 and of ϵ from $-1{\cdot}0$ to $+1{\cdot}0$ at *intervals* of 0·1.

$$\left[\phi(n, \epsilon)=1-\frac{n}{n+1}\epsilon+\frac{n(n-1)}{(n+1)(n+2)}\epsilon^2-\dots(-1)^n\frac{n!}{(n+1)(n+2)\dots(2n)}\epsilon^n.\right]$$

FACIAL SPASM INHERITED THROUGH FOUR GENERATIONS.

By PERCY STOCKS, M.D.

A CASE of facial spasm which was of considerable interest from the hereditary standpoint recently came to the notice of the Anthropometric Laboratory, University College, London, and is here recorded. As there has been much confusion in the classification and nomenclature of affections of this type, a few preliminary notes on the modern views regarding facial spasm are necessary. The confusion which has in the past existed between so-called "tics" and "spasms," both of which are evidenced by twitching movements of the face muscles, has been chiefly responsible for the large number of names used to describe these affections, such as "facial tic," "spasmodic tic," "convulsive tic," "habit spasm," "mimic spasm," "tic non-douloureux," and others. Confusion with choreic movements has also led to such names as "habit chorea."

The distinction between "tic" and "spasm" has been made clear in recent years by the researches of Charcot and other French workers, leading up to the excellent work of Meige and Feindel (2). *Spasm* is now defined by modern writers as a motor reaction which results from irritation at some point in a reflex spinal or bulbo-spinal arc, and neither depends on consciousness nor is under the influence of the will. *Tic*, on the other hand, is essentially a psychical process in which a co-ordinated muscular movement is repeated intermittently for no useful purpose, and in which consciousness is essential and the exercise of the will plays an important part.

Facial tic (also called habit spasm, convulsive tic, &c.) is therefore characterised by frequent or intermittent clonic contractions of a co-ordinated group of facial muscles (not strictly limited to those supplied by any one nerve), occurring commonly in children or young persons with some neuropathic tendency or heredity. It is never entirely beyond the control of the will, never occurs during sleep, and each paroxysm is usually preceded by an impulse, and followed by a certain feeling of satisfaction. These tics are of great variety, but they need not be enumerated here. Heredity is not infrequently manifest in this affection, but owing to its very nature, it is difficult to eliminate the element of imitation, which may be the starting point of the habit in a child of unstable nervous temperament, whose parent or brother or sister is the subject of a tic. Several cases of apparent direct heredity have been recorded; thus Sir W. Gowers (1) mentions the case of a father and two children with identical tics; and Piedagnel (2) a case where mother and daughter had similar tics. Tissié (2) describes ocular tic occurring in mother and two sons, and Letulle (2) a similar tic in father and

two sons. Strauch (3) mentions two cases; in one a boy, his mother, two maternal aunts and one maternal uncle all had tics in their youth; in the other a girl, both her parents and one uncle were affected. Other cases, where two or more children in a family developed identical tics, have been mentioned by Gintrac (2) (two brothers), Blacho (4) (3 children), Delasiauve (2) (brother and sister), Meige and Feindel (2) (two sisters) and Flatau (5) (two sisters). Similar tics in two brothers are also recorded by Rudler and Chomel (11). Whilst direct heredity is not very often traced, a neuropathic heredity is frequent; thus it is common for example to find that the mother of a ticqueur is hysterical, a brother epileptic and a grandparent insane. Flatau (5) describes two such cases; in one the mother and her sister had tics and the son was mentally abnormal; in the other the mother of a ticqueur had impulsive insanity. It is also common to find mental instability in one parent combined with intellectual brilliance in the other.

Facial Spasm is characterised by spasms, usually clonic but occasionally tonic, occurring in muscles supplied by the facial nerve. These are most often, but not always, unilateral, and may affect all the muscles or only those supplied by one or more branches of the nerve. The affection may be secondary to paralysis; or result from direct irritation of the seventh (facial) nerve; or from reflex irritation of the fifth (trigeminal) nerve by a carious tooth, injury, ocular defect, ear disease or tumour; or from injury to the facial area in the cortex; or from a tumour pressing on the pons. In many cases (labelled "idiopathic") no definite cause can be assigned, but shock, nervous emotion, exposure to cold, prolonged migraine or neuralgia, pregnancy, and the climacteric have all been mentioned as precursors, whilst a few have been attributed to heredity. The idiopathic form is more frequent in females than males and usually occurs between 45 and 60 years of age, and rarely before 30.

Pathological evidence shows that the facial nucleus in the pons is the starting point of the motor discharges resulting in spasm, and that these are a pure reflex in which the cortex takes no part (6).

Paroxysms may occur at intervals of several hours, or recur every minute or two; they generally consist of clonic muscular contractions which begin more or less slowly and become quicker and more pronounced until they may be so rapid as to have a quivering character; the paroxysm then slows down and finally ceases. Emotional excitement, cold, or a bright light often make the spasms more pronounced. In the early stages a single muscle, or part of a muscle (7), is usually involved, but the spasm usually tends to become more generalised.

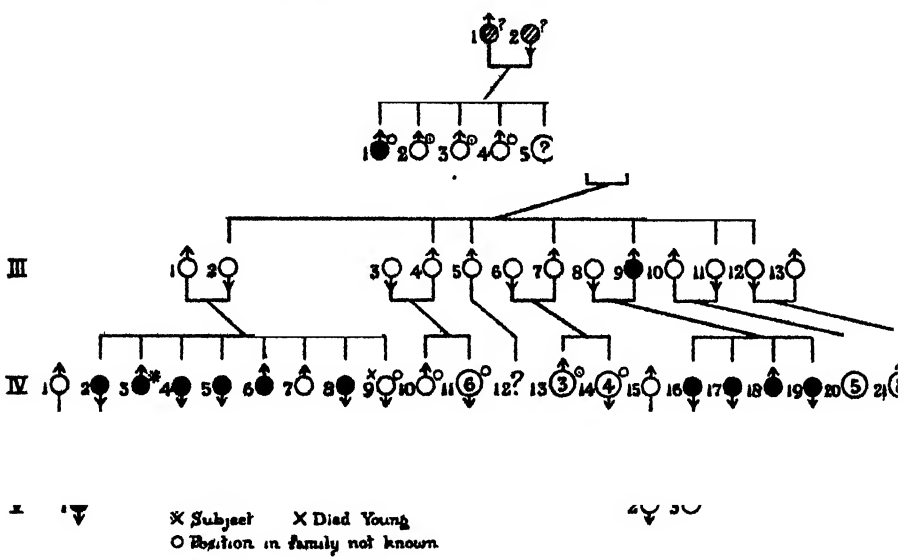
Gowers (1) states that direct heredity is very rare, but mentions a case where mother and daughter were both affected in late life. Rosenthal (8) speaks of a family where mother, son, daughter and two maternal relatives were affected by more or less extensive facial spasm, but this is not described in detail. Mayer (9) describes the case of a man with infra-orbital neuralgia giving place after the lapse of five years to spasm of the left eyelid and finally of the whole side of the face; his mother had been similarly affected, but in her case the spasm

preceded the neuralgia. Probably the last were really cases of tic douloureux or painful facial spasm.

The hereditary cases described by Piedagnel, Blache, Delasiauve and Gintrac, which are sometimes alluded to under the heading of facial spasm, seem to have been most probably true cases of tic, and have been referred to above as such. Gowers states that an inherited neurotic tendency to insanity or epilepsy can sometimes be traced.

The case we shall now describe is unusual in several respects, and appears to demonstrate the hereditary transmission of the affection more completely than any case yet described.

Pedigree of Familial Facial Spasm.



The subject who came under our notice was a male of 18 years of age, of Polish parentage on both sides, and the second child in a family of 8. His general health had been excellent; he weighed 87 kilograms, and his stature was 178 cm. His respiratory measurements were high and his circulatory system showed nothing abnormal. In a series of mental speed tests he was uniformly brilliant and in tests of muscular power was also in the first grade, whilst both visual and auditory acuity were much above normal. In muscular precision and discrimination of weights he made average scores, but was distinctly subnormal in tests of sensory discrimination, and failed completely in a balancing test designed to measure steadiness of hand and arm. These results confirmed an impression of a good physique and unusually quick intellect combined with a somewhat restless and highly strung temperament.

Throughout the examination there was noticeable a very rapid clonic spasm localised to the levator menti muscle between the chin and lower lip. This spasm

was often so rapid as to become a quivering movement or tremor, and ceased for short intervals, becoming more pronounced during the performance of psychological tests requiring sustained effort. It was never observed to spread to any other muscle, and according to his own account no other parts of the face or body were ever affected. The whole muscle was apparently involved and there was no sign of any predominant lateral distribution of the spasm.

On inquiring as to heredity, he asserted definitely that an identical affection was present in one of his brothers and four sisters, in a daughter of one of those sisters, in a maternal uncle and four children of the latter (one male and three female), and in a brother of his maternal grandmother. His parents and grandparents were themselves free. From further details the accompanying pedigree was constructed, from which it will be noticed that:

(i) The affection was manifested in 13 individuals in 4 generations. About his great grandparents the subject could give no information.

(ii) Of these 13, 8 were females and 5 males, a proportion agreeing closely with Gowers' statement that the disease occurs predominantly in females in the proportion of about two to one.

(iii) Transmission occurs both through affected males and females, and there is an apparent tendency for its prevalence to increase with succeeding generations; at any rate the proportions of total children affected in generations II to V were successively $\frac{1}{8}$, $\frac{1}{7}$, $\frac{1}{4}$, $\frac{1}{3}$. Where an affected male transmitted the disease, all his children were affected.

(iv) Transmission through an unaffected female occurs twice in succession, an unaffected mother and her daughter carrying the disease to six grandchildren. This of course occurs in many hereditary anomalies, but in the case of nervous affections has rarely been demonstrated. Thus Huntingdon's Chorea is stated by Huntingdon never to skip a generation (10), and I am not aware that it has been demonstrated in Paramyoclonus multiplex or other hereditary nervous affections.

The occurrence of insanity, epilepsy or any other neuropathic tendency in this family was denied. Another unusual feature is the appearance of the affection at an early age in some members of the family. A few instances have, however, been met with where the onset was noticed as early as 20. This case presented none of the features characteristic of a tic or habit spasm, and there appears little doubt as to the diagnosis. Gowers states that where the spasm is localised to one muscle, the orbicularis palpebrarum or zygomatici are most commonly affected, but that the depressor anguli oris or levator menti are sometimes implicated. According to the information supplied, the affection remained entirely confined to the levator menti region in all the affected members of this family.

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A BIOMETRIC STUDY OF THE INTER-RELATIONS OF "VITAL CAPACITY," STATURE, STEM LENGTH AND WEIGHT IN A SAMPLE OF HEALTHY MALE ADULTS.

By LUCY D. CRIPPS, MAJOR GREENWOOD AND ETHEL M. NEWBOLD.

[*Being a paper read before the Society of Biometricians and Mathematical Statisticians, October* 23, 1922.]

INTRODUCTION.

In any study of physical fitness or efficiency the first difficulty to be encountered is that of the definition of "Fitness." A more or less satisfactory definition having been framed, one seeks for a variable magnitude highly correlated with the complex defined as "fitness," so highly that it may itself be deemed a measure of "fitness." This point having been reached, we have to ascertain whether "fitness" so measured is correlated with other measurable physical or mental attributes to such an extent that, the latter measurements of an individual being assigned, his or her "fitness" can be estimated within reasonably narrow limits of accuracy. Eighty years ago, Dr John Hutchinson* concluded that "vital capacity," that is, the volume of the maximum expiration following a maximum inspiration, was a valuable measure of "fitness" and that its variations were highly correlated with certain bodily dimensions. More recent investigators have endorsed Hutchinson's contentions in principle, although they have modified the form of the relation between "vital capacity" and other variables, and made a different choice of the latter.

This memoir is wholly concerned with the second part of Hutchinson's and his successors' problem. We shall not discuss the question whether "vital capacity" be or be not an adequate measure of "fitness," but confine ourselves exclusively to the simple biometric question—with what success can "vital capacity" be expressed as a function of the variables, stature, stem length (or sitting height), and body weight when either linear equations or simple exponential functions of the independent variables are employed.

PREVIOUS INVESTIGATIONS.

We shall not attempt to give a complete historical study of the work stimulated by the publication of Hutchinson's results, but an abstract of the original work and a short reference to subsequent investigations will probably interest the reader.

* "On the Capacity of the Lungs and on the Respiratory Functions with a view of establishing a precise and easy method of detecting Disease by the Spirometer." *Transactions of the Royal Medico-Chirurgical Society of London*, Vol. xxix.

TABLE I.

Table of Mean Vital Capacity of 15 *Different Classes considered as Healthy.* (HUTCHINSON)

	ft. in. ft. in. 0 0—5 0		ft. in. ft. in. 5 0—5 1		ft. in. ft. in. 5 1—5 2		ft. in. ft. in. 5 2—5 3		ft. in. ft. in. 5 3—5 4		ft. in. ft. in. 5 4—5 5		ft. in. ft. in. 5 5—5 6	
	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases
nen	2474	5	3376	1	3146	7	3589	1	3572	10	3490	9	3556	15
Brigade ...	—	—	—	—	—	—	—	—	3441	1	3408	2	3572	20
ce (Met.) ..	—	—	-	—	—	—	—	—	—	—	—	—	—	—
ce (Thames)	2589	1	—	—	—	—	—	—	3064	6	3376	9	3730	9
pers	2474	7	2720	3	2655	10	2950	10	2851	21	3130	20	3097	19
d Classes ...	1311	1	3032	1	2655	5	2966	5	3032	17	3130	16	3146	20
adier Guards	—	—	—	—	—		2753	1	—	—	3572	1	3261	2
positors ...	-		—	—	2884	3	2704	2	3212	5	3081	6	3408	7
smen ..		-	2491	1	—	-	—	—	—	—	3490	2	3326	8
men ...	—	—		-		—	-	—	—		-	—	3146	1
lemen ..	—	—	—	—	2376	1	2638	1	2556	7	2900	9	3097	14
ists, etc. ...	—	—	--	-	3310	1	3572	2	3572	1	3458	4	3556	3
e Guards .	—	—	—	-	--	—	—	—	—	—	—	—	--	--
of 1st series (weighted)	2212	14	2900	6	2835	27	3015	22	3163	68	3408	78	3343	118
iam recruits	—	—	2737	1	2966	1	—	—	3097	1	—	—	3818	19
wich marines	—	—	—	—		—	—	—	—	—	3540	3	3654	7
llaneous ..		—	2950	1	--	—	3179	4	3245	4	2950	4	3212	10
mean under height (unweighted)	2212	14	2868	8	2900	28	3097	26	3163	73	3294	85	3507	154
corrected weighting	2399	—	2843	-	2827	-	3022	—	3049	—	3208	—	3406	—

	ft. in. ft. in. 5 6—5 7		ft. in. ft. in. 5 7—5 8		ft. in. ft. in. 5 8—5 9		ft. in. ft. in. 5 9—5 10		ft. in. ft. in. 5 10—5 11		ft. in. ft. in. 5 11—6 0		ft. in. ft. in. 6 0—6 1		Total	Weighted Mean
	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	cases	c.c.
... ...	3703	14	3753	15	3916	11	4228	18	4471	12	4124	6	4031	2	126	3790
igade ..	3523	17	3785	26	3785	20	3884	3	4261	1	4080	2	-	—	92	3693
Met.) .	3834	4	3736	33	3703	46	4064	22	3834	13	4293	12	4605	11	141	3903
Thames)	3638	15	4031	17	4097	10	3933	5	4211	3	—			—	75	3751
... ..	3441	10	3064	9	3261	10	4293	1	3933	3	—	—	-	—	123	3037
lasses .	3441	20	3638	28	3900	16	4031	14	3900	7	4408	9			159	3481
er Guards	—	—	3736	7	3818	22	3933	16	3802	11	4146	9	4228	14	83	3907
tors .	3720	5	3523	8	3507	6	3785	3	—	-	4146	1	-	—	46	3387
n ...	3343	3	3654	7	4015	1	3916	4	4046	2	—	—	--		28	3552
ı .	3949	1	3572	3	3654	4	4015	1	4277	6	4064	4	—	—	20	3918
en ...	3408	10	3408	18	3408	16	3867	8	4162	12	4097	5	4293	5	106	3445
, etc. ...	4375	3	3376	1	3982	2	4474	3	4457	5	4064	2		-	27	3957
uards ...	—	—	—	—	—	—	—	—			4179	30	4506	26	56	4331
st series (weighted)	3671	102	3605	172	3753	164	4031	98	4162	75	4179	80	4261	58	—	—
recruits	3900	67	4048	28	4113	22	4359	16	3867	2	4277	5	4654	3	175	3999
marines	3818	99	3851	192	3933	130	4031	75	4097	39	4310	18	4523	9	572	3925
eous ...	3638	18	3490	9	3769	13	3703	12	—	—	4244	7	4687	6	88	3645
n under height (un-)	3753	286	3736	411	3884	329	4031	201	4048	116	4244	110	4523	76	1917	—
rected ...ting	3739	—	3786	—	3836	—	4042	—	4094	—	4239	--	4465	—	—	—

Hutchinson took as his chief criterion vital capacity, i.e. the volume of the greatest expiration following the deepest inspiration. This differs from the total lung volume by the amount of the residual air, i.e. the quantity of air which always remains in the lungs and over which we have no control. Having chosen his measure, he next proceeded to investigate its normal value in healthy people as a standard of measurement. In examining the work of his predecessors on vital capacity, Hutchinson found various statements as to its mean value (varying from 1637 c.c. to 4916 c.c.), but few direct experiments and these on very small numbers. Hutchinson's great advance on previous work was that, besides using direct experiment on large numbers, he brought in the idea of proportion and tried to establish normal relations between vital capacity and other physical measurements. He says:

"One of the fundamental rules in architecture is proportion, the relation that the whole fabric has to its constituent parts and which each part has to the complete idea of the whole, for in buildings that are perfect in their kind, from any particular part an architect may form a tolerable judgment of the whole: just in like manner the physiologist from a portion of the viscera, say an organ, should be able to form a tolerable judgment of the whole man from whom it was taken."

The other observations that Hutchinson took besides vital capacity were:

Power of Expiratory Muscles.
Power of Inspiratory Muscles.
Circumference of the Chest over the nipples.
Height.
Weight.
Pulse (sitting).
Respirations per minute (sitting).
Age.
Temperature of air breathed into the spirometer.
Remarks on the occupation and general appearance.

TABLE II.

Height and Vital Capacity in Arithmetical Progression. (HUTCHINSON.)

Height ft. in.	Series from 1012 cases cubic inches	Series from 1917 cases cubic inches	Series from Arithmetical Progression cubic inches	Arithmetical Progression cubic cms.
5 1	175·0	176·0	174·0	2851
5 3	188·5	191·0	190·0	3113
5 5	206·0	207·0	206·0	3376
5 7	222·0	228·0	222·0	3638
5 9	237·5	241·0	238·0	3900
5 11	254·5	258·0	254·0	4162
Means of all heights	214·0	217·0	214·0	3507

These records were made on 2130 persons of the following classes:

Sailors (Merchant Service)	121
Fire Brigade of London	82
Metropolitan Police	144
Thames Police	76
Paupers	129
Mixed Class (artisans)	370
1st Battalion Grenadier Guards ...	87
Royal Horse Guards (Blues)	59
Chatham Recruits	185
Woolwich Marines	573
Pugilists and Wrestlers	24
Giants and Dwarfs	4
Printers (Pressmen 30, Compositors 43)	73
Draymen	20
Girls	26
Gentlemen	97
Diseased cases	60
Total	2130

The mean values of the vital capacities that he observed for these at different heights are given in Table I (cubic inches are converted into cubic centimetres for comparison with others), which is typical of the way he presents his data. The means for each height (at the bottom of the table), as Dreyer has pointed out, are wrong as Hutchinson had not weighted his groups. (To the last column of means has been added the corrected means for each height.)

To get the relation between vital capacity and height he prepared two similar tables grouping his classes of people together, calculated the mean of vital capacity every two inches of height, and so obtained Table II. The last column in this is the nearest Arithmetical Progression obtained apparently by inspection. His result was:

"For every inch of height from 5 feet to 6 feet eight additional cubic inches of air at 60° are given out by a forced expiration."

He treated his weight table in the same manner, sub-dividing for height at the same time, but found that the effect of weight was not so definite as that of height. He concluded that weight did not affect vital capacity until it was in excess of what it should be for a given height; but the difficulty arises, that there exists no standard of weight for given heights. He found that the relation between vital capacity and weight was not linear throughout. Vital capacity ascended up to 160 lbs. and then remained nearly stationary up to 200 lbs. More exactly, vital capacity increased with weight in the ratio of 1 cubic inch to 1 lb. from 105 to 155 lbs. and from 155 to 200 lbs. there is a loss of 39·5 cubic inches. This

refers to heights of 5 feet 6 inches and he modified it for other heights. Dreyer has re-calculated this, weighting the various groups properly, and finds that when this is done vital capacity ascends with weight.

Hutchinson next tried to find a standard relation between weight and height, he added other observations and so brought his males up to 3000 of ages from 15 to 40 years. Table III shews his scale of weights for different heights. He says that instead of varying as the cube of the height as it should do if the body were symmetrical, weight does in fact vary as the 2·75th power of the height.

TABLE III.

Mean Weight (including clothes) in Relation to Height of 3000 *Males. Ages* 15 *to* 40 *years.* (HUTCHINSON.)

Heights ft. in. to ft. in.	Number of Cases	Mean Weight in lbs.	Mean Weight in kilograms
4 6 to 5 0	26	92·26	41·85
5 0 5 1	17	115·52	52·40
5 1 5 2	36	124·33	56·40
5 2 5 3	43	127·86	58·00
5 3 5 4	88	138·01	62·60
5 4 5 5	126	139·17	63·13
5 5 5 6	214	144·93	65·74
5 6 5 7	316	144·29	65·45
5 7 5 8	379	152·59	69·21
5 8 5 9	468	157·76	71·56
5 9 5 10	368	166·40	75·48
5 10 5 11	348	170·86	77·50
5 11 6 0	245	177·45	80·49
6 0 6 1	326	218·66	99·18
Total ...	3000	147·86	67·07

He is here probably quoting Quetelet, as he did when he corrected for clothing by taking its weight to be 1/18 of the total weight in the case of males and 1/24 in the case of females. Hutchinson's conclusion was that an excess of 7 °/ₒ or more over the normal weight for any given height would influence the vital capacity, but that a smaller excess would have no effect.

He treated age in the same way as weight for different height groups. He first examined 1088 subjects with no definite result. Then from a series of 1775 healthy subjects he found that from 15 to 35 years the vital capacity increased but after the age of 35 decreased by 19 cu. in., 11 cu. in., and 13 cu. in. respectively in the three succeeding 10-year periods, i.e. approximately at the rate of 117 c.c. per five years.

As regards chest, Hutchinson said:

"I do not find there exists any direct relation between the circumference of

the chest and the vital capacity; for the height of 5 feet 6½ inches the difference in vital capacity for an increase of ½ inch in the circumference is quite irregular."

He therefore rejected absolutely the circumference of the chest as a guide to vital capacity. This is in spite of the fact that he asserted that the most remarkable relation of the circumference of the chest was to weight, with which it increased in an *exact* arithmetical progression of 1 inch to every 10 lbs. How such a definite relation could exist between chest and weight, in conjunction with the relation he found between weight and vital capacity, without any relation between chest and vital capacity, Hutchinson did not attempt to explain.

To sum up his conclusions. In healthy cases vital capacity is chiefly affected by height, weight and age thus:

Height. An increase of 8 cubic inches at 60° for every inch of height.

Weight. At height 5 feet 6½ inches vital capacity is not affected under 161 lbs., but above this point it diminishes in the relation of 1 cubic inch per lb. up to 196 lbs. At other heights 10 °/ₒ may be added to the mean height before applying this rule.

Age. A decrease of rather more than 1 cubic inch per year.

Chest. No relation.

Hutchinson also asserted that standing height was not correlated with sitting height, but that the latter was constant. He only gave figures for one pair of cases, but said that others gave the same result.

He considered that vital capacity was governed by the mobility of the chest rather than its absolute volume, but did not go into this point statistically beyond examining the lung capacities of 20 corpses.

He found that position might make a difference of 655 cubic centimetres in vital capacity. In support of his claim that vital capacity might be a useful measure to detect progress of disease, he gave the vital capacity in 22 early cases of phthisis and 9 advanced compared with the theoretical healthy values, calculated from his means of people of the same physical development (see Table IV).

Not many observations appear to have been published concerning the vital capacity of diseased persons, but Hutchinson's differences were larger than those found by other writers quoted below.

Hutchinson's work was of much statistical value. He did not, unfortunately, present his data in a form to allow correlation coefficients to be calculated, for one cannot get the standard deviation of his vital capacity, but, as we shall see later, in spite of his faulty method of getting means, the regression coefficients at which he arrived differ very little from those we have found in a sample of about 1000 Air Force cadets.

Shortly after the publication of Hutchinson's work, Fabius of Amsterdam, anticipating a criticism reiterated in our own time, objected that stem length and not height was the appropriate independent variable to which vital capacity

ought to be related. We have not seen Fabius' paper, which was apparently a degree thesis, but his statistics are re-printed in the monograph of Arnold.

TABLE IV.

Vital Capacities of Tuberculous Subjects.

MALES.

HUTCHINSON.

	Vital Capacity: Diseased (Observed) c.c.	Vital Capacity: Healthy (Calculated) c.c.	Vital Capacity: Difference c.c.
Early stage	1852	3605	1753
	1884	2835	951
	1721	2835	1114
	2130	3343	1213
	2098	3605	1507
	1966	3753	1787
	1639	3163	1524
	2294	4031	1737
	1639	3343	1704
	1803	3605	1802
	2229	3753	1524
	2212	3343	1131
	3146	3769	623
	3687	4916	1229
	2376	3605	1229
	3277	3933	656
	3032	3769	737
	3572	3933	361
	2114	3605	1491
	5637	7112	1475
	3605	4261	656
	3212	4162	950
Advanced stage	967	2212	1245
	1458	3671	2213
	1770	4162	2392
	1180	2212	1032
	1311	3753	2442
	1229	4162	2933
	557	4031	3474
	2802	4424	1622
	983	3884	2901

[illegible]

	Vital Capacity: Diseased (Observed) c.c.	Class	Vital Capacity: Healthy (Calculated) from Dreyer c.c.	Vital Capacity: Difference c.c.
Nine incipient cases	3813	B	3734	+ 79
	3494	B	3991	497
	3879	B	4012	133
	3611	B	3991	380
	3838	B	3970	132
	3649	B	4054	405
	4295	B	4137	+158
	5104	B	4361	740
	4095	B	4054	+ 41
Thirteen moderately advanced cases	2405	B	3537	1132
	2986	B	4033	1047
	3600	B	3884	281
	2894	B	4075	1181
	3401	B	4262	861
	3013	B	4096	1083
	3741	B	4075	334
	2749	B	4075	1326
	2952	B	4075	1123
	4516	A	4825	279
	3501	B	3842	341
	3894	A	4780	886
	3950	B	4179	229
Nine advanced cases	2106	B	3514	1408
	2606	B	3603	997
	2696	B	3927	1231
	2607	B	3777	1170
	2100	A	4486	2386
	4256	B	4033	+223
	2598	B	3906	1308
	2449	A	4262	1813
	2400	B	4012	1612

From an analysis of these (Table V) it is evident that Fabius had no justification for his conclusion that stem length was a better variable to choose, and it also appears that stem length is not even approximately constant

In 1858, Arnold of Heidelberg published a full monograph on the question, confirming generally Hutchinson's views, but attaching more importance than the latter did to chest circumference and expansion. From Arnold's tabulation of 216 adults (observations of his own, together with those of Fabius and Simon) we

find the correlation of stature and vital capacity to be $0{\cdot}7118 \pm 0{\cdot}0226$. The mean vital capacity was 3588 c.c. with a standard deviation of 487 c.c., giving a coefficient of variation of 13·6 %. The correlation is higher, but not significantly higher, than we have found. The variability of vital capacity is remarkably constant in all the series we have examined, despite differences both of methods of measurement and of mean values; Arnold's, Schuster's and our own results are nearly identical. Arnold also published data respecting women, with these we shall not deal in this paper.

TABLE V.

Fabius' Amsterdam Students.

	Mean	Standard Deviation	Coefficient of Variation	Number of Observations
Vital Capacity .	3717·74 c.c.	505·90 c.c.	13·61	116
Standing Height .	173·43 cms.	6·68 cms.	3·85	116
Sitting Height ...	75·22 cms.	4·38 cms.	5·82	116

Correlation Coefficients and Ratios.

	r	η	$\eta^2 - r^2$
Vital Capacity and Standing Height...	·661 ± ·035	·715 ± ·031	·0738 ± ·034
Vital Capacity and Sitting Height ...	·604 ± ·040	·652 ± ·036	·0600 ± ·031
Standing Height and Sitting Height.	·732 ± ·029	·740 ± ·028	·0124 ± ·011

Regression Equations	Equation	Partial Standard Deviation
Vital Capacity in c.c. from Standing Height in cms	$x = 50{\cdot}0094z - 4955{\cdot}19$	379·71
Vital Capacity in c.c. from Sitting Height in cms.	$x = 69{\cdot}8229y - 1534{\cdot}28$	403·10

After Arnold's time several other papers were published, but we have given typical results and now pass to Dr Edgar Schuster's paper in *Biometrika*, VIII. 1911, p. 40), wherein the vital capacity and other dimensions of 959 Oxford undergraduates are analysed.

Schuster studied the correlations of vital capacity with stature, weight and strength of pull. (These are given in Table VI for comparison.) Schuster does not give any measure of the linearity of these correlations. The mean of these Oxford undergraduates is greater than that of any of Hutchinson's healthy classes, and especially greater than that of the class he calls "gentlemen." Schuster's vital capacities are the *mean* of each individual's attempts, most other investigators take the *maximum*. Schuster's results agree closely with those we obtained

from the Air Force data. The size of the standard deviations is to be noted in view of the suggested use of vital capacity as a test for the early diagnosis of disease. Though it might be argued that these data contain unfit persons as well as fit, it will be seen that standard deviations of the same size were obtained from a perfectly fit sample of Air Force cadets.

TABLE VI.

Schuster's Results. (Oxford Undergraduates, *Biometrika*, Vol. VIII.)

(Weight taken fully dressed except boots and coats.)
(Lung Capacity = Vital Capacity = Mean of 3 trials.)

Lung Capacity			
Age	No.	Mean c.c.	Standard Deviation c.c.
18	129	4184 ± 34	566·2 ± 23·8
19	330	4278 ± 22	602·5 ± 15·8
20	209	4268 ± 29	611·6 ± 20·2
21	137	4438 ± 39	683·1 ± 27·8
22	95	4301 ± 37	540·9 ± 26·5
23 and over	59	4418 ± 59	674·8 ± 41·9
General Average	959	4315	613·2

Weight	Mean st. lbs. oz.	Standard Deviation st. lbs. oz.
General Average	10 11 15	1 2 6

Stature	Mean	Standard Deviation
General Average	1765·0 mm.	66·08 mm.

Correlation Coefficients.

Age	Lung Capacity and Stature	Lung Capacity and Weight	Lung Capacity and Strength of Pull	Stature and Weight
18	·44 ± ·05	·55 ± ·04	·44 ± ·05	·50 ± ·04
19	·65 ± ·02	·62 ± ·02	·40 ± ·03	·63 ± ·02
20	·55 ± ·03	·52 ± ·03	·37 ± ·04	·68 ± ·03
21	·64 ± ·03	·66 ± ·03	·32 ± ·05	·76 ± ·02
22	·59 ± ·04	·62 ± ·04	·31 ± ·06	·72 ± ·03
General Average	·57	·59	·37	·66
Pearson's 1000 Cambridge Undergraduates, 1899				·49 ± ·02

Garvin, Lundsgaard and van Slyke. These three authors published a paper entitled "Studies of Lung Volume,"* in 1918, which contains some detailed results on tuberculous subjects. The vital capacities for males deduced from their figures are given in Table IV. In the second column has been added the normal vital capacity from weight according to Dreyer's Tables, and the third column

* *Journal of Experimental Medicine*, XXVII. p. 87.

shews the differences. The loss of vital capacity shewn by all the incipient cases is well within the sampling error given by Schuster, and it is only in the more advanced cases, which have well developed physical signs of disease, that the difference can have any significance.

In 1919, Professor Georges Dreyer entered the field. He maintained that Hutchinson's work was fundamentally sound, but that (*a*) stem length should be substituted for standing height, (*b*) that the variables used should be connected by equations of the form $y = ax^b$, where a and b are constants. Dreyer was originally led to test this form of relationship by biological considerations. From a study of the relationship between body weight and aortic and tracheal area in animals, he was led to surmise that the constants were proportional to body surface. His position appears to be that, if a formula capable of a biological interpretation is found to describe a range of facts with tolerable efficiency—even if not actually so efficient as an *ad hoc* method—if, further, the range covered is wide, then such a method deserves the preference.

The practical results of Dreyer's investigations are summarised in a small volume of tables entitled "The Assessment of Physical Fitness" by G. Dreyer and G. F. Henson. The precise data and groupings used to compute the constants have not, so far as we know, been published, but are stated to depend upon "a number of observations sufficient to ensure a degree of accuracy that should prove entirely satisfactory." Tables for the computation of:

Normal Weight from stem length,
Normal Weight from circumference of the chest,
Normal Vital Capacity from weight,
Normal Vital Capacity from stem length,
Normal Vital Capacity from chest circumference,

are provided; there are separate tables for males and females, and both imperial and metric units are entered.

The instructions for using the tables state that one should first ascertain whether an individual's weight is normal, judged by his chest circumference and stem length. If it is so, the vital capacity-weight table should be used; if it is not, the values from the vital capacity-stem length and vital capacity-chest circumference are to be averaged. It may be remarked that, according to these tables, a variation of 10 °/ₒ from the mean weight is probably, and of 15 °/ₒ, certainly abnormal. We have found that in a sample of 950 passed fit candidates for the Air Force, the coefficient of variation was 11·4 °/ₒ, while for constant chest circumference it was only reduced to 8·4 °/ₒ.

Dreyer's results have found support and criticism in various papers; we have no concern here either with supporters or critics, our problem being a purely biometric one, viz. whether in fact the formulae used for the tables or similar formulae were better descriptions of a sample population than those based upon elementary biometric considerations.

We knew that, owing to the energy of Wing Commander Martin Flack, C.B.E., a considerable mass of data suitable for our purpose had been collected in the Royal Air Force. By the kind permission of the Director of Medical Services of the Royal Air Force, and with the cordial co-operation of Wing Commander Flack—to whose expert advice and criticism we owe much—we have been able to use a large number of records, to which we now turn.

The data were measurements of 1238 candidates for commissions in the Royal Air Force, supplied by Wing Commander Flack. This population, distinguished as A in the tables, was analysed and the whole analysis repeated upon the 950 measurements relating to candidates passed as fit, tables relating to the latter being marked B. The selection has made very little difference in the arithmetical values of the constants. It will be seen that the mean and standard deviation of vital capacity are slightly larger than and the coefficient of variation almost identical with Dr Schuster's findings on Oxford undergraduates (see Tables VI and VII). There is a similarly close agreement in respect of stature, our men are slightly shorter but have the same variability. Dr Schuster's coefficient of variation was 3·74, ours 3·64. It may be said that, in respect of the variables studied, the Air Force candidates are in *pari materia* with the Oxford undergraduates, who provided some of the evidence upon which Dreyer relied.

From the table of coefficients of correlation (Table VIII) we may infer that few of the total regressions are strictly linear, but that the departures from linearity although significant are not large, excepting those instances into which weight and age enter. Passing to the effect upon variability of vital capacity produced by allowance for the other variables, it is found that the fourth order standard deviation is for A 496 c.c., 77·4 °/$_{\circ}$ of the standard deviation of zero order, for B data it is 76·6 °/$_{\circ}$ of the zero order value. This result is itself enough to shew that predictions from all four measurements, standing height, sitting height, weight and chest circumference, are subject to a very large error of sampling, roughly 10 or 12 °/$_{\circ}$ of the average value.

Returning to the deviation from linearity, we remark that all the deviations from the linear trend of importance occurred where observations were scanty. Apart from age (not at first considered owing to the limited range) standing height, chest circumference, and weight were least linear and, in the two latter cases, equations of type $y = ax^b$ were applied, but neither shewed a better graduation; vital capacity in terms of weight was, however, better graduated by a cubic than by a linear equation. The trend of means of vital capacity in terms of weight, actually shews the same apparent inflexion at higher weights mentioned by Hutchinson, but we can put little stress upon our result since the frequency of heavy weights amongst the cadets was small. It is, however, possible, as Hutchinson thought, that after a certain point increase of weight—due to mere adiposity—is unfavourable to vital capacity. The American actuaries found that "over weights" were worse lives on the average than "under weights."

When exponential functions were fitted to the data, we first obtained an

TABLE   *F*   *Candida*

	M	St lard De	Coefficient of Vi riation		
A (All Candidates) 1238 men					
Vital Capacity	·72 c.c.	640·69 c.c.	14·0	·02835 ± ·011	3·1295 ± ·127
Weight	63·44 kg.	7·40 kg.	11·6	·63302 ± ·143	4·3434 ± ·520
Standing Height	74·82 cms.	6·36 cms.	3·6	·000299 ± ·000016	3·4933 ± ·208
Sitting Height	90·39 „	3·12 ..	3·4	·000046 ± ·000016	3·0654 ± ·105
Chest	86·25 „	4·74 „	5	·1857 ± ·055	3·5411 ± ·256
Age	23·92 yrs.	5·07 yrs.	21	1·8657	5·8886
B (Fit Candidates) 950 men.					
Vital Capacity ...	04·82 c.c.	614·72 c.c.	13	·0737 ± ·034	3·3746 ± ·230
Weight	63·48 kg.	7·24 kg	11	6098 ± ·179	4·4919 ± ·823
Standing Height ...	74·81 cms	6·28 cms.	3	·000557 ± ·00030	3·3876 ± ·202
Sitting Height ...	90·34 „	3·06	3	·001447 ± ·000039	3·144 ± ·136
Chest	86·29 „	4·65	5	·1805 ± ·0058	3·4518 ± 4
Age	·80 yrs.	4·60	19	1 7925	6·4425

LE

	e		η^2 –		*B*		*x* .	
Vital Capacity and St ing Heigh	·568 ± ·013	·588 ± ·013	·29 ±		·586 ± ·014	·615 ± ·014	·0	± ·0068
Vital Capacity and Sitting Height	·527 ± ·014	·527 ± ·014	·047 ±		·549 ± ·015	·554 ± 015	·0	± ·0031
Vital Capacity and Weight ...	·483 ± ·015	·528 ± ·014	·0454 ±		·495 ± ·017	·531 ± ·016	·0	± ·0085
Vital Capacity and Chest ...	·413 ± ·016	·436 ± ·016	·012 ±		·395 ± ·019	·435 ± ·018	·0	± ·0080
Vital Capacity and Age ...	– ·018 ± ·019	·211 ± ·019	·0440 ±		– ·0119 ± ·022	·190 ± ·021	·0	± ·0083
Weight and Chest	·750 ± ·009	·751 ± ·009	·00198 ± ·001		·7405 ± ·009	·7406 ± ·009	·0	± ·00045
Sitting Height and Standing Heigh	·676 ± ·010	·700 ± ·010	·0336 ± ·00		·720 ± ·011	·736 ± ·010	·0	± ·0065
Weight and Sitting Height	·528 ± ·014	·528 ± ·014	·00 ± ·00		·540 ± ·016	·540 ± ·016	·0	± ·000
Standing Height and Weig	·547 ± ·013	·582 ± ·013	·03 ± ·00		·544 ± ·015	·580 ± ·015	0	± ·0088
Chest and Sitting Height	·30 ± ·017	·367 ± ·017	·00 ± ·001		·367 ± ·019	371 ± ·019	·0	± ·0023
Standing Heigh , and Chest	·3 ± ·017	·357 ± ·017	·00 ± ·00		·336 ± ·019	·429 ± ·018	·0	± ·0
Chest and Age	·3 ± ·017	·331 ± ·018	·01 ± ·00		·272 ± ·020	·312 ± ·020	·0	±
Weight and Ag	·2 ± ·018	·241 ± ·019	·01 ± ·004		·178 ± ·021	·229 ± ·021	·0	±
Sitting Height nd Age	·0 ± ·019	·150 ± ·019	·01 ± ·0049		·053 ± ·022	·163 ± ·021	·0	±
Standing Heigl , and Age	– ·0 ± ·019	·179 ± ·019	·03 ± ·0067		– ·0546 ± ·022	·218 ± ·021	·0446	±

approximate value by fitting the logarithmic line by least squares and then improved the value by the use of one or other of the following methods, the use of which was suggested respectively by Dr Isserlis* and Mr Trachtenberg.

(I) To fit $y = \alpha x^{\beta}$. First find b an approximation to β by fitting the logarithmic equation by least squares. Then find the least square equations for making $\Sigma_{r=1}^{n} (y_r - \alpha x_r^{\beta})^2$ a minimum, eliminate α from them and in the resulting equation for β, replace β by $b+u$, expand x^{β} in powers of $u \log x$ and solve for u neglecting u^2 and higher powers. Having got β, α can be obtained from the least square equations.

Thus notation

$$p_1 = \sum_{r=1}^{n} \log y_r, \qquad q_1 = \Sigma x^b, \qquad s_1 = \Sigma x^{\beta},$$

$$q_2 = \Sigma x^{2b}, \qquad s_2 = \Sigma x^{2\beta},$$

$$p_2 = \sum_{r=1}^{n} \log x_r, \qquad q_3 = \Sigma y x^b, \qquad s_3 = \Sigma y x^{\beta},$$

$$q_4 = \Sigma y x^b \log x, \qquad s_4 = \Sigma y x^{\beta} \log x.$$

$$p_3 = \sum_{r=1}^{n} (\log x_r)^2, \qquad q_5 = \Sigma x^{2b} \log x,$$

$$q_6 = \Sigma y x^b (\log x)^2,$$

$$p_4 = \sum_{r=1}^{n} (\log x_r \log y_r). \qquad q_7 = \Sigma x^{2b} (\log x)^2.$$

Make $\quad \sum_{r=1}^{n} (\log y_r - \log a + b \Sigma \log x_r)^2$ a minimum.

This gives $\quad b = \dfrac{np_4 - p_1 p_2}{np_3 - p_2^2} \quad \log \alpha = \dfrac{p_1 p_3 - p_2 p_4}{np_3 - p_2^2} = \dfrac{p_1 - bp_2}{n} = \dfrac{p_4 - bp_3}{p_2}.$

Now fit $y = \alpha x^{\beta}$, i.e. make $\sum_{r=1}^{n} (y_r - \alpha x_r^{\beta})^2$ a minimum.

$$\therefore \quad \Sigma x_r^{\beta} (y_r - \alpha x_r^{\beta}) = 0 \quad \ldots\ldots\ldots\ldots (1),$$

$$\Sigma x_r^{\beta} \log x_r (y_r - \alpha x_r^{\beta}) = 0 \quad \ldots\ldots\ldots\ldots (2),$$

$$\alpha \Sigma x_r^{2\beta} = \Sigma y_r x_r^{\beta} \text{ and } \alpha \Sigma (x_r^{2\beta} \log x_r) = \Sigma (y_r x_r^{\beta} \log x_r) \quad \ldots\ldots\ldots\ldots (3).$$

Eliminate α. $\quad \therefore \quad \dfrac{\Sigma (x_r^{2\beta} \log x_r)}{\Sigma (x_r^{2\beta})} = \dfrac{\Sigma (y_r x_r^{\beta} \log x_r)}{\Sigma (y_r x_r^{\beta})} \quad \ldots\ldots\ldots\ldots (4).$

Take $\beta = b + u$ and solve (4) for u neglecting u^2 and expanding x^{β}.

$$\therefore \quad x^{\beta} = x^b (1 + u \log x),$$

$$x^{2\beta} = x^{2b} (1 + 2u \log x).$$

$$\therefore \quad \frac{\Sigma (x^{2b} \log x) + 2u \Sigma \{x^{2b} (\log x)^2\}}{\Sigma (x^{2b}) + 2u \Sigma \{x^{2b} \log x\}} = \frac{\Sigma \{y x^b \log x\} + u \Sigma \{y x^b (\log x)^2\}}{\Sigma (y x^b) + u \Sigma (y x^b \log x)},$$

$$\frac{q_5 + 2uq_7}{q_2 + 2uq_5} = \frac{q_4 + uq_6}{q_3 + uq_4}, \quad \therefore u = \frac{q_3 q_5 - q_2 q_4}{q_2 q_6 + q_4 q_5 - 2q_3 q_7}.$$

From (3) $\alpha = \dfrac{s_3}{s_2} = \dfrac{s_4}{s_5}$, and if these two values of α agree, then the values of α and β are correct.

[* The process is Gauss' fundamental method applied to a special case. Ed.]

more accurate values are wanted, repeat the process, using β instead of b.

(II) To fit $y = bc^x$ etc. or similar index forms.

Let y_x = observed value, $f(x, a, b, \ldots)$ = the theoretical value.

Let $$y_x - f(x, a, b, \ldots) = \epsilon_x.$$

Then $$\log f(x, a, b, \ldots) = \log (y_x - \epsilon_x)$$

$$= \log y_x + \log \left(1 - \frac{\epsilon_x}{y_x}\right)$$

$$= \log y_x - \frac{\epsilon_x}{y_x} - \frac{1}{2}\left(\frac{\epsilon_x}{y_x}\right)^2 - \ldots \text{ if } \frac{\epsilon_x}{y_x} < 1.$$

TABLE IX A.

Air Force Candidates. Total and Partial Correlation Coefficients.

A (All Candidates).

	Vital Capacity and Standing Height	Vital Capacity and Sitting Height	Vital Capacity and Weight	Vital Capacity and Chest	Vital Capacity and Age
Total Coefficients	·568 ± ·013	·527 ± ·014	·483 ± ·015	·413 ± ·016	− ·0182 ± ·019
Partial Coefficients					
1st *order. Variables kept constant:*					
Standing Height		·235 ± ·018	·251 ± ·018	·282 ± ·018	
Sitting Height	·338 ± ·017		·284 ± ·018	·279 ± ·018	
Weight	·414 ± ·016	·365 ± ·017		·0879 ± ·019	
Chest	·498 ± ·014	·443 ± ·015	·288 ± ·018	—	
Age	·568 ± ·013	·530 ± ·014	·498 ± ·014	·439 ± ·015	
2*nd order:*					
Standing Height and Sitting Height			·202 ± ·018	·248 ± ·018	
Standing Height and Weight		·183 ± ·019	—	·152 ± ·019	
Standing Height and Chest		·192 ± ·018	·073 ± ·019	-	
Standing Height and Age		·238 ± ·018	·260 ± ·018	·299 ± ·017	
Sitting Height and Weight	·276 ± ·018		—	·116 ± ·019	
Sitting Height and Chest	·315 ± ·017		·128 ± ·019	—	
Sitting Height and Age	·333 ± ·017		·304 ± ·017	·315 ± ·017	
Weight and Chest	·430 ± ·016	·372 ± ·017		—	
Weight and Age	·400 ± ·016	·361 ± ·017		·123 ± ·019	
Chest and Age	·485 ± ·015	·144 ± ·015	·287 ± ·018		
3*rd order:*					
Standing Height, Sitting Height and Weight	—			·153 ± ·019	
Standing Height, Sitting Height and Chest..			·041 ± ·019		
Standing Height, Sitting Height and Age			·217 ± ·018	·273 ± ·018	
Sitting Height, Weight and Chest ...	·292 ± ·018			—	
Sitting Height, Weight and Age ...	·258 ± ·018			·134 ± ·019	
Sitting Height, Chest and Age ...	·295 ± ·018		·127 ± ·019	—	
Weight, Chest and Age	·404 ± ·016	·367 ± ·017			
Standing Height, Weight and Chest		·183 ± ·019			
Standing Height, Weight and Age ...		·189 ± ·018		·138 ± ·019	
Standing Height, Chest and Age ...		·202 ± ·018	·079 ± ·019		
4*th order:*					
Standing and Sitting Height, Weight and Age				·141 ± ·019	
Standing and Sitting Height, Chest and Age			·0453 ± ·019		
Standing Height, Weight, Chest and Age ...		·191 ± ·018			
Sitting Height, Weight, Chest and Age ...	·261 ± ·018				

TABLE IX B.

Air Force Candidates. Total and Partial Correlation Coefficients.

B (Fit only).

	Vital Capacity and Standing Height	Vital Capacity and Sitting Height	Vital Capacity and Weight	Vital Capacity and Chest	Vital Capacity and Age
Total Coefficients	·586 ± ·014	·549 ± ·015	·495 ± ·017	·395 ± ·019	− ·0119 ± ·022
Partial Coefficients					
1st order. Variables kept constant:					
Standing Height	—	·227 ± ·021	·259 ± ·020	·259 ± ·020	—
Sitting Height	·328 ± ·020	—	·282 ± ·020	·248 ± ·021	—
Weight	·434 ± ·018	·386 ± ·019	—	·048 ± ·022	—
Chest	·524 ± ·016	·473 ± ·017	·328 ± ·020	—	—
2nd order:					
Standing and Sitting Height	—	—	·214 ± ·021	·225 ± ·021	—
Standing Height and Weight...	—	·172 ± ·021	—	·111 ± ·022	—
Standing Height and Chest	—	·187 ± ·021	·111 ± ·022	—	
Sitting Height and Weight	—	—	—	·0763 ± ·022	—
Sitting Height and Chest	—	—	·158 ± ·021	—	—
Weight and Chest	·444 ± ·018	·390 ± ·019		—	—
3rd order:					
Standing Height, Sitting Height and Weight	—	—	—	·110 ± ·022	
Standing Height, Sitting Height and Chest...	—	—	·082 ± ·022	—	—
Sitting Height, Weight and Chest	·284 ± ·020	—	—	—	—
Standing Height, Weight and Chest... ...	—	·171 ± ·021	—		—

Neglecting squares of $\dfrac{\epsilon_x}{y_x}$, $\log f(x, a, b, \ldots) = \log y_x - \dfrac{\epsilon_x}{y_x}$.

$$\therefore \quad \epsilon_x = y_x \{\log f(x, a, b, \ldots) - \log y_x\}.$$

We now deviate from the general case, which would minimise $\Sigma \dfrac{\epsilon_x^2}{\sigma_{y_x}^2}$, and make $\Sigma\epsilon_x^2$ a minimum,

i.e. $$\Sigma [y_x \{\log f(x, a, b, \ldots) - \log y_x\}]^2 \text{ a minimum*}.$$

Differentiate with respect to a, b, c, etc. and get the equations for the constants.

The results of these are shewn in Table XI (p. 332), the two methods give practically the same result in the case we have tested. Method II has the advantage of being much shorter in working, but Method I can be applied repeatedly if a still better value is wanted.

We only tested those cases where the departure from the linearity was such as to suggest that a logarithmic function might be more suitable; in other cases, graphs gave no reason to expect any improvement; further, our main object was to institute a comparison with Dreyer's published method. For this particular purpose a simpler method, suggested to us by Professor Karl Pearson, leads to

[* This method therefore amounts to fitting logarithms by least squares and weighting with the *squares* of the ordinates. If $Y_x = \log y_x$, we have $1/\sigma_{Y_x}^2 = \bar{y}_x^2/\sigma_{y_x}^2$ where $\bar{y}_x$ is mean y_x for given x. Hence if we could consider σ_{y_x} constant for all values of x, this result would be, *without regard to the approximations by which it was deduced*, the weighted logarithmic least square equation and might be written down at once. The assumption σ_{y_x} = constant would need justification. Ed.]

TABLE X.

Multiple Correlation Coefficients.

	A. All Candidates	*B.* Fit Candidates
One Variable.		
Vital Capacity from :		
Standing Height	·5679 ± ·013	·5858 ± ·014
Sitting Height	·5266 ± ·014	·5439 ± ·015
Weight	·4831 ± ·015	·4947 ± ·017
Chest	·4131 ± ·016	·3946 ± ·019
Age	− ·01818 ± ·019	− ·0119 ± ·022
Two Variables.		
Vital Capacity from :		
Standing Height and Chest	·6136*	·6246
Standing Height and Weight	·6042	·6222
Standing Height and Sitting Height	·6000	·6139
Sitting Height and Weight	·5792	·5977
Sitting Height and Chest	·5775	·5871
Standing Height and Age	·5679	—
Sitting Height and Age	·5300	-
Weight and Age	·4982	—
Weight and Chest	·4892	·4965
Chest and Age	·4395	·4098
Three Variables.		
Vital Capacity from :		
Standing Height, Sitting Height and Chest	·6321	·6392
Standing Height, Sitting Height and Weight	·6215	·6367
Standing Height, Chest and Age	·6189	—
Standing Height, Chest and Weight	·6163	·6283
Standing Height, Weight and Age	·6069	—
Standing Height, Sitting Height and Age	·6006	—
Sitting Height, Chest and Age	·5933	—
Sitting Height, Weight and Age	·5893	—
Sitting Height, Weight and Chest	·5869	·6008
Chest, Weight and Age	·5095	—
Four Variables.		
Vital Capacity from :		
Standing Height, Sitting Height, Chest and Age ...	·6390	
Standing Height, Sitting Height, Chest and Weight ...	·6329	·6423
Standing Height, Weight, Chest and Age	·6325	—
Standing Height, Sitting Height, Weight and Age ...	·6252	—
Sitting Height, Weight, Chest and Age	·6017	—
Five Variables.		
Vital Capacity from :		
Standing Height, Sitting Height, Weight, Chest and Age	·6400	—

* The significance of these values of ρ of the second and higher orders may be judged by comparison with the following root-mean-square values of ρ from independent variables, calculated from Yule's formula $\rho_0 = \frac{(n-1)^{\frac{1}{2}}}{N^{\frac{1}{2}}}$, where n = number of variables and N = number of observations.

		A	*B*
Vital Capacity from	2 other variables	·0402	·0459
,, ,,	3 ,, ,,	·0492	·0562
,, ,,	4 ,, ,,	·0568	·0649
,, ,,	5 ,, ,,	·0636	·0725

instructive results. If the exponential relation be a more adequate expression the facts than a linear relation, it follows that the correlation between y and x^n should be significantly greater than the correlation between x and y, where y is vital capacity and x a physical dimension. We have computed the correlation

TABLE XI.

Equations for finding Vital Capacity from Chest (see Table XIII A).
Different Methods of Fitting $x_2 = ax_1^{\beta}$.

x_2 = Vital Capacity in cubic centimetres. x_1 = Chest in centimetres.

	χ^2	P
(1) Equation obtained by fitting a straight line to the logs $x_2 = 39{\cdot}41684x_1^{1{\cdot}004471}$	39·021	·000977
(2) Equation obtained by Method I applied to (1) $x_2 = 44{\cdot}488878x_1^{1{\cdot}030177}$ ...	33·507	·005045
(3) Equation obtained by Method II $x_2 = 44{\cdot}14914x_1^{1{\cdot}041038}$	33·548	·005015

and correlation ratio on a random sample of 100 fit candidates with the results shown in Table XII. It will be seen that the constants are sensibly equal. This result might have been deduced from the size of the coefficient of variation, since the correlation between y and x^n only differs from that between y and x by terms of the order $\left(\frac{\sigma_y}{y}\right)^2$.

TABLE XII.

100 *Fit Air Force Candidates Chosen at Random.*

	r	η	$\eta^2 - r^2$
Vital Capacity and Weight ...	·4733 ± ·052	·5132 ± ·050	·0394 ± ·027
Vital Capacity and (Weight)$^{0{\cdot}72}$ ...	·4830 ± ·052	·5193 ± ·049	·0364 ± ·026

It has also been suggested that the high variability of the sample population is due to heterogeneity, i e. to a mixture of two populations having different means and standard deviations for the characters considered. A mixed population so constituted would exhibit a greater relative variability than either constituents, it would also in this case, we think, exhibit a higher arithmetical correlation. Hence the correlations within the homogeneous sub-groups should be lower than the values we have obtained and, since the latter are inconsistent with the existence of a relation sufficiently stringent to give much weight to individual predictions, we are not, on the basis of the evidence before us, disposed to attach much importance to the point, although it should receive further consideration. It must also be noticed that the coefficient of variability of vital capacity is sensibly the same in the samples derived from such different populations as those of Schuster, Arnold and ourselves.

Returning to the main problem, we have applied the following experimental test.

All the regression equations of all four orders in both A and B (age not included) have been tested by substituting a random sample of 50 Air Force candidates *not* included in those from which the constants were calculated. The vital capacities of the same 50 were also calculated from each of Dreyer's Tables. The root-mean-square error of these results is shewn in Tables XIII A and B together with the theoretical standard deviations for each regression equation. In every case, the regression equations give a slightly smaller error than Dreyer's formulae. The observed errors of the regression lines are very near the theoretical ones, standing height and sitting height on the whole having more weight than weight or chest, but in one or two cases, owing to errors of sampling, a higher order equation gives a slightly larger observed error than a lower order one. The observed errors from the B equations, as would be expected, are on the whole smaller than those from the corresponding A equations. In both sets if only one variable is used, standing height gives the smallest theoretical error, if two are used, standing height and chest, if three are used, standing height, sitting height and chest The theoretical errors in B range from 565 c.c. to 471 c.c. according to the different variables used to determine the vital capacity, and the corresponding observed mean errors range from 610 c.c. to 490 c c. while those from Dreyer's equation range from 801 c.c. to 586 c.c.

Incidentally if we translate our total regression coefficients into English units to compare with Hutchinson, we find that, in spite of his errors, his final conclusions (except for sitting height) do not differ greatly from those obtained here, e.g.

Height. Hutchinson says an increase of 1 inch in standing gives an increase of 8 cubic inches in vital capacity.

By A an increase of 1 inch in standing height gives an increase of 8·88 cubic inches in vital capacity.

By B an increase of 1 inch in standing height gives an increase of cubic 8·87 inches in vital capacity.

Weight. Hutchinson says an increase of 1 lb. in weight gives an increase of 1 cubic inch in vital capacity.

By A an increase of 1 lb in weight gives an increase of 1·16 cubic inches in vital capacity.

By B an increase of 1 lb. in weight gives an increase of 1·16 cubic inches in vital capacity.

Chest. Hutchinson says there is no direct relation between chest and vital capacity, but if we combine his rates for weight and chest, and weight and vital capacity it follows that an increase of 1 inch in chest gives increase of 10 cubic inches in vital capacity.

TABLE XIII A.

Force Data. *ession Equations. A (All Candidates).*

x_1 = Chest in centimetres.
x_2 = Vital Capacity in cubic centimetres.
x_3 = Weight in kilogrammes.
x_4 = Standing Height in centimetres.
x_5 = Sitting Height in centimetres.

Total St dard Deviation of Vital Capacity $= \sigma_2 = 640{\cdot}69$ c.c.

ital Capacity from :		Vital Capacity Theoretical Standard Deviation c.c.		Square Root of Mean Square Error observed from a random 50: Regression Equation c.c.	Dreyer's Tables c.c.
1st order :					
Standing Height	$_2 = 57{\cdot}239x_4 - 5436{\cdot}92$ ... ... al	$\sigma_{2\cdot4}$	527·36	527	—
Sitting Height	$_2 = 108{\cdot}109x_5 - 5202{\cdot}51$	$\sigma_{2\cdot5}$	544·67	527	586
Weight	$_2 = 41{\cdot}807x_3 + 1917{\cdot}30$	$\sigma_{2\cdot3}$	560·98	561	726
Chest	$_2 = 55{\cdot}784x_1 - 241{\cdot}77$	$\sigma_{2\cdot1}$	583·46	605	801
2nd order :					
Sta ding Height and Chest	$= 33{\cdot}396x_1 + 48{\cdot}682x_4 - 6821{\cdot}39$	$\sigma_{2\cdot14}$	505·92	523	—
Sta ding Height and Weight	$_2 = 21{\cdot}321x_3 + 43{\cdot}674x_4 - 4418{\cdot}10$	$\sigma_{2\cdot34}$	510·52	513	—
Sta ding Height and Sitting Height ...	$_2 = 39{\cdot}336x_4 + 53{\cdot}968x_5 - 7185{\cdot}46$	$\sigma_{2\cdot45}$	512·53	503	—
Sitting Height and Weight	$_2 = 24{\cdot}592x_3 + 77{\cdot}280x_5 - 3975{\cdot}99$	$\sigma_{2\cdot35}$	522·28	510	646
Sitting Height and Chest	$_2 = 34{\cdot}407x_1 + 89{\cdot}005x_5 - 6443{\cdot}38$	$\sigma_{2\cdot15}$	523·04	514	670
Weight and Chest	$= 15{\cdot}701x_1 + 34{\cdot}262x_3 + 1041{\cdot}76$	$\sigma_{2\cdot13}$	558·81	566	760
3rd order :					
Standing Height, Sitting Height and Chest	$_2 = 2\ \ 3x_1 + 35{\cdot}476x_4 + 43{\cdot}117x_5 - 8040{\cdot}82$ ...	$\sigma_{2\cdot145}$	496·47	514	—
Standing Height, Sitting Height and Weight	$_2 = 1\ \ 7x_3 + 32{\cdot}318x_4 + 41{\cdot}933x_5 - 5968{\cdot}49$ ...	$\sigma_{2\cdot345}$	501·94	493	—
Standing Height, Weight and Chest .	$_2 = 2\ \ 7x_1 + 8{\cdot}516x_3 + 45{\cdot}434x_4 - 6063{\cdot}37$ ..	$\sigma_{2\cdot134}$	504·57	517	—
Sitting Height, Weight and Chest	$_2 = 1\ \ 8x_1 + 15{\cdot}092x_3 + 78{\cdot}549x_5 - 5152{\cdot}81$ ...	$\sigma_{2\cdot135}$	518·74	510	683
h order :					
Standing Height, Sitting Height, Weight and Chest	$= 2\ \ 7x_1 + 4{\cdot}721x_3 + 34{\cdot}164x_4 + 41{\cdot}525x_5 - 7575{\cdot}83$	$\sigma_{2\cdot1345}$	496·06	507	—

TABLE XIII B.

Air Force Data. Regression Equations. B (Fit Candidates).

x_1 = Chest in centimetres.
x_2 = Vital Capacity in cubic centimetres.
x_3 = Weight in kilogrammes.
x_4 = Standing Height in centimetres.
x_5 = Sitting Height in centimetres.

Total Standard Deviation of Vital Capacity $= \sigma_2 = 614{\cdot}72$ c.c.

Vital Capacity from:			Vital Capacity Theoretical Standard Deviation c.c.	E[illegible]	Dreyer's Tables c.c.
1st order:					
Standing Height	$x_2 = 57{\cdot}314x_4 - 5588{\cdot}5$[illegible]		[illegible]$\cdot 20$	[illegible]	—
Sitting Height ...	$x_2 = 110{\cdot}251x_5 - 5355{\cdot}29$		3·67	528	586
Weight	$x_2 = 41{\cdot}997x_3 + 1938{\cdot}69$		[illegible]4·23	530	726
Chest	$x_2 = 52{\cdot}122x_1 + 107{\cdot}4$[illegible]		[illegible]4·85	610	801
2nd order:					
Standing Height and Chest	$x_2 = 29{\cdot}429x_1 + 49{\cdot}986x_4 - 6672{\cdot}76$	$\sigma_{2\cdot14}$	481·21	517	—
Sitting Height and Weight ...	$x_2 = 21{\cdot}220x_3 + 44{\cdot}008x_4 - 4435{\cdot}43$	$\sigma_{2\cdot34}$	481·23	512	—
Standing Height and Sitting He[illegible]	$x_2 = 38{\cdot}664x_4 + 53{\cdot}122x_5 - 6953{\cdot}214$	$\sigma_{2\cdot45}$	485·24	500	—
Sitting Height and Weight ...	$x_2 = 23{\cdot}753x_3 + 79{\cdot}952x_5 - 4125{\cdot}94$	$\sigma_{2\cdot35}$	492·83	510	646
Sitting Height and Chest ...	$x_2 = 29{\cdot}434x_1 + 93{\cdot}821x_5 - 6410{\cdot}67$	$\sigma_{2\cdot15}$	497·63	510	670
Weight and Chest	$x_2 = 8{\cdot}251x_1 + 38{\cdot}07$[illegible]$ - 1475{\cdot}70$	$\sigma_{2\cdot13}$	533·60	559	760
3rd order:					
Standing Height, Sitting Height and Chest	$x_2 = 25{\cdot}426x_1 + 35{\cdot}862x_4 + 43{\cdot}068x_5 - 7749{\cdot}10$	$\sigma_{2\cdot145}$	472·76	491	—
Standing Height, Sitting Height and Weigh[illegible]	$x_2 = 17{\cdot}641x_3 + 32{\cdot}093x_4 + 40{\cdot}327x_5 - 5768{\cdot}61$	$\sigma_{2\cdot345}$	474·04	494	—
Standing Height, Weight and Chest	$x_2 = 17{\cdot}277x_1 + 12{\cdot}424x_3 + 45{\cdot}221x_4 - 5579{\cdot}95$	$\sigma_{2\cdot134}$	478·33	500	—
Sitting Height, Weight and Chest ...	$x_2 = 12{\cdot}037x_1 + 17{\cdot}834x_3 + 80{\cdot}782x_5 - 4863{\cdot}83$	$\sigma_{2\cdot135}$	491·41	506	
4th order:					
Standing Height, Sitting Height, Weight and Chest	$x_2 = 16{\cdot}790x_1 + 9{\cdot}136x_3 + 33{\cdot}4136x_4 + 39{\cdot}860x_5 - 6866{\cdot}09$	$\sigma_{2\cdot1345}$		490	

By *A* an increase of 1 inch in chest gives an increase of 8·65 cubic inches in vital capacity.

By *B* an increase of 1 inch in chest gives an increase of 8·08 cubic inches in vital capacity.

Chest on Weight. Hutchinson says an increase of 10 lbs. in weight gives *exact* increase of 1 inch in chest.

By *A* an increase of 10 lbs. in weight gives an increase of ·858 inch (and is very nearly linear).

By *B* an increase of 10 lbs. in weight gives an increase of ·850 inch (and is very nearly linear).

From the results above described we draw the following conclusions.

(1) The relation between vital capacity taken as dependent and weight, height—however measured—and chest circumference taken as independent variables is rather better described by multiple linear regression formulae than by the "rational formulae" of Professor Dreyer, when the data under study are measurements of ostensibly normal adult males.

(2) The systematic use of the method of multiple regression has the advantage over Professor Dreyer's method of bringing out the extent to which the accuracy of an estimate of vital capacity is improved, by the introduction of more and more independent variables.

(3) It appears from such analysis that the variability of the vital capacity of normal male adults is very great, and remains very great, being reduced by less than 25 °/ₒ of its initial value, when as many as four independent variables are used for estimation.

(4) Upon the evidence now available, it does not appear that either biometric or "rational" formulae can deduce from the non-physiological or ordinary anthropometric constants, estimates of normal vital capacity confined within sufficiently narrow limits to possess real prognostic value in individual cases.

(5) The substitution of stem length for height does not improve the accuracy of prediction and, with the minor exceptions noticed above, Hutchinson's method is as good as or better than its recent modifications.

In conclusion, we desire to express our sincere thanks to various friends, including Wing Commander Flack, Professor Karl Pearson and Professor Dreyer for advice and criticism.

It is likewise our painful duty to record that a valued friend and colleague, whose strenuous work powerfully contributed to the execution of what has been a laborious task, has not been spared to collaborate in the published paper. The death of Miss E. C. C. Allen has not only deprived us of a valued friend and colleague, it has deprived biometric science of a brilliant worker.

A STUDY IN HUMAN FERTILITY.

By G. D. MAYNARD, M.D.

THE solution of many social and eugenic problems is dependent on an accurate knowledge of the laws of human fertility, but unfortunately data from which such laws may be directly determined have been scanty, and indirect methods of approaching the problem were of necessity, therefore, often adopted. The New Zealand Census Fertility Report of 1918 added a new and important set of figures, and it is an examination of some of these which is undertaken in this paper.

The following discussion is concerned chiefly with an attempt to discover the optimum age for marriage. The actual fertility-rate for each age at marriage, that is number of living children born to each married woman, is directly available from the tables. From the social point of view it is, no doubt, of equal importance to determine the effective fertility-rate, that is, the average number of children who survive to adult life. Unless, however, children born of women at certain ages are less likely to survive to adult life than those born of either younger or older mothers, the actual fertility-rate will probably be closely correlated with the effective rate, where the grouping is by age of mother at marriage, and class distinctions are not made. The age of maximum fecundity*, clearly, need not be identical with the best age for marriage in order that either the actual or the effective fertility-rate should be a maximum.

Marriage *de facto* is the social relationship which in this connection is of concern to the biologist, but for obvious reasons it is only the *de jure* marriages of which we get direct records. To overcome this difficulty Mr A. O. Powys (*Biometrika*, Vol. I) excluded from his figures women who bore a child within 9 months from the date of the recorded marriage, and called such unions "prejudiced" marriages. The problem he discussed was the "age of maximum fertility in Anglo-Saxon women in Australia," and he found it to be at about 25 years of age. He writes, "They show quite clearly that at any rate for the Anglo-Saxon race the view of Körösi that fertility is greater the younger the wife is not correct. The woman reaches her greatest reproductive vigour between 24·5 and 27 years, according to climate, and the man at about 32 years." I am unable to refer to Körösi's writings, but so far as the above extract is concerned, it seems possible that here two different meanings are attached to the word fertility, and this may explain the difference in views. Mr Powys very definitely states what he means in this sentence by the use of the phrase "her greatest reproductive vigour," but Körösi's view is

* Fecundity is used with the meaning of potential fertility, and fertility the degree of success in child-bearing. The "age of maximum fecundity" would connote the period in the life of a woman when exposure to risk would be most likely to be followed by pregnancy.

not quoted in sufficient detail to make it clear that he was referring to the same matter. It might be true, that all women reach an age of greatest reproductive vigour, say at 25 years, but this would not necessarily give us any clue to fertility in the sense of largest resultant family. In this latter sense of the term, it seems possible that the words accredited to Körösi in the above extract are approximately correct even for groups of Anglo-Saxon women, as the figures studied in this paper suggest.

The New Zealand Census Fertility Report (1918) gives valuable information in regard to this problem, but is necessarily defective in certain respects. While it is possible to reconstruct the tables, so that the number of children born to women at various ages at marriage can be determined, and the number surviving at the date of the Census (1916), clearly no information is available in regard to such proportion of each group as did not survive to the census year. Although, as will be shewn later, European women in New Zealand aged 15 at marriage have the greatest fertility; information as to what percentage of the original group the survivors represent is not available. It is possible that the survivors are a highly selected sample, selected in respect to the character here measured, and such selection may not be uniform with that of women married at other ages. This and other sources of fallacy, referred to later, must be borne in mind when deductions are drawn from the figures recorded.

The distributions given in Table I were compiled by extracting, for example, from the printed Tables arrays of families for women at ages 45, 46, etc. to 65 at date of census, and who had been married 30, 31, etc. years, and in this case giving when summed the distribution according to size of family for women all aged 15 at marriage, and who had lived to complete the average child bearing period, namely, to 45.

On plotting the data thus obtained, it was seen that no simple curve would give a good representation of the distributions of families for given age at marriage, unless some modification was made in the group "no children," and this becomes more and more evident as the distributions consist of women married at higher ages. It is clear that whether or no a *relative* sterility in the male be reflected in the number of children a woman bears, a complete male sterility must affect the zero group, and, therefore, this group will be a measure of both male and female sterility. While, therefore, it is certain that if the husband be sterile the woman to whom he is mated must fall into the "zero" group, it is not equally evident, having regard to the physiology of the human male reproductive process, and the frequent repetition of risk of conception, that relative sterility in the male will be a factor of much importance in modifying the fertility distribution of wives when completed families are considered. If this assumption be correct, then excluding the zero group, the distribution according to size of family is in the main a measure of female fertility, while adventitious factors are undoubtedly introduced into the group "no children." To include this group in the distribution without modification would therefore lead to erroneous values for the constants, if the object aimed at be

to obtain a measure of the fertility of women. Assuming as would seem probable, that the distribution according to size of family (including wives with no children) would be represented by one of the unimodal curves of the Pearson types, it is clear that the number of sterile women as recorded requires adjustment by removing those cases of sterility which are not due to the woman. No direct measure of the reduction could of course be obtained, but by inspection of a graph of the whole distribution an idea could be formed of what might be a reasonable number of zero families so that a simple curve would be a close representation of the data. The zero group was therefore adjusted by inspection and the constants of the distribution obtained from the recorded values of the other groups, plus the assumed value for the zero group. The equation to this curve having been found, and the values of the ordinates corresponding to 1, 2, 3, etc. children in a family calculated, the "goodness of fit" test was applied to all groups excepting the zero group. If the value of P, so obtained, did not give a reasonably satisfactory value, further adjustments to the data for the sterile wives was tried, previous results being used as a guide to subsequent alterations. The equation to the best fitting curve as found by this method was then adopted, and the value of the calculated ordinate for the zero group assumed to be the true measure of female sterility for the distribution of women under consideration. The constants of the distributions so treated are given in Table II, and in addition the mean size of the family calculated from the unadjusted data.

There are other factors which may affect the size of the 'zero' group, and which would not strictly be due to constitutional variation in degrees of female fertility on which in the main the type of the distribution probably depends; for example, certain surgical operations, acquired disease, and of course voluntary prevention of conception, all of which are in relation to a potential female fertility adventitious conditions.

There is little doubt that, in at least some of the distributions, slightly better fitting curves might be obtained by further modification in this group, but I do not think that such alterations are likely to affect the general deductions which can legitimately be drawn from the table as presented.

In the higher age groups, namely marriage at 37 and 40, some modification in the tail of the distribution was also necessary before a reasonable fit could be obtained. The unaltered distributions gave J-shaped curves with impossible starting points. Further justification for such alterations was provided by inspection of the returns for late marriages, as it was evident that misstatements occurred. For example, one woman is recorded as having been married at 47 and having had a family of 10 living children, a most unlikely event. It is probable that either the marriage recorded was a second marriage, or that the date of marriage recorded was merely the legal recognition of a pre-existing state. The earlier the age of marriage the less likely are misstatements of this kind to occur.

An inspection of Table II shews: first, that, within the range studied, whichever series of mean values be adopted, the younger the age at marriage the larger

the family; and secondly, that the type of the fertility curves changes from the symmetrical Type II for ages 15 and 17 to skew curves of Types I, III or VI at the higher ages. The correlation of age at marriage with size of family as obtained from Table I is

$$r = -\cdot 5068$$

and the correlation ratio

$$\eta = \cdot 5114.$$

Thus the regression is not strictly linear, although as seen in the left-hand bottom diagram of Fig. 2, p. 353, the divergency is not great.

That early marriage is detrimental to the woman and results in a restricted family and unhealthy children is a view widely held, although, so far as I can ascertain, one based rather on what are called "general principles" than on ascertained facts. On general principles, however, the reverse might equally be expected, for if it were really detrimental to a race that early conceptions should occur, the age of puberty should have become delayed through the process of evolution. Among the animals, and in some human societies, desire and fulfilment wait only on opportunity, so that it is not unreasonable to expect that the appearance of the sexual passions should coincide with the optimum age for marriage. An experienced stock breeder informs me that if the mating of Friesland heifers be delayed the calves are inferior and the mothers have smaller families, while among Shorthorn cattle early mating is detrimental to both the mother and offspring. If this be so, then it is clear that "general principles" are of little value as a guide in such problems, and that even within the limits of a genus a uniform law may not hold.

The analysis of the figures in the above tables seems to indicate that from the age of 15 onward the postponement of marriage results in a decrease in size of resultant family. In some figures very kindly sent me by Prof. Pearson dealing with similar data from England and Scotland, it appears that the largest families are associated in England with marriage at 16, and in Scotland at 18. The mean family of girls married at 16 in the New Zealand Returns is 8·178, slightly lower than that found for age 15.

Certain possible sources of fallacy, however, must be considered. First, if the girls who marry at the earlier ages are drawn from a super-fertile section of the population a disturbing factor of importance would have been introduced. It is generally believed that the lower social ranks are more fertile, and also marry at an earlier age, than the better educated classes. If the values here found can be accepted at their face value then increase in fertility is closely associated with age at marriage, and the higher fertility as a class characteristic may be more apparent than real. The comparatively undifferentiated character of the New Zealand population would, I think, for general population statistics, reduce such a disturbing factor to a minimum. Secondly, if strong development of sexual desire be correlated with fertility the early marriages may consist of a highly selected group. It is possible that this may be a factor of some importance, although one I do not see how to estimate or to check. Thirdly, if there be a selective mortality among the less fertile of the child-bearing women, those surviving to 45 or over will be

progressively less highly selected the later the age of marriage. If this be so a study of the number of children born within a limited period, after marriage at various ages, might indicate the importance of such a factor.

In Table VI is recorded the distribution of families of women who have completed 15 years of married life, and no more. Three groups are given, age at marriage 15—20; 21—25; and 26—30. Owing to the comparatively small numbers available for treatment in this way, the ages have been grouped so that errors of sampling may be reduced. Table VII gives the constants of these distributions, and it will be seen that the same order is maintained. It would seem, therefore, unlikely that a selective death-rate of the less fertile women is a factor of sufficient importance in any of the groups to disturb the general results obtained.

An inspection of the tables from which the family arrays for age at marriage were derived, brings to light another point of interest, namely, that the older women have on the average the larger families Table III is given as an example, although the groups at all ages at marriage exhibit the same feature.

This table gives the distribution of families of women at ages 45 to 65 at date of census, all of whom were married at 20. Reference to the last column giving the mean family for each array, shews the general trend of the family to increase as the age of the mother increases, and this cannot to any material extent be due to additional births after the age of 45. The correlation of age at census with size of family for this group is

$$r = \cdot 2804 \pm \cdot 0092,$$

and the regression is closely linear,

$$y = \cdot 1743x.$$

Similar values for age 15 at marriage are

$$r = \cdot 1306 \pm \cdot 0385 \quad \text{and} \quad y = \cdot 0973x.$$

For age 25 at marriage,

$$r = 2506 \pm \cdot 0107 \quad \text{and} \quad y = \cdot 1306x.$$

There are at least four possible factors which must be considered in relation to this phenomenon:

(*a*) That the value found for "r" is the measure of the correlation of longevity and fertility.

(*b*) That there has been a progressive increase in prevalence of artificial restriction of families.

(*c*) That a biological decrease in fertility has been in operation, other than an intentional restriction.

(*d*) That improved hygienic conditions leading to a reduced infantile death-rate have lowered the birth-rate.

Professor Pearson* and his co-workers have shewn that, in certain groups of mothers studied by them, fertility was correlated with longevity. The correlation found for mothers in Series I was $r = \cdot 5009$ and in Series II $r = \cdot 2374$. Further, that

* *Proceedings of Royal Society*, Vol. LXVII. p. 159 et seq.

after the age of 50 the regression was approximately linear. These authors write: "fertility is correlated with longevity after the fecund period is passed."

The correlation values as recorded above fall into line with that found for Series II quoted above.

The statistical determination of the extent of an effective artificial restriction in the size of family will be difficult to obtain until the extent of other factors as agents in effecting maternal-fertility have been assessed. The linearity of the regression line of age at census on size of family, not only for age 20 at marriage, but for other ages, renders it, I think, improbable that artificial restriction is a factor of much importance in producing this result.

It is possible that as a result of change in social or other conditions a racial decrease in fertility may be taking place, apart from a conscious restriction of the family in certain classes of the people. If the group, aged 20 at marriage, be divided at age 54 the constants of the distribution of families for the older half of the table resemble those obtained from the whole table for the age of 17 at marriage. There is thus for this subgroup of women an apparent gain of nearly three years, which does, I think, suggest the possibility of a biological change in fertility having taken place. Change of type of the distribution, owing to elimination of the less fertile at the higher ages, would presumably be equally operative throughout both groups, 17 and 20. No doubt a deliberate restriction of the family might produce the same result, provided it be reasonable to assume that effective methods were sufficiently widely known and their use progressively desired by the general population of New Zealand wives, married in 1871 to 1891.

Improved hygienic conditions and care of the baby, if successful in decreasing the infantile mortality, might conceivably be an important factor in reducing the number of children born to a woman. A high infantile mortality in the first months of life reduces appreciably the duration of lactation, and therefore decreases the period of immunity to pregnancy which is usual during lactation. If, however, this were of importance in raising the mean family as determined by the number of children born, its extent would be indicated by comparing the mean number of children surviving infancy for each age of mother at marriage. Tables IV and V give the data in respect of children who survived to date of census, and therefore, in regard to the older mothers, will on the whole represent adults, and thus provide too stringent an elimination for this purpose. Nevertheless, even in these tables the older women are seen to have the larger families. Table IX corresponds to Table III, but records children living at date of census, $r = \cdot 1696 \pm \cdot 0097$ and $y = \cdot 0933x$. It does not seem probable therefore that a hygienic factor is one of much importance from this point of view.

Powys, in the paper already referred to, attempted to estimate the age of greatest fertility for both males and females in the Australian population. The method adopted is not perhaps ideal, but the best available with the data at his disposal. He assumed that the ratio of women who bore children in the 9th to the 12th month after marriage for groups of women married at different ages would be

an index of fertility at these ages. He had therefore to eliminate from his tables all cases of women who bore children before the 9th month, on the very reasonable assumption that in the great majority of such cases conception had antedated marriage. While it is clear that to have included "prejudiced" marriages would have been unsound practice, it is not equally clear that their exclusion did not leave selected groups of the relatively infertile at the earlier ages. Powys writes: "It must be borne in mind that most of the marriages contracted under the age of 20 are compulsory, i.e. were contracted after conception had followed illicit intercourse." If this be true of the Australian figures, it is not unlikely that a considerable proportion of those marriages considered as unprejudiced were nevertheless also compulsory owing to discovery, fear of pregnancy, and so forth, but as pregnancy did not follow they constitute to some extent a selection of the less fertile of the group. Pregnancy in the early months, especially in the unmarried girl, is extremely difficult for the untrained to recognise, and many mistakes would be sure to occur. An unmarried girl who has exposed herself to risk will probably regard herself as pregnant if she has missed a menstrual period, whereas false alarms of this nature are not uncommon. There are other objections that may be raised to these returns which are equally, or nearly equally, applicable to those discussed in this paper. The homogeneity of the population in regard to class fertility must be assumed, but should there be a variable class fertility and also a tendency for different classes to marry at different ages, an additional unmeasured complication will be introduced.

Further it may be doubted whether the 3 months period adopted is entirely satisfactory as a measure of age of greatest fecundity. There are many psychological and other factors which appear to influence the chance of conception taking place in the first few months of marriage, and such factors may not be uniform at all ages so that we may in reality be measuring the relationship of these factors to age, and therefore, a longer period would be preferable in order to obtain the true "age of greatest fertility."

The figures in the New Zealand Report do not enable a table similar to that of Powys to be constructed, but the subject can be approached in a slightly different way. Table VIII gives for marriages at each year of age from 20 to 45, the number of women married, the number who bore no children, one, two or three children, within a period of 24 months. In this way prejudiced marriages can probably be ignored, as so small a proportion (2·4 °/₀) of women bear more than one child within this period, that the results are unlikely to be significantly affected by this factor. Unfortunately ages below 20 were not separated in the returns, and can only be given for the group 15—19. If, at the age of 20 and upwards, prejudiced marriages occur to any considerable extent, and if such premarital conceptions are most frequent amongst the more fertile of the group, a disturbing selection may be present in this table; also if fecundity and social class be correlated, and the mean age of marriage in various classes differs, error will be introduced.

Accepting, however, these returns as they stand, the regression line of age at marriage (from 20 upwards) on percentage of women who have borne a child during

this 24 months period is closely linear; a cubic parabola has also been fitted to the data and approximates to the best fitting straight line.

The group 15—19 indicates a falling off in fecundity, and in this corresponds to figures similarly obtained from English and Scotch data kindly sent me by Professor Pearson. The English figures give highest rate at age 18 and the Scotch at age 19, a rapid fall occurring from these ages to that of marriage at 15 years. The sterility rate has been adopted, that is, the percentage of women who did not bear a child during this period of 24 months is shewn in the chart on the bottom right-hand corner of p. 353. It does not seem probable that either of the possibly disturbing factors mentioned above would entirely mask the true trend of the events or result in a regression so nearly linear.

Two other curves are shewn on this chart, p. 353; the first of these gives the percentage of recorded zero families, for marriage at each year from 15 to 27, and thereafter for every second year. This distribution will represent sterility from whatever cause it may arise, either in the husband or the wife. The distribution was fitted by a cubic parabola to the logarithms of the rates per 10,000.

The remaining curve on this chart is that obtained from what is called earlier in this paper the "theoretical zero group," and it is suggested that these groups represent the sterility due only to the female, and form an integral portion of the fertility curves. These three curves form the converse of the fertility curves. The highest mean family is produced at the earliest recorded ages, and the younger the age at marriage within this range the lower the rate of sterility. The purely female sterility curve rises slowly in the earlier ages at marriage and differs in this respect from the almost linear regression of fertility on age at marriage.

The results of the analysis of these figures are, I think, somewhat unexpected, and either, it is true, that in a comparatively homogeneous population, mainly of British origin, the fertility of marriage is directly related to the age at marriage, and steadily diminishes with increasing age at marriage from 15 years upwards, or the groups recorded are not random samples of the women at these ages. If the latter be the case then the selection must be of sufficient intensity to mask completely the true trend of the events. While this is possible, it does not seem probable, having regard to the character of the regression curves obtained.

If, on the other hand, it be true for the race in general that the earlier the marriage the greater the fertility of the wife, then the disadvantage of a rising mean age of marriage in the socially fitter sections of the population is evident. Moreover, class fertility rates will have little biological meaning unless the mean age at marriage for the class be taken into consideration. Accepting for the moment the above figures at their face value, a class with a mean age at marriage of 17 would have a mean family per married woman of nearly 8 children, as against just under 5 for a class with a mean age at marriage of 25 years, a fall of 38 per cent. in the fertility due only to postponing the age of marriage.

From a physiological point of view, the decreasing fertility with advancing age

at marriage is of interest. As is shewn in Table VII the younger women maintain their superiority when the period considered is fifteen years of married life for each group, and also to a modified extent in Table VIII when the limit is under 24 months. One is tempted to speculate as to whether early pregnancy has some beneficial effect from this point of view on the female constitution, or whether the male secretion contains some hormone which is essential to the full development of the female reproductive powers. Professor A. Thomson, in a different connection, writes:

If this be so, then there is absolute proof that the ejaculate contains other ingredients than those alone concerned with fertilization. May there not be some hormone or endoctrine secretion, call it what you will, which by rapid absorption through the tissues of the female, sets agoing, through the agency of the thyroid, the complex mechanism involved in the elaborate preparation of the sexual system to meet its reproductive obligations?

Possibly the earlier such a process is started the more complete its effect.

Dr John W. Harris, in the John Hopkins Hospital *Bulletin*, January 1922, from a study of 160 confinements in white young primiparæ and 340 young coloured primiparæ, of ages from 12 to 16 years, concludes as follows:

Based upon the study of 500 patients comprised in this report, it seems permissible to conclude that pregnancy and labor are attended by no greater danger to the young primipara than in older women. On the other hand the duration of labor is actually shorter. As our figures show that the size of the children is not inferior to that noted in older women, and that abnormal pelves occur quite as frequently, this result must be attributed to the greater elasticity of the soft parts. Consequently, speaking from a purely obstetrical point of view, the ages under consideration appear to be the optimum time for the occurrence of the first labor.

SUMMARY.

1. That if the fertility data here discussed be reasonably homogeneous, it is probable that in the European population of New Zealand over the age of 15, the younger the wife at marriage the larger will be the mean family of children born alive, unless girls who marry at the earlier ages are drawn from a super-fertile section of the population.

2. That a similar observation is true of the family which survives to adult age.

3. That within the same range of age, namely, 15 and upwards, the percentage of women who bear no children is smaller the younger the age of the wife at marriage.

4. That if fecundity be measured by the percentage of women who bear children within a short period after marriage (24 months) the age of maximum fecundity for New Zealand wives lies probably between the ages 15 to 20.

TABLE I.

Children in Family.

Age at Marriage.	*0*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*N*
15	10	4	8	9	12	19	23	17	20	30	27	31	26	25	11	11	7	5	2	—	—	—	290
17	38	17	41	42	72	99	128	126	125	140	117	112	109	93	63	53	24	13	2	1	1	2	1380
20	175	70	175	258	393	421	465	462	413	428	381	336	237	155	108	70	25	14	3	4	3	—	4421
25	300	100	209	393	433	474	438	354	310	212	165	96	58	31	14	8	4	1	—	1	—	—	3302
30	200	40	174	269	277	266	190	123	69	37	20	10	4	4	1	—	1	—	—	—	—	—	1485
35	180	44	142	155	99	57	24	13	6	3	2	1	1	—	—	—	—	—	—	—	—	—	547
37	195	72	138	106	72	35	18	5	2	1	1	—	—	—	—	—	—	—	—	—	—	—	450
40	223	100	119	59	21	8	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	311

Table giving distribution of families for completed family histories, i e. women who are 45 years of age and over. Distributions from which constants of Table II were obtained. First "*0*" group records actual observation, second "*0*" group number used in calculation. Tail of array at 37, and 40 also modified. At age 37 observations were as above except that there were 4 cases of 8 children and 3 of 9 and 10. At age 40 after group 3 the numbers of cases of families of 4, 5, etc. were recorded as 13, 7, 6, 4, 4, 3, 3, 0, 1, 1, the last being therefore in group 13.

TABLE II.

Constants of Distributions of Table I.

Age at Marriage	Mean with obs. O group	Mean from adjusted O group	Mode			β_2
15	8·2162	8·3862 ± ·1501		3·8132 ± ·1068	·0117	2·3671
17	7·7485	7·8667 ± ·0672		3·7052 ± ·0476	·0105	2·4544
20	6·5075	6·6469 ± ·0346	5·8387	3·4275 ± ·0246	·0835	2·5841
25	4·4888	4·9116 ± ·0329	3·8686	2·8031 ± ·0233	·3220	3·1197
30	3·3277	3·6862 ± ·0375	2·7828	2·1433 ± ·0265	·7107	4·1551
35	1·8430	2·3400 ± ·0479	1·3903	1·6597 ± ·0338	1·7493	5·9451
	1·4802	1·9044 ± ·0477	1·0405	1·4988 ± ·0337	1·0376	4·3427
40	0·8901	1·1350 ± ·0424	0·3054	1·1087 ± ·0300	1·4158	4·7273
		7·4294 ± ·0525	—	3·4521 ± ·0371	·0044	2·5347

		Per cent. of O families as recorded	Per cent. of O families as obt. from equation to curve	Equation to Curve
15	·89	3·39	1·03	$y=27{\cdot}07\left(1+\frac{x^2}{109{\cdot}8791}\right)^{2{\cdot}2399}$
17	·64	2·71	1·23	$y=135{\cdot}8\left(1+\frac{x^2}{123{\cdot}6832}\right)^{2{\cdot}0988}$
20	·10	3·87	1·57	$y=479{\cdot}9\left(1+\frac{x}{7{\cdot}4085}\right)^{2{\cdot}1757}\left(1-\frac{x}{14{\cdot}1347}\right)^{4{\cdot}147}$
	·91	8·56	1·47	$y=470{\cdot}6\left(1+\frac{x}{5{\cdot}2737}\right)^{2{\cdot}0027}\left(1-\frac{x}{20{\cdot}7528}\right)^{10{\cdot}2424}$
30	·97	12·16	2·75	$y=299{\cdot}0\,e^{-1{\cdot}1009x}\left(1+\frac{x}{4{\cdot}1814}\right)^{4{\cdot}6284}$
35	·98	26·16	7·95	$y=y_0\,(x-24{\cdot}8342)^{2{\cdot}4489}\,x^{-34{\cdot}3023}$
37	·97	33·56	17·42	$y=133{\cdot}4\left(1+\frac{x}{1{\cdot}7352}\right)^{1{\cdot}6762}\left(1-\frac{x}{29{\cdot}6928}\right)^{28{\cdot}0837}$
40	·83	53·38	37·90	$y=138{\cdot}0\left(1+\frac{x}{0{\cdot}7442}\right)^{0{\cdot}7202}\left(1-\frac{x}{15{\cdot}2652}\right)^{14{\cdot}7794}$
**20*	·20	3·93	1·22	$y=211{\cdot}4\left(1+\frac{x^2}{129{\cdot}8461}\right)^{3{\cdot}0480}$

* Subgroup of group 20 consisting of women over 54 years of age in 1916.

TABLE

Women age 20 at Marriage.

Number of living children born.

Age of Mother at date of Census.	Number of Marriages	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	Mean
45	319	18	21	22	36	43	39	38	26	25	13	11	15	9	1	1	1	—	—	—	—	5·30
46	308	13	21	23	37	40	29	32	32	25	17	12	10	3	7	2	2	—	2	1	—	5·59
47	219	10	7	17	29	25	29	23	21	15	8	10	15	3	4	1	1	—	—	—	1	5·72
48	341	7	23	31	44	41	40	34	28	23	29	17	8	6	4	4	1	1	—	—	—	5·56
49	227	2	10	16	23	14	22	27	22	23	25	18	10	8	3	2	1	—	—	—	1	6·56
50	386	20	15	34	45	42	43	50	39	24	20	14	19	10	7	2	1	—	—	1	—	5·61
51	190	4	8	7	14	19	30	18	14	27	14	14	11	2	2	4	1	1	—	—	—	6·51
52	276	13	13	11	25	26	30	28	26	28	26	17	13	8	7	4	—	1	—	—	—	6·31
53	245	11	7	18	23	20	19	30	19	27	18	18	19	9	3	3	1	—	—	—	—	6·41
54	242	8	10	12	24	15	31	18	21	20	24	18	13	13	6	6	3	—	—	—	—	6·77
55	202	3	9	9	14	25	24	24	22	21	20	11	7	6	4	3	—	—	—	—	—	6·37
56	212	5	6	9	15	17	22	25	13	20	22	18	12	11	10	5	1	—	—	1	—	7·18
57	178	6	2	4	10	19	20	12	18	18	23	17	11	4	4	4	2	3	—	—	1	7·35
58	197	9	5	5	15	14	22	14	25	24	25	15	8	6	6	3	—	1	—	—	—	6·85
59	149	2	3	7	11	10	13	10	17	14	16	21	12	7	5	—	1	—	—	—	—	7·35
60	277	29	5	18	14	15	22	26	22	23	16	37	13	19	7	7	2	2	—	—	—	7·80
61	121	4	—	2	3	7	3	10	14	15	16	15	10	11	5	4	1	1	—	—	—	8·44
62	100	2	4	3	4	8	6	11	5	9	13	19	4	3	7	1	—	1	—	—	—	7·66
63	129	4	1	5	4	9	8	8	10	17	17	14	14	5	4	4	3	—	1	1	—	8·12
64	102	2	3	2	2	7	6	12	7	17	11	9	2	6	7	7	1	1	—	—	—	8·14
65	106	3	2	3	1	5	7	12	12	13	8	11	11	6	5	3	2	2	—	—	—	8·21
	4526	175	175	258	393	421	465	462	413	428	381	336	237	155	108	70	25	14	3	4	3	6·51

TABLE IV.

Distribution of Families. Mothers over 44 Years of Age at date of Census

Children living at Census Year.

Age at Marriage	*0*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*N*
17	48	25	56	57	103	132	174	144	137	153	113	108	67	52	31	18	3	1	—	1374
20	215	80	234	353	474	536	536	532	454	393	305	217	137	70	34	21	2	1	4	4383
25	331	120	274	455	510	510	462	348	256	144	105	63	26	10	3	1	1	—	—	3288

TABLE V.

Constants of Distributions in Table IV.

Age at Marriage	Mean adjusted *0* group	Mode		β_1	β_2
17	$6{\cdot}6215 \pm {\cdot}0589$	6·2117	$3{\cdot}2334 \pm {\cdot}0416$	0·0214	2·4550
20	$5{\cdot}6610 \pm {\cdot}0303$	4·8646	$2{\cdot}9739 \pm {\cdot}0214$	0·1266	2·7106
	$4{\cdot}3020 \pm {\cdot}0291$	3·3868	$2{\cdot}4755 \pm {\cdot}0206$	0·2949	3·0414

Equation to Curve

17 $$y = 154{\cdot}7\left(1 + \frac{x}{7{\cdot}9084}\right)^{2{\cdot}2776}\left(1 - \frac{x}{11{\cdot}0256}\right)^{3{\cdot}1753}$$

20 $$y = 559{\cdot}3\left(1 + \frac{x}{6{\cdot}3704}\right)^{2{\cdot}4436}\left(1 - \frac{x}{14{\cdot}2676}\right)^{5{\cdot}4727}$$

$$y = 525{\cdot}7\left(1 + \frac{x}{4{\cdot}4965}\right)^{2{\cdot}4050}\left(1 - \frac{x}{16{\cdot}5532}\right)^{8{\cdot}0011}$$

TABLE VI.

Distribution of Size of Family. Duration of Marriage for each age group, 15 *years.*

Number of Children born.

Age at Marriage	*0*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*N*
15--20	42	16	76	145	147	172	178	134	94	67	32	17	8	3	—	—	—	1089
21—25	204	44	231	465	490	436	347	262	164	96	54	15	9	2	—	1	1	2617
26—30	170	40	165	219	230	190	146	100	63	24	16	1	1	1	—	—	—	1196

TABLE VII.

Constants of Distribution in Table VI.

Age at Marriage	Mean adjusted *O* group	Mode			β_2
15—20	4·5592 ± ·0478	3·8767	2·3383 ± ·0338	0·1541	2·7591
21—25	3·9687 ± ·0285	2·9290	2·1607 ± ·0201	0·4311	3·3318
26—30	3·5059 ± ·0398	2·6405	2·0421 ± ·0281	0·3229	2·9908

		Equation to Curve
15—20	·83	$y = 177{\cdot}6\left(1+\frac{x}{4{\cdot}8294}\right)^{2\cdot3831}\left(1-\frac{x}{11{\cdot}7945}\right)^{6}$
21—25		$y = 494{\cdot}0\left(1+\frac{x}{3{\cdot}0991}\right)^{1\cdot5401}\left(1-\frac{x}{14{\cdot}1039}\right)^{7\cdot0302}$
26—30	·92	$y = 230{\cdot}9\left(1+\frac{x}{3{\cdot}2841}\right)^{1\cdot7419}\left(1-\frac{x}{12{\cdot}0571}\right)^{6\cdot3960}$

TABLE VIII.

Women Married One but under Two Years.

Age at Marriage	Number of Marriages	Number of children *0*	*1*	*2*	*3*	% Marriages with no children	Mean Family
15—19	279	70	198	11		25·1	0·789
20	313	71	227	15	—	22·7	0·821
21	376	102	265	9	—	27·1	0·753
22	603	207	379	16	1	34·3	0·687
23	654	244	391	19	—	37·3	0·656
24	700	292	398	10		41·7	0·597
25	689	282	391	16	-	40·9	0·614
26	629	288	321	20		45·8	0·574
27	514	214	284	16	—	41·6	0·615
28	472	208	254	10	—	44·1	0·581
29	452	237	210	5		52·4	0·487
30	311	151	150	9	1	48·6	0·550
31	247	128	115	4	—	51·8	0·498
32	222	120	100	2	—	54·1	0·468
33	197	107	84	5	1	54·4	0·492
34	123	55	66	2	—	44·7	0·569
35	120	68	51	1	-	56·6	0·442
36	106	63	43	—	—	59·4	0·406
37	83	51	31	1		61·4	0·398
38	68	49	16	3	-	72·1	0·324
39	64	49	15	—	—	76·6	0·234
40	60	42	17	1	-	70·0	0·317
41	45	37	8	—	—	82·2	0·178
42	43	35	7	—	1	81·4	0·233
43	19	15	3	-	1	79·0	0·316
44	22	18	4	—	—	81·8	0·182
45	34	32	1	1	—	94·2	0·088
46	29	25	4	—	—	86·3	0·138
47	15	12	3	—	—	80·0	0·200
48	15	13	2	—		86·6	0·133
49	14	13	1	—	—	93·9	0·071
50	12	12	—	-	—	100·0	0·000

TABLE IX.

Women aged 20 at Marriage.

Number of Children living at Census 1916.

Age of Mothe date Census.	Number of Marriages	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	Mean
45	319	21	23	33	44	45	34	45	25	20	11	5	7	6	—	—	—	—	—	4·621
46	308	18	28	23	43	44	33	33	27	21	10	14	7	4	1	1	—	—	1	4·864
47	219	11	11	23	31	30	30	23	19	14	6	9	6	1	4	1	—	—	—	4·982
48	341	13	26	38	54	35	56	33	27	18	16	14	3	5	3		—	—	—	4·783
49	226	4	14	23	24	18	26	28	22	23	20	10	5	5	3	—	–	1	—	5·708
50	385	26	14	50	48	50	57	39	30	25	18	12	7	5	1	2	—	—	1	4·842
51	190	7	10	10	18	25	35	15	18	18	14	9	6	3	—	2	—		–	5·605
52	276	15	17	16	32	29	34	36	25	23	22	9	8	7	2	—	1	—	—	5·460
53	243	12	11	21	27	28	26	27	22	25	20	10	8	4	1	1	—	—	–	5·494
54	242	11	13	19	26	29	25	26	25	16	18	18	9	3	2	2	—	—	—	5·620
55	201	5	9	17	20	31	22	31	23	15	10	9	5	2	2	—	–	–	—	5·418
56	211	6	10	7	23	31	27	19	22	24	14	11	8	5	2	1	—	—	1	5·938
57	178	7	3	8	16	29	13	21	29	19	7	6	9	3	3	4	–	—	1	6·174
58	196	8	6	11	16	23	22	31	27	20	13	13	3	1	1	1	—	—	—	5·776
59	149	2	7	11	12	7	18	17	21	17	14	13	7	3	–	—			—	6·289
60	277	33	16	16	12	29	25	31	27	19	25	25	11	4	1	3	–	–	—	5·599
61	122	5	—	4	6	11	11	15	14	19	15	5	10	4	2	1	–	—		6·951
62	100	2	7	3	8	12	10	11	15	12	12	4	2	1	1	—	—	–	—	5·970
63	128	4	3	6	8	11	8	16	16	17	21	10	2	1	2	2	1	—	–	6·711
64	101	2	3	7	3	9	14	13	12	15	5	5	9	2	2	—	—	—	–	6·515
65	106	3	3	7	3	10	10	22	8	13	14	6	5	1	1	—	–	—	–	6·358
	4518	215	234	353	474	536	536	532	454	393	305	217	137	70	34	21	2	1	4	

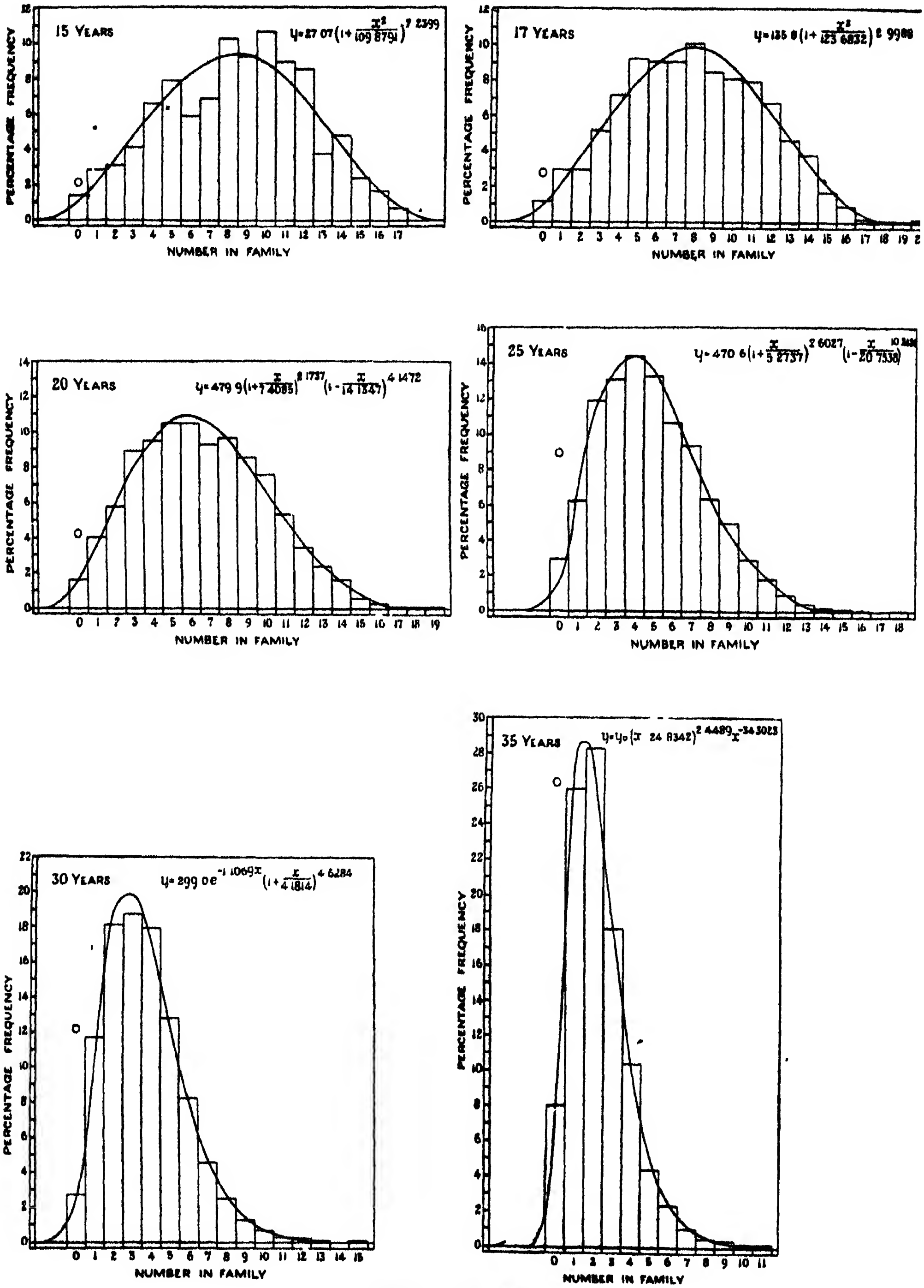
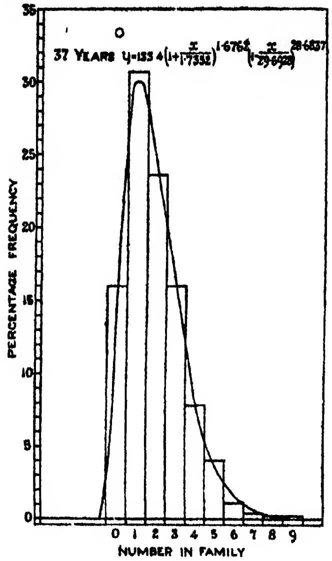
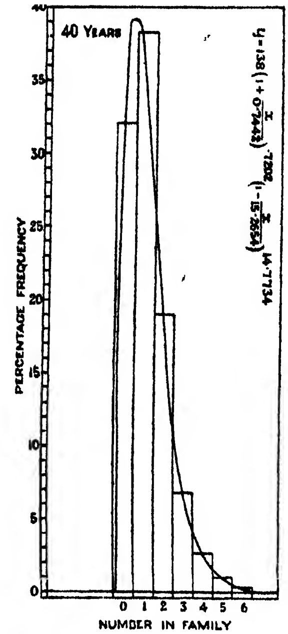


Fig. 1. Age at Marriage of Wife.





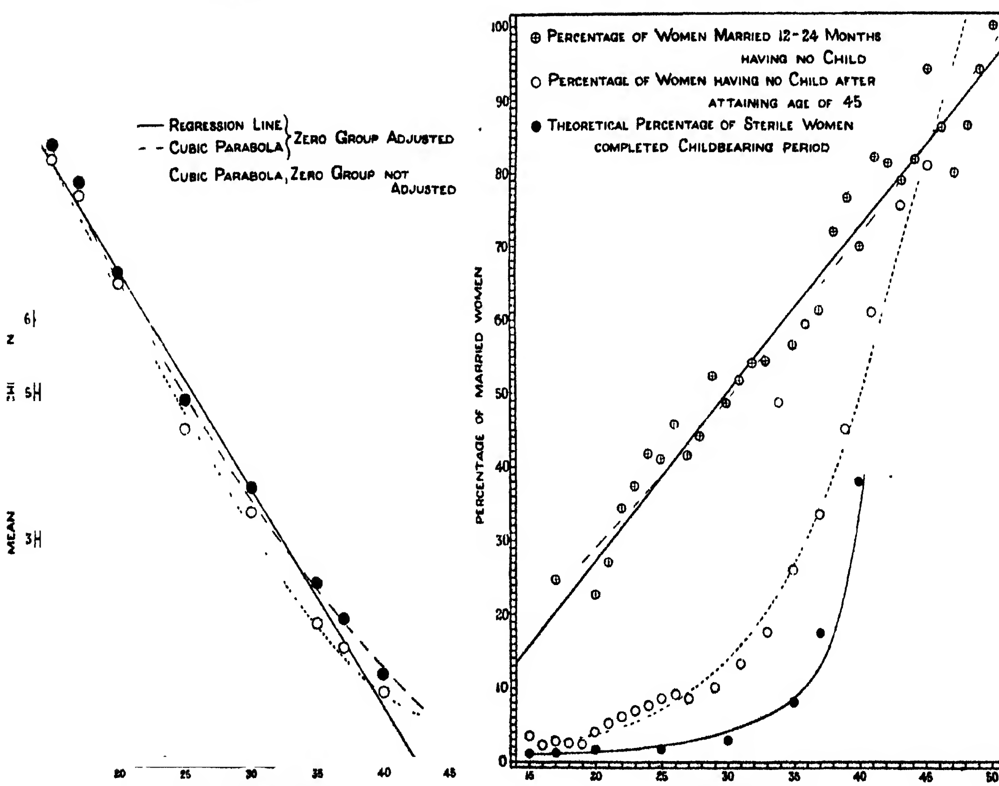


Fig. 2. Age at Marriage of Wife.

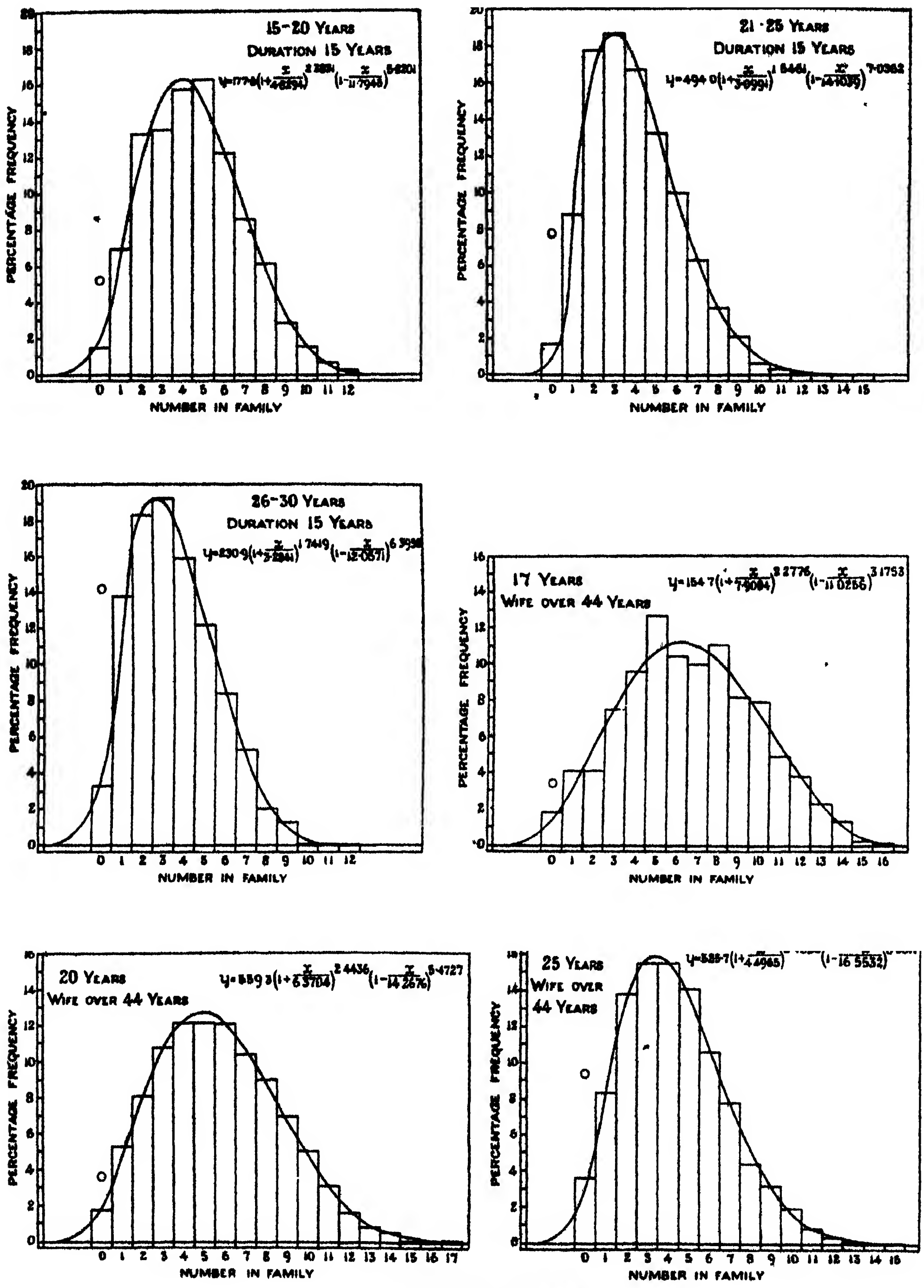


Fig. 3. Age at Marriage of Wife.

ON A CERTAIN SKEW CORRELATION SURFACE.

By E. C. RHODES.

Professor Karl Pearson has discussed in the *Philosophical Transactions of the Royal Society*, and in *Biometrika*, the development of a series of Frequency Curves from a fundamental differential equation which was obtained by considering a certain problem in probability. I have endeavoured, as others have, to obtain from another problem in probability, two general fundamental differential equations, from which to develop a family of Frequency Surfaces, which would extend the Normal Frequency Surface as Professor Pearson's Curves extend the Normal Curve. The two differential equations have been obtained, but so far they have defied general integration. They are of the form

$$\frac{1}{z}\frac{dz}{dx} = \frac{\text{Cubic in } x, y}{\text{Quartic in } x, y},$$

$$\frac{1}{z}\frac{dz}{dy} = \frac{\text{Another Cubic in } x, y}{\text{Same Quartic in } x, y},$$

where x, y are the independent variables and z is the dependent variable or $z\,\delta x\,\delta y$ the frequency between x and $x + \delta x$, y and $y + \delta y$.

Particular simple cases were then considered where the denominator included only terms such as

$$c_0 + c_1 x + c_2 y; \quad c_0 + c_1 x + c_2 y + c_3 xy,$$

and so on, but these did not lead to surfaces which were of any real value; the form of these surfaces, however, indicated that just as the general Type I Curve of Professor Pearson represents a frequency distribution of limited range, so the types of surface we wish to consider will be limited to represent frequency for a certain restricted area of the (xy) plane.

Further when I imposed conditions on the differential equations to make the surface unimodal, I obtained a surface which was of little value. Now if we do not impose such conditions, although we find that the surface has more than one mode, yet actually we can arrange that the surface has only one mode as far as its form for the restricted part of the (xy) plane is concerned; the other modes which occur in that part of the (xy) plane, where the ordinate to the surface is either negative or imaginary, we are not interested in.

The numerical illustration I had before me was a frequency distribution of the Barometric Heights at Southampton and Laudale. These data had been used by Professor Pearson and Dr A. Lee in a paper in *Phil. Trans.* A. Vol. 190 (1897), p. 423. This material was definitely skew and seemed a good example of skew correlation.

A glance at Table I will shew that roughly the frequency is contained between two straight lines. I decided therefore to experiment with the surface

$$z = z_0 e^{-lx-my} (1 - x/a + y/b)^p (1 + x/a' - y/b')^{p'};$$

the two straight lines

$$1 - x/a + y/b = 0,$$
$$1 + x/a' - y/b' = 0,$$

being boundaries to the surface, and the exponential term being introduced to ensure that, for x, y large, z should tend to zero.

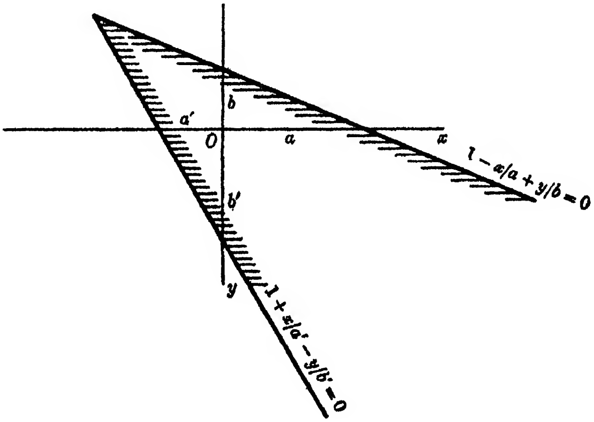


Fig. 1.

Frequency exists in the shaded region.

The equation to this surface may seem artificial, but it is the reasonable type of surface to consider, once the fact that frequency exists only for certain values of x and y is realised, and that in the particular material considered the boundary to the frequency in the (xy) plane consists approximately of two straight lines.

A further point of interest to note is the fact that no effort has been made to use a surface whose arrays shall be Pearson Curves. This question will be referred to later.

Discussion of the surface

$$z = z_0 e^{-lx-my} (1 - x/a + y/b)^p (1 + x/a' - y/b')^{p'}.$$

1. *The Mode.* Differentiating with regard to x and y we have

$$\frac{1}{z}\frac{dz}{dx} = -l - \frac{p/a}{1 - x/a + y/b} + \frac{p'/a'}{1 + x/a' - y/b'} \quad \ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{1}{z}\frac{dz}{dy} = -m + \frac{p/b}{1 - x/a + y/b} - \frac{p'/b'}{1 + x/a' - y/b'} \quad \ldots\ldots\ldots\ldots\ldots(2).$$

The mode is given by

$$-\frac{p/a}{L} + \frac{p'/a'}{M} - l = 0,$$

$$+\frac{p/b}{L} - \frac{p'/b'}{M} - m = 0 \text{ where } L \equiv 1 - x/a + y/b,$$
$$M \equiv 1 + x/a' - y/b',$$

i e by $$\frac{\dfrac{1}{L}}{-mp'/a'-lp'/b'}=\frac{\dfrac{1}{M}}{-lp/b-mp/a}=\frac{1}{pp'(1/ab'-1/a'b)},$$

i.e. by $$1-\frac{x}{a}+\frac{y}{b}=\frac{pp'(1/ab'-1/a'b)}{-(mp'/a'+lp'/b')} \quad\text{.......................(3)},$$

$$1+\frac{x}{a'}-\frac{y}{b'}=\frac{pp'(1/ab'-1/a'b)}{-(lp/b+mp/a)} \quad\text{.......................(4)}.$$

There is therefore one mode to this surface, which is at the origin if

$$\left.\begin{aligned} l &= -p/a+p'/a' \\ m &= p/b-p'/b' \end{aligned}\right\} \quad\text{..............................(5)}.$$

We will assume that these conditions are satisfied and that the mode of the surface is the origin.

2. *Discussion of the arrays of the surface.*

Equations (1) and (2) give us easily

$$\frac{1}{z}\frac{dz}{dx}\frac{1}{b'}+\frac{1}{z}\frac{dz}{dy}\frac{1}{a'}=-\frac{l}{b'}-\frac{m}{a'}-\left(\frac{p/ab'-p/a'b}{1-x/a+y/b}\right),$$

$$\frac{1}{z}\frac{dz}{dx}\frac{1}{b}+\frac{1}{z}\frac{dz}{dy}\frac{1}{a}=-\frac{l}{b}-\frac{m}{a}+\left(\frac{p'/a'b-p'/ab'}{1+x/a'-y/b'}\right).$$

But (3) and (4) shew us that when the origin is the mode,

$$l/b'+m/a'=-p(1/ab'-1/a'b),$$
$$l/b+m/a=-p'(1/ab'-1/a'b).$$

We have therefore, calling

$$\frac{1}{a'b}-\frac{1}{ab'}=X,$$

$$\frac{1}{z}\frac{dz}{dx}\frac{1}{b'}+\frac{1}{z}\frac{dz}{dy}\frac{1}{a'}=-pX+\frac{pX}{1-x/a+y/b}=\frac{pX(x/a-y/b)}{1-x/a+y/b} \quad\text{.........(6)},$$

$$\frac{1}{z}\frac{dz}{dx}\frac{1}{b}+\frac{1}{z}\frac{dz}{dy}\frac{1}{a}=-p'X+\frac{p'X}{1+x/a'-y/b'}=\frac{p'X(-x/a'+y/b')}{1+x/a'-y/b'} \quad\text{......(7)}.$$

Therefore: $\left(\dfrac{1}{b'}\dfrac{dz}{dx}+\dfrac{1}{a'}\dfrac{dz}{dy}\right)(1-x/a+y/b)=pXz\,(x/a-y/b).$

Let us integrate this equation throughout the range of x, say from x_1 to x_2, keeping y constant.

Now $$\int_{x_1}^{x_2}\frac{dz}{dx}\,dx=[z]_{x_1}^{x_2}=0,$$

since z vanishes at the boundary lines of the surface;

$$\int_{x_1}^{x_2}\frac{dz}{dx}\,x\,dx=[zx]_{x_1}^{x_2}-\int_{x_1}^{x_2}z\,dx=-z_y,$$

z_y being the area of the array;

$$\int_{x_1}^{x_2} \frac{dz}{dx} y\,dx = y\,[z]_{x_1}^{x_2} = 0,$$

$$\int_{x_1}^{x_2} \frac{dz}{dy}\,dx = \frac{d}{dy}\int_{x_1}^{x_2} z\,dx = \frac{d}{dy}(z_y),$$

$$\int_{x_1}^{x_2} \frac{dz}{dy} x\,dx = \frac{d}{dy}\int_{x_1}^{x_2} zx\,dx = \frac{d}{dy}(z_y\bar{x}_y),$$

$\bar{x}_y$ being the mean of the array;

$$\int_{x_1}^{x_2} \frac{dz}{dy} y\,dx = y\frac{d}{dy}\int_{x_1}^{x_2} z\,dx = y\frac{d}{dy}(z_y).$$

We have then

$$\frac{1}{b'}\left(+\frac{1}{a}z_y\right) + \frac{1}{a'}\left(\frac{d}{dy}(z_y) - \frac{1}{a}\frac{d}{dy}(z_y\bar{x}_y) + \frac{y}{b}\frac{d}{dy}(z_y)\right) = pX\left(\frac{1}{a}\bar{x}_y - \frac{1}{b}y\right)z_y,$$

i.e. $$\frac{1}{a'z_y}\left(\frac{dz_y}{dy}\right)\left[1 - \frac{\bar{x}_y}{a} + \frac{y}{b}\right] + \frac{1}{ab'} = pX\left(\frac{\bar{x}_y}{a} - \frac{y}{b}\right) + \frac{1}{aa'}\frac{d}{dy}(\bar{x}_y) \quad \text{.........(8).}$$

Similarly equation (7) gives us

$$\frac{1}{az_y}\left(\frac{dz_y}{dy}\right)\left[1 + \frac{\bar{x}_y}{a'} - \frac{y}{b'}\right] - \frac{1}{a'b} = p'X\left(-\frac{\bar{x}_y}{a'} + \frac{y}{b'}\right) - \frac{1}{aa'}\frac{d}{dy}(\bar{x}_y) \quad \text{......(9).}$$

Adding these two equations we obtain

$$\frac{1}{z_y}\frac{dz_y}{dy}\left(\frac{1}{a} + \frac{1}{a'} + yX\right) = X\left(p\left(\frac{\bar{x}_y}{a} - \frac{y}{b}\right) - p'\left(\frac{\bar{x}_y}{a'} - \frac{y}{b'}\right) + 1\right).$$

If we call $$1 - \frac{\bar{x}_y}{a} + \frac{y}{b} = Z,$$

$$1 + \frac{\bar{x}_y}{a'} - \frac{y}{b'} = Y,$$

then $$\frac{Z}{a'} + \frac{Y}{a} = \frac{1}{a} + \frac{1}{a'} + Xy = u \text{ (say)},$$

and the equation above becomes

$$\frac{1}{z_y}\frac{dz_y}{dy}\,.\,u = X\left[p(1 - Z) - p'(Y - 1) + 1\right]$$

$$= X\left[p + p' + 1 - p'Y - p\left(u - \frac{Y}{a}\right)a'\right]$$

$$= X(R - 1 - pua' - a'Yl) \quad \text{..............................(10),}$$

where $p + p' + 2 = R$.

Also we can write equation (9) in the form

$$\frac{1}{az_y}\frac{dz_y}{dy}\,.\,Y = \frac{1}{a'b} + p'X(1 - Y) - \frac{1}{a}\left(\frac{dY}{dy} + \frac{1}{b'}\right)$$

$$= X\left(p' + 1 - p'Y - \frac{1}{a}\frac{dY}{X\,dy}\right).$$

But we have $Xdy = du$; we can therefore write this last equation

$$\frac{1}{a}\frac{Y}{z_y}\frac{dz_y}{du} = p' + 1 - p'Y - \frac{1}{a}\frac{dY}{du} \quad \text{.......................(11)}.$$

Now if we call

$$\frac{1}{z_y}\frac{dz_y}{du} + pa' = \frac{dv}{du} \quad \text{.............................(12)}$$

we can write equation (10):

$$\frac{u}{z_y}\frac{dz_y}{du} = R - 1 - pa'u - a'lY,$$

as

$$u\frac{dv}{du} = R - 1 - a'lY \quad \text{.............................(13)},$$

and equation (11) as

$$\frac{Y}{a}\left(\frac{dv}{du} - pa'\right) = p' + 1 - p'Y - \frac{1}{a}\frac{dY}{du},$$

i.e.

$$\frac{Y}{a}\frac{dv}{du} = p' + 1 + a'Y\left(-\frac{p'}{a'} + \frac{p}{a}\right) - \frac{1}{a}\frac{dY}{du}$$

$$= p' + 1 - a'lY - \frac{1}{a}\frac{dY}{du} \quad \text{..........................(14)}.$$

Write

$$\frac{dv}{du} = \frac{1}{w}\frac{dw}{du} \quad \text{...................................(15)}.$$

Differentiate (13): we get

$$\frac{d}{du}\left(u\frac{dw}{du}\right) = \frac{d}{du}((R-1)w - a'lYw),$$

i.e.

$$u\frac{d^2w}{du^2} + \frac{dw}{du} = (R-1)\frac{dw}{du} - a'lY\frac{dw}{du} - a'lw\frac{dY}{du}$$

$$= (R-1)\frac{dw}{du} - a'lY\frac{dw}{du} - a'la\left[(p'+1)w - a'lYw - \frac{Y}{a}\frac{dw}{du}\right] \quad \text{from (14)}$$

$$= (R-1)\frac{dw}{du} - aa'lw(p' + 1 - a'lY)$$

$$= (R-1)\frac{dw}{du} - aa'lw\left(p' + 1 + \frac{u}{w}\frac{dw}{du} - R + 1\right), \quad \text{from (13)}$$

$$u\frac{d^2w}{du^2} = \frac{dw}{du}(R - 2 - aa'lu) + aa'lpw,$$

or

$$u\frac{d^2w}{du^2} + \frac{dw}{du}(aa'lu - (R-2)) - aa'lpw = 0 \quad \text{..............(16)}.$$

This is the differential equation to obtain w which, from (15) and (12), is given by

$$\log w = \text{const.} + \log z_y + pa'u,$$

or

$$w = z_y e^{pa'u} \times \text{const.}$$

Thus from (16) we can obtain the total of the x-array—in other words this is the differential equation giving the y-margin curve.

The solution to this equation may be obtained in a series of powers of u in the usual way. We obtain

$$w = C\left(u^{R-1} - \frac{aa'ls'}{R} u^R + \frac{(aa'l)^2}{2!} \frac{s'.s'+1}{R.R+1} u^{R+1} - \frac{(aa'l)^3}{3!} \frac{s'.s'+1.s'+2}{R.R+1.R+2} u^{R+2} + \ldots\right) \quad \ldots\ldots\ldots(17),$$

or the curve to the y-margin is

$$z_y = Ce^{\,pa'\left(\frac{1}{a}+\frac{1}{a'}+Xy\right)}\left[\left(\frac{1}{a}+\frac{1}{a'}+Xy\right)^{R-1} - \frac{aa'ls'}{R}\left(\frac{1}{a}+\frac{1}{a'}+Xy\right)^R + \ldots\right],$$

putting in the expression for u, where $s' = p' + 1$.

From (13) we have
$$a'lY = R - 1 - \frac{u}{w}\frac{dw}{du};$$

and since
$$Y = 1 + \frac{x_y}{a'} - \frac{y}{b'},$$

this enables us to obtain the mean (x_y) of an array for any y: in other words this equation is that to the regression line of x on y.

Since

$$\frac{dw}{du} = C\left((R-1)u^{R-2} - \frac{aa'ls'}{R}.R.u^{R-1} + \frac{(aa'l)^2}{2!}\frac{s'.s'+1}{R.R+1}(R+1)u^R - \ldots\right.$$

$$C\frac{(R-1)}{u}\left(u^{R-1} - \frac{aa'l}{R-1}.s'.u^R + \frac{(aa'l)^2}{2!}\frac{s'.s'+1}{R-1.R}.u^{R+1} - \ldots\right),$$

that

$$a'lY = (R-1)\left[1 - \frac{u^{R-1} - \frac{aa'l}{R-1}s'u^R + \frac{(aa'l)^2}{2!}\frac{s'.s'+1}{R-1.R}u^{R+1} - \ldots}{u^{R-1} - \frac{aa'l}{R}s'u^R + \frac{(aa'l)^2}{2!}\frac{s'.s'+1}{R.R+1}u^{R+1} - \ldots}\right]$$

$$= \frac{aa'ls'}{R}\,\frac{\left[u^R - aa'l\frac{s'+1}{R+1}u^{R+1} + \frac{(aa'l)^2}{2!}\frac{s'+1.s'+2}{R+1.R+2}u^{R+2} - \ldots\right]}{u^{R-1} - \frac{aa'l}{R}s'u^R + \frac{(aa'l)^2}{2!}\frac{s'.s'+1}{R.R+1}u^{R+1} - \ldots}.$$

If we call
$$S_{R\,s'} = u^{R-1} - \frac{aa'l}{R}s'u^R + \ldots,$$

$$S_{R+1\,s'+1} = u^R - aa'l\frac{(s'+1)}{R+1}u^{R+1} + \ldots,$$

then
$$Y = \frac{as'}{R}\frac{S_{R+1\,s'+1}}{S_{R\,s'}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

and
$$w = CS_{R\,s'}.$$

We see that the regression line is an infinite series in u or y.

Referring back to equations (13) and (14), we have

$$\frac{Y}{au} = \frac{p'+1 - a'lY - \frac{1}{a}\frac{dY}{du}}{R - 1 - a'lY},$$

i.e.
$$\frac{1}{a}\frac{dY}{du} + a'lY - (p'+1) + ((R-1) - a'lY)\frac{Y}{au} = 0 \quad \ldots\ldots\ldots(19)$$

is the differential equation for Y, the solution to which is not as simple an expression as that already obtained for Y from w.

But we can note an interesting point here. If the regression is linear, equation (18) shews that

$$Y = \frac{as'}{R}(u + A) \text{ where } A \text{ is some constant.}$$

(19) then gives

$$\frac{1}{a}\frac{as'}{R} + a'l\frac{as'}{R}(u+A) - s' + \left[R - 1 - a'l\frac{as'}{R}(u+A)\right]\frac{s'}{R}\left(1 + \frac{A}{u}\right) = 0.$$

This shews that $A = 0$ and $\frac{s'}{R} = 1$; i.e. $s = p + 1 = 0$, and the regression line is

$$1 + \frac{x_y}{a'} - \frac{y}{b'} = a\left(\frac{1}{a} + \frac{1}{a'} + Xy\right),$$

i.e.

$$\frac{x_y}{a'} = \frac{a}{a'} + y\left(\frac{a}{a'b} - \frac{1}{b'} + \frac{1}{b'}\right)$$

$$= \frac{a}{a'} + \frac{ay}{a'b},$$

i.e.

$$\frac{x_y}{a} = 1 + \frac{y}{b}.$$

(13) gives now

$$u\frac{dv}{du} = R - 1 - a'lau,$$

whence

$$v = \text{const.} + (R-1)\log u - aa'lu,$$

$$z_y e^{pa\,u} = Ce^{-aa'lu}u^{R-1},$$

or the margin is a Type III Curve.

We see that the regression line is $\frac{x_y}{a} = 1 + \frac{y}{b}$; i.e. it coincides with one of the bounding lines; the surface has therefore been reduced to a plane, and the correlation is perfect.

Thus we see that this type of surface has only linear regression when the correlation coefficient is unity; so we shall expect that for surfaces of this type the regression will be reasonably linear only when there is high correlation between the two characters. This happens in the case we are considering—the Barometric Height material—for the regression is apparently sensibly linear and the correlation coefficient is + ·78.

We have so far obtained the curve to the y-margin, and the equation to the regression line of x on y. In the same way we may obtain the curve to the x-margin and the regression line of y on x.

3. *The moments of the surface.*

We may write equations (6) and (7)

$$\left(\frac{1}{b'}\frac{dz}{dx} + \frac{1}{a'}\frac{dz}{dy}\right)(1 - x/a + y/b) = pXz\,(x/a - y/b) \quad \text{............(20)},$$

$$\left(\frac{1}{b}\frac{dz}{dx} + \frac{1}{a}\frac{dz}{dy}\right)(1 + x/a' - y/b') = p'Xz\,(-x/a' + y/b') \quad \text{.........(21)}.$$

Integrate equation (20) throughout the whole range of x and y. We have

$$\iint \frac{dz}{dx}\,dx\,dy = \int [z]\,dy = 0,$$

$$\iint \frac{dz}{dx}\,x\,dx\,dy = \int [zx]\,dy - \iint z\,dx\,dy = -N, \text{ the total frequency,}$$

$$\iint \frac{dz}{dx}\,x^m y^n\,dx\,dy = \int [zx^m]\,y^n\,dy - \iint zmx^{m-1}y^n\,dx\,dy = -Nmp'_{m-1,\,n},$$

$$\iint \frac{dz}{dy}\,x^m y^n\,dx\,dy = \int [zy^n]\,x^m\,dx - \iint znx^m y^{n-1}\,dx\,dy = -nNp'_{m,\,n-1},$$

where $p'_{m,n}$ is the m, nth product-moment about an origin not at the mean.

We get from (20)

$$\frac{1}{b'}\left[-\frac{1}{a}(-N)\right] + \frac{1}{a'}\left[\frac{1}{b}(-N)\right] = pX\left(\frac{p'_{10}}{a} - \frac{p'_{01}}{b}\right)N,$$

i.e.
$$pX\left(\frac{p'_{10}}{a} - \frac{p'_{01}}{b}\right) + X = 0,$$

or
$$p\left(\frac{p'_{10}}{a} - \frac{p'_{01}}{b}\right) + 1 = 0.$$

Similarly from (21)

$$p'\left(-\frac{p'_{10}}{a'} + \frac{p'_{01}}{b'}\right) + 1 = 0,$$

whence
$$\left.\begin{aligned} p'_{10} &= \frac{1}{X}\left(\frac{1}{pb'} + \frac{1}{p'b}\right) \\ p'_{01} &= \frac{1}{X}\left(\frac{1}{pa'} + \frac{1}{p'a}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21\,(\mathrm{i})),$$

which gives the distance of the mean from the mode.

Change the origin of the surface from the mode to the mean, i.e. put

$$x = x + p'_{10} \text{ and } y = y + p'_{01}.$$

Then (20) and (21) become

$$\left(\frac{1}{b'}\frac{dz}{dx} + \frac{1}{a'}\frac{dz}{dy}\right)\left(1 - \frac{x}{a} + \frac{y}{b} - \frac{p'_{10}}{a} + \frac{p'_{01}}{b}\right) = pXz\left(\frac{x}{a} - \frac{y}{b} + \frac{p'_{10}}{a} - \frac{p'_{01}}{b}\right),$$

i.e.
$$\left(\frac{1}{b'}\frac{dz}{dx} + \frac{1}{a'}\frac{dz}{dy}\right)\left(1 - \frac{x}{a} + \frac{y}{b} + \frac{1}{p}\right) = pXz\left(\frac{x}{a} - \frac{y}{b} - \frac{1}{p}\right),$$

and
$$\left(\frac{1}{b}\frac{dz}{dx} + \frac{1}{a}\frac{dz}{dy}\right)\left(1 + \frac{x}{a'} - \frac{y}{b'} + \frac{1}{p'}\right) = p'Xz\left(-\frac{x}{a'} + \frac{y}{b'} - \frac{1}{p'}\right),$$

whence multiplying these by $x^m y^n$ and integrating we have

$$\frac{1}{b'}\left(\frac{s}{p}\,mp_{m-1,n} - \frac{1}{a}(m+1)\,p_{m,n} + \frac{m}{b}\,p_{m-1,n+1}\right)$$
$$+ \frac{1}{a'}\left(\frac{s}{p}\,np_{m,n-1} - \frac{1}{a}\,np_{m+1,n-1} + \frac{1}{b}(n+1)\,p_{m,n}\right) = -pX\left(\frac{p_{m+1,n}}{a} - \frac{p_{m,n+1}}{b} - \frac{p_{m,n}}{p}\right) \quad .(22),$$

and

$$\frac{1}{b}\left(\frac{s'}{p'}\,mp_{m-1,n}+\frac{1}{a'}(m+1)\,p_{m,n}-\frac{m}{b'}\,p_{m-1,n+1}\right)$$
$$+\frac{1}{a}\left(\frac{s'}{p'}\,np_{m,n-1}+\frac{1}{a'}\,np_{m+1,n-1}-\frac{1}{b'}(n+1)\,p_{m,n}\right)=-p'X\left(-\frac{p_{m+1,n}}{a'}+\frac{p_{m,n+1}}{b'}-\frac{p_{m,n}}{p'}\right) \quad \text{.........(23),}$$

where $p_{m,n}$ is a product-moment about the mean.

When $m=1$, $n=0$, (22) gives

$$\frac{1}{b'}\frac{s}{p}=-pX\left(\frac{p_{20}}{a}-\frac{p_{11}}{b}\right) \quad \text{..........................(24).}$$

(23) gives

$$\frac{1}{b}\frac{s'}{p'}=-p'X\left(-\frac{p_{20}}{a'}+\frac{p_{11}}{b'}\right) \quad \text{..........................(25),}$$

$$\therefore \quad \frac{s}{b'^2p^2}+\frac{s'}{b^2p'^2}=-Xp_{20}\left(\frac{1}{ab'}-\frac{1}{a'b}\right)=X^2p_{20},$$

$$\therefore \quad p_{20}=\frac{1}{X^2}\left(\frac{s}{p^2b'^2}+\frac{s'}{p'^2b^2}\right) \quad \text{.....................(25 bis),}$$

and

$$p_{02}=\frac{1}{X^2}\left(\frac{s}{p^2a'^2}+\frac{s'}{p'^2a^2}\right) \text{ similarly.}$$

Also

$$\frac{s}{p^2a'b'}+\frac{s'}{p'^2ab}=Xp_{11}\left(\frac{1}{a'b}-\frac{1}{ab'}\right)=X^2p_{11},$$

$$\therefore \quad p_{11}=\frac{1}{X^2}\left(\frac{s}{p^2a'b'}+\frac{s'}{p'^2ab}\right),$$

and we have here equations which give us σ_x, σ_y and r in terms of the constants of the surface.

Note that

$$p_{20}p_{02}-p_{11}^2=\frac{ss'}{X^4}\left(\frac{1}{a^2b'^2}+\frac{1}{a'^2b^2}-\frac{2}{aa'bb'}\right)\frac{1}{p^2p'^2}=\frac{ss'}{p^2p'^2X^2},$$

whence

$$1-r^2=\frac{ss'X^2}{p^2p'^2\left(\frac{s}{p^2b'^2}+\frac{s'}{p'^2b^2}\right)\left(\frac{s}{p^2a'^2}+\frac{s'}{p'^2a^2}\right)} \quad \text{...............(26).}$$

When $m=1$, $n=1$, (22) gives

$$\frac{1}{b'}\left(-\frac{2}{a}p_{11}+\frac{1}{b}p_{02}\right)+\frac{1}{a'}\left(-\frac{1}{a}p_{20}+\frac{2}{b}p_{11}\right)=-pX\left(\frac{p_{21}}{a}-\frac{p_{12}}{b}-\frac{p_{11}}{p}\right),$$

i.e.

$$\frac{p_{02}}{bb'}-\frac{p_{20}}{aa'}+2p_{11}X=-pX\left(\frac{p_{21}}{a}-\frac{p_{12}}{b}\right)+p_{11}X,$$

i.e.

$$\frac{p_{02}}{bb'}-\frac{p_{20}}{aa'}+p_{11}X=-pX\left(\frac{p_{21}}{a}-\frac{p_{12}}{b}\right).$$

(23) gives

$$\frac{1}{b}\left(\frac{2}{a'}p_{11}-\frac{1}{b'}p_{02}\right)+\frac{1}{a}\left(\frac{1}{a'}p_{20}-\frac{2}{b'}p_{11}\right)=-p'X\left(-\frac{p_{21}}{a'}+\frac{p_{12}}{b'}-\frac{p_{11}}{p'}\right),$$

$$-\frac{p_{02}}{bb'}+\frac{p_{20}}{aa'}+p_{11}X=-p'X\left(-\frac{p_{21}}{a'}+\frac{p_{12}}{b'}\right).$$

But

$$\frac{p_{02}}{bb'} - \frac{p_{20}}{aa'} + p_{11}\left(\frac{1}{a'b} - \frac{1}{ab'}\right) = \left(\frac{p_{02}}{b} - \frac{p_{11}}{a}\right)\frac{1}{b'} + \frac{1}{a'}\left(\frac{p_{11}}{b} - \frac{p_{20}}{a}\right) = \frac{s}{a'p^2Xb'} + \frac{s}{b'p^2Xa'},$$

and $$-\frac{p_{02}}{bb'} + \frac{p_{20}}{aa'} + p_{11}X = \frac{1}{b}\left(\frac{p_{11}}{a'} - \frac{p_{02}}{b'}\right) + \frac{1}{a}\left(\frac{p_{20}}{a'} - \frac{p_{11}}{b'}\right) = \frac{s'}{ap'^2Xb} + \frac{s'}{bp'^2Xa}.$$

Therefore $$-pX\left(\frac{p_{21}}{a} - \frac{p_{12}}{b}\right) = \frac{2s}{Xp^2a'b'},$$

$$-p'X\left(-\frac{p_{21}}{a'} + \frac{p_{12}}{b'}\right) = \frac{2s'}{Xp'^2ab},$$

whence $$p_{21}X = \frac{2s}{X^2p^3a'b'^2} + \frac{2s'}{X^2p'^3ab^2},$$

$$\therefore\ p_{21} = \frac{2}{X^3}\left(\frac{s}{p^3a'b'^2} + \frac{s'}{p'^3ab^2}\right) \quad\text{.......................(27)},$$

and $$p_{12} = \frac{2}{X^3}\left(\frac{s}{p^3a'^2b'} + \frac{s'}{p'^3a^2b}\right) \quad\text{.......................(28)}.$$

When $m = 2$, $n = 0$, (22) gives

$$\frac{1}{b'}\left(-\frac{3}{a}p_{20} + \frac{2}{b}p_{11}\right) + \frac{1}{a'}\left(\frac{1}{b}p_{20}\right) = -pX\left(\frac{p_{30}}{a} - \frac{p_{21}}{b} - \frac{p_{20}}{p}\right),$$

i.e. $$\frac{1}{b'}\left(-\frac{2}{a}p_{20} + \frac{2}{b}p_{11}\right) = -pX\left(\frac{p_{30}}{a} - \frac{p_{21}}{b}\right),$$

i.e. $$\frac{1}{b'^2}\frac{2s}{p^2X} = -pX\left(\frac{p_{30}}{a} - \frac{p_{21}}{b}\right) \quad\text{..............(29)}.$$

(23) gives $$\frac{2s'}{b^2p'^2X} = -p'X\left(-\frac{p_{30}}{a'} + \frac{p_{21}}{b'}\right) \quad\text{......(30)}.$$

Therefore $$p_{30}X = \frac{2s'}{b^3p'^3X^2} + \frac{2s}{b'^3p^3X^2},$$

i.e. $$p_{30} = \frac{2}{X^3}\left(\frac{s}{p^3b'^3} + \frac{s'}{p'^3b^3}\right) \quad\text{.........................(31)},$$

and similarly $$p_{03} = \frac{2}{X^3}\left(\frac{s}{p^3a'^3} + \frac{s'}{p'^3a^3}\right) \quad\text{.........................(32)}.$$

When $m = 3$, $n = 0$, (22) gives

$$\frac{1}{b'}\left(\frac{3s}{p}p_{20} - \frac{4}{a}p_{30} + \frac{3}{b}p_{21}\right) + \frac{1}{a'}\frac{1}{b}p_{30} = -pX\left(\frac{p_{40}}{a} - \frac{p_{31}}{b} - \frac{p_{30}}{p}\right).$$

(23) gives

$$\frac{1}{b}\left(\frac{3s'}{p'}p_{20} + \frac{4}{a'}p_{30} - \frac{3}{b'}p_{21}\right) - \frac{1}{a}\frac{1}{b'}p_{30} = -p'X\left(-\frac{p_{40}}{a'} + \frac{p_{31}}{b'} - \frac{p_{30}}{p'}\right).$$

The first equation is

$$\frac{3s}{pb'}p_{20} - \frac{3}{b'}\left(\frac{p_{30}}{a} - \frac{p_{21}}{b}\right) + p_{30}X = -pX\left(\frac{p_{40}}{a} - \frac{p_{31}}{b}\right) + p_{30}X,$$

i.e. $$\frac{3s}{pb'}p_{20}+\frac{3}{b'}\frac{2s}{p^3X^2b'^2}=-pX\left(\frac{p_{40}}{a}-\frac{p_{31}}{b}\right) \text{ from (29).}$$

The second equation is

$$\frac{3s'}{p'b}p_{20}+\frac{3}{b}\left(\frac{p_{30}}{a'}-\frac{p_{21}}{b'}\right)+p_{30}X=-p'X\left(-\frac{p_{40}}{a'}+\frac{p_{31}}{b'}\right)+p_{30}X,$$

i.e. $$\frac{3s'}{p'b}p_{20}+\frac{3}{b}\frac{2s'}{p'^3X^2b^2}=-p'X\left(-\frac{p_{40}}{a'}+\frac{p_{31}}{b'}\right) \text{ from (30).}$$

Eliminate p_{31} from these equations. We have

$$3p_{20}\left(\frac{s}{p^2b'^2}+\frac{s'}{p'^2b^2}\right)+\frac{6}{X^2}\left(\frac{s}{p^4b'^4}+\frac{s'}{p'^4b^4}\right)=X^2p_{40},$$

$$\therefore\ p_{40}-3p_{20}^2=\frac{6}{X^4}\left(\frac{s}{p^4b'^4}+\frac{s'}{p'^4b^4}\right) \text{ from (25 bis)} \quad\ldots\ldots\ldots\ldots(33).$$

Similarly $$p_{04}-3p_{02}^2=\frac{6}{X^4}\left(\frac{s}{p^4a'^4}+\frac{s'}{p'^4a^4}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(34).$$

Now if we call $$\frac{ap'}{a'p}=\theta;\quad \frac{bp'}{b'p}=\phi;\quad \frac{s'}{s}=\lambda,$$

we have $$p_{20}=\frac{s}{X^2p'^2b^2}(\phi^2+\lambda),$$

$$p_{02}=\frac{s}{X^2p'^2a^2}(\theta^2+\lambda),$$

$$p_{30}=\frac{2s}{X^3p'^3b^3}(\phi^3+\lambda),$$

$$p_{03}=\frac{2s}{X^3p'^3a^3}(\theta^3+\lambda),$$

$$p_{11}=\frac{s}{X^2p'^2ab}(\theta\phi+\lambda),$$

$$p_{21}=\frac{2s}{X^3p'^3ab^2}(\theta\phi^2+\lambda),$$

$$p_{12}=\frac{2s}{X^3p'^3a^2b}(\theta^2\phi+\lambda),$$

$$p_{40}-3p_{20}^2=\frac{6s}{X^4p'^4b^4}(\phi^4+\lambda),$$

$$p_{04}-3p_{02}^2=\frac{6s}{X^4p'^4a^4}(\theta^4+\lambda).$$

Calling $$\beta_{10}=\frac{p_{30}^2}{p_{20}^3};\quad \beta_{01}=\frac{p_{03}^2}{p_{02}^3};\quad \beta_{20}=\frac{p_{40}}{p_{20}^2};\quad \beta_{02}=\frac{p_{04}}{p_{02}^2};$$

$$q_{21}=\frac{p_{21}}{p_{20}\sqrt{p_{02}}};\quad q_{12}=\frac{p_{12}}{p_{02}\sqrt{p_{20}}};\quad r=\frac{p_{11}}{\sqrt{p_{20}p_{02}}},$$

we have
$$r = \frac{(\theta\phi + \lambda)}{\sqrt{(\theta^2+\lambda)(\phi^2+\lambda)}} \quad \text{.........(35)},$$

$$\beta_{10} = \frac{4}{s}\frac{(\phi^3+\lambda)^2}{(\phi^2+\lambda)^3}; \quad \beta_{01} = \frac{4}{s}\frac{(\theta^3+\lambda)^2}{(\theta^2+\lambda)^3} \quad \text{.........(36)},$$

$$q_{21} - r\sqrt{\beta_{10}} = \frac{2(\theta\phi^2+\lambda)}{\sqrt{s}(\phi^2+\lambda)\sqrt{\theta^2+\lambda}} - \frac{2(\theta\phi+\lambda)(\phi^3+\lambda)}{\sqrt{s}(\phi^2+\lambda)^2\sqrt{\theta^2+\lambda}}$$
$$= \frac{2}{\sqrt{s}}\frac{(\theta\phi^2+\lambda)(\phi^2+\lambda)-(\theta\phi+\lambda)(\phi^3+\lambda)}{(\phi^2+\lambda)^2\sqrt{\theta^2+\lambda}}$$
$$= \frac{2}{\sqrt{s}}\frac{\lambda(\phi^2+\theta\phi^2-\phi^3-\theta\phi)}{(\phi^2+\lambda)^2\sqrt{\theta^2+\lambda}} = \frac{2\lambda\,\phi(1-\phi)(\phi-\theta)}{\sqrt{s}\,(\phi^2+\lambda)^2\sqrt{\theta^2+\lambda}}.$$

Now
$$\sqrt{1-r^2} = \frac{(\phi-\theta)\sqrt{\lambda}}{\sqrt{(\theta^2+\lambda)(\phi^2+\lambda)}} \text{ from (26) above},$$

$$\therefore \frac{q_{21} - r\sqrt{\beta_{10}}}{\sqrt{1-r^2}} = \frac{2\sqrt{\lambda}}{\sqrt{s}}\frac{\phi(1-\phi)}{(\phi^2+\lambda)^{\frac{3}{2}}}.$$

Similarly
$$\frac{q_{12} - r\sqrt{\beta_{01}}}{\sqrt{1-r^2}} = \frac{2\sqrt{\lambda}}{\sqrt{s}}\frac{\theta(1-\theta)}{(\theta^2+\lambda)^{\frac{3}{2}}} \quad \text{.(37)}.$$

These functions $q_{21} - r\sqrt{\beta_{10}}$; $q_{12} - r\sqrt{\beta_{01}}$ have entered elsewhere.

Again from (33) and (34):
$$\beta_{20} - 3 = \frac{6}{s}\frac{\phi^4+\lambda}{(\phi^2+\lambda)^2}, \qquad \beta_{02} - 3 = \frac{6}{s}\frac{\theta^4+\lambda}{(\theta^2+\lambda)^2}.$$

Let us consider
$$2\beta_{20} - 3\beta_{10} - 6.$$

This is
$$\frac{12}{s(\phi^2+\lambda)^3}\left[(\phi^4+\lambda)(\phi^2+\lambda) - (\phi^3+\lambda)^2\right]$$
$$= \frac{12\lambda\phi^2(1-\phi)^2}{s(\phi^2+\lambda)^3},$$

$$\therefore \frac{q_{21} - r\sqrt{\beta_{10}}}{\sqrt{1-r^2}}\sqrt{3} = \sqrt{2\beta_{20} - 3\beta_{10} - 6}.$$

Similarly
$$\frac{q_{12} - r\sqrt{\beta_{01}}}{\sqrt{1-r^2}}\sqrt{3} = \sqrt{2\beta_{02} - 3\beta_{01} - 6}.$$

These relations therefore hold amongst the moments of the surface, and are of interest because $2\beta_2 - 3\beta_1 - 6 = 0$ is the condition that the margins should be Type III Curves; $q_{21} = r\sqrt{\beta_{10}}$, $q_{12} = r\sqrt{\beta_{01}}$ are the conditions that the regression should be linear; and $r = 1$ means absolute causation. We have pointed out above, that we shall only get approximately linear regression with this type of surface, when there is high correlation between the characters, and we saw that when this happens the margin curves will approach Type III Curves.

4. *To find z_0.*

The total frequency N is $\iint z\,dx\,dy$, where the integration is taken over the whole of the (xy) plane for which frequency exists, i.e. over that part contained between the bounding lines
$$1 - x/a + y/b = 0, \quad 1 + x/a' - y/b' = 0.$$

Let us integrate first with regard to x. We have

$$z_y = \int_{x_1}^{x_2} z_0 e^{-lx-my} (1 - x/a + y/b)^p (1 + x/a' - y/b')^p \, dx,$$

the limits x_1, x_2 being given by

$$1 - \frac{x_2}{a} + \frac{y}{b} = 0, \quad 1 + \frac{x_1}{a'} - \frac{y}{b'} = 0,$$

y being regarded as constant.

Let us change the origin of x to x_1, i.e. replace x by $x + x_1$ in the above; then we have

$$z_y = \int_0^{x_2 - x_1} z_0 e^{-l(x+x_1)-my} \left(\frac{x}{a'}\right)^p \left(1 - \frac{x + x_1}{a} + \frac{y}{b}\right)^p dx.$$

Now
$$x_1 = \left(\frac{y}{b'} - 1\right) a',$$

$$x_2 = \left(1 + \frac{y}{b}\right) a,$$

$$\therefore \quad x_2 - x_1 = a + a' + y\left(\frac{a}{b} - \frac{a'}{b'}\right) = aa'\left(\frac{1}{a} + \frac{1}{a'} + yX\right)$$

$$= aa' . u. \quad \text{(See p. 358.)}$$

And
$$1 - \frac{x_1}{a} + \frac{y}{b} = 1 + \frac{y}{b} - \frac{a'}{a}\left(\frac{y}{b'} - 1\right) = a'\left(\frac{1}{a} + \frac{1}{a'} + yX\right) = a'u.$$

Our integral is therefore

$$z_y = \int_0^{aa'u} z_0 e^{-lx-lx_1-my} \left(\frac{x}{a'}\right)^p \left(a'u - \frac{x}{a}\right)^p dx$$

$$= z_0 e^{-lx_1-my} \int_0^{aa'u} e^{-lx} \left(\frac{x}{a'}\right)^p (a'u)^p \left(1 - \frac{x}{aa'u}\right)^p dx.$$

Call
$$\frac{x}{aa'u} = t.$$

Then
$$z_y = z_0 e^{-lx_1-my} \int_0^1 e^{-laa'ut} (aut)^p (a'u)^p (1-t)^p \, aa'u\,dt,$$

i.e.
$$z_y = z_0 e^{-lx_1-my} a^{p+1} a'^{p+1} u^{p+p+1} \int_0^1 e^{-laa'ut} t^{p'} (1-t)^p \, dt$$

$$= z_0 e^{-lx_1-my} a^{p+1} a'^{p+1} u^{p+p'+1} \int_0^1 t^{p'} (1-t)^p \, dt \left(1 - laa'ut + \frac{(laa'ut)^2}{2!} - \dots\right) dt$$

We can obtain this integral as an infinite series, viz.

$$z_y = z_0 e^{-lx_1-my} a^s a'^s u^{K-1} \Big[B(p+1, p'+1) - laa'u B(p+1, p'+2) + \frac{(laa'u)^2}{2!} B(p+1, p'+3) - \dots \Big],$$

where the B-functions are the ordinary Beta functions.

Therefore

$$z_y = z_0 e^{-lx_1 - my} a^{s} a'^{s} u^{R-1} \left[\frac{\Gamma(s)\Gamma(s')}{\Gamma(s+s')} - laa'u \frac{\Gamma(s)\Gamma(s'+1)}{\Gamma(s+s'+1)} + \frac{(laa'u)^2}{2!} \frac{\Gamma(s)\Gamma(s'+2)}{\Gamma(s+s'+2)} - \dots \right]$$

$$= z_0 e^{-lx_1 - my} a^{s'} a'^{s} u^{R-1} \frac{\Gamma(s)\Gamma(s')}{\Gamma(R)} \left[1 - laa'u \frac{s'}{R} + \frac{(laa'u)^2}{2!} \frac{s'.s'+1}{R.R+1} - \dots \right],$$

which agrees with the result on p. 360, for

$$lx_1 + my = \frac{la'}{b'} y - la' + my$$

$$= y\left(\frac{p}{b} - \frac{p'}{b'} - \frac{pa'}{ba} + \frac{p'}{b'}\right) - la'$$

$$= ypa'X + \frac{pa'}{a} - p'a = pa'\left(u - \frac{1}{a'}\right) - p'$$

$$= pa'u - (R-2).$$

Therefore

$$z_y = z_0 a^{s} a'^{s} \frac{\Gamma(s)\Gamma(s')}{\Gamma(R)} e^{R-2} e^{-pa'u} u^{R-1} \left(1 - laa'u \frac{s'}{R} + \frac{(laa'u)^2}{2!} \frac{s'.s'+1}{R.R+1} - \dots\right).$$

Then we wish to integrate z_y from the value of y given by $1 - x/a + y/b = 0$, $1 + x/a' - y/b' = 0$ to ∞. The value of y where these lines intersect is given by

$$1/a + 1/a' + yX = 0, \quad \text{i.e. } u = 0,$$

and we can therefore effect our summation by considering this integration with regard to u instead of y, viz.:

$$N = \int z_y dy,$$

the limits of integration being as stated above; this gives

$$N = \int_0^\infty z_y \frac{du}{X}$$

$$= \int_0^\infty z_0 a^{s} a'^{s} \frac{\Gamma(s)\Gamma(s')}{X\Gamma(R)} e^{R-2} e^{-pa'u} u^{R-1} \left(1 - laa'u \frac{s'}{R} + \frac{(laa'u)^2}{2!} \frac{s'.s'+1}{R.R+1} - \dots\right) du.$$

Let us write now $pa'u = v$.

$$N = \int_0^\infty z_0 a^{s'} a'^{s} \frac{\Gamma(s)\Gamma(s')}{X\Gamma(R)} \frac{e^{R-2} e^{-v}}{(pa')^R} v^{R-1} \left\{1 - \frac{la}{p}\frac{s'}{R} v + \left(\frac{la}{p}\right)^2 \frac{s'.s'+1}{R.R+1} \frac{v^2}{2!} - \dots\right\} dv.$$

We can write down this integral, it gives

$$N = z_0 a^{s} a'^{s} \frac{\Gamma(s)\Gamma(s')}{X\Gamma(R)} \frac{e^{R-2}}{(pa')^R} \left\{\Gamma(R) - \frac{las'}{pR} \Gamma(R+1) + \left(\frac{la}{p}\right)^2 \frac{s'.s'+1}{R.R+1} \frac{\Gamma(R+2)}{2!} - \dots\right\}$$

$$= z_0 a^{s} a'^{s} \frac{\Gamma(s)\Gamma(s')}{X\Gamma(R)} \frac{e^{R-2}}{(pa')^R} \Gamma(R) \left\{1 - \frac{las'}{p} + \left(\frac{la}{p}\right)^2 \frac{s'.s'+1}{2!} - \dots\right\}$$

$$z_0 a^{s'} a'^{s} \frac{\Gamma(s)\Gamma(s') e^{R-2}}{X(pa')^R} \left(1 + \frac{la}{p}\right)^{-s'},$$

and $$1+\frac{la}{p}-\frac{ap'}{a'p}$$

$$\therefore\quad N = z_0 a^s a'^{s'} \frac{\Gamma(s)\Gamma(s')e^{R-2}}{X(a'p)^s(ap')^{s'}}$$

$$= z_0 e^{R-2}\frac{\Gamma(s)\Gamma(s')}{Xp^s p'^{s'}}.$$

$$\therefore\quad z_0 = \frac{NXp^s p'^{s'}}{e^{R-2}\Gamma(s)\Gamma(s')} \quad\text{.............................(38)},$$

from which we can obtain z_0.

5. We finally require an approximation to the integral $I=\int_{x-h/2}^{x+h/2}\int_{y-k/2}^{y+k/2} z\,dx\,dy$ where $$z = z_0 e^{-lx-my}(1-x/a+y/b)^p(1+x/a'-y/b')^{p'},$$ when h, k are subranges of x and y, small compared with σ_x, σ_y respectively.

We can express $\iint z\,dx\,dy$, where the integration is between the limits shewn above, in the form

$$\iint\left\{z+\xi\frac{dz}{dx}+\eta\frac{dz}{dy}+\frac{1}{2}\left(\xi^2\frac{d^2z}{dx^2}+2\xi\eta\frac{d^2z}{dx\,dy}+\eta^2\frac{d^2z}{dy^2}\right)+\dots\right\}d\xi\,d\eta$$

where z, $\frac{dz}{dx}$, $\frac{dz}{dy}$, etc., in the above are understood to refer to the mid point (x, y) of the region of x and y enclosed in the rectangle $x-\frac{h}{2}$, $y-\frac{k}{2}$ to $x+\frac{h}{2}$, $y+\frac{k}{2}$, over which the summation is to take place, so that ξ, η vary from $-\frac{h}{2}$ to $\frac{h}{2}$; $-\frac{k}{2}$ to $\frac{k}{2}$ respectively.

We have $$\iint \xi^{2r}\eta^{2s}d\xi\,d\eta = \left[\frac{\xi^{2r+1}}{2r+1}\right]_{-h/2}^{h/2}\left[\frac{\eta^{2s+1}}{2s+1}\right]_{-k/2}^{k/2}$$

$$=4\left(\frac{h}{2}\right)^{2r+1}\left(\frac{k}{2}\right)^{2s+1}\frac{1}{(2r+1)(2s+1)}.$$

$$\iint \xi^{2r}\eta^{2s-1}d\xi\,d\eta = \left[\frac{\xi^{2r+1}}{2r+1}\right]_{-h/2}^{h/2}\left[\frac{\eta^{2s}}{2s}\right]_{-k/2}^{k/2}=0.$$

$$\iint \xi^{2r-1}\eta^{2s}d\xi\,d\eta = \left[\frac{\xi^{2r}}{2r}\right]_{-h/2}^{h/2}\left[\frac{\eta^{2s+1}}{2s+1}\right]_{-k/2}^{k/2}=0, \text{ etc.}$$

And we have for the integral I,

$$hk\left[z+\frac{1}{24}\left(h^2\frac{d^2z}{dx^2}+k^2\frac{d^2z}{dy^2}\right)+\frac{1}{192}\left(\frac{h^4}{10}\frac{d^4z}{dx^4}+\frac{h^2k^2}{3}\frac{d^4z}{dx^2dy^2}+\frac{k^4}{10}\frac{d^4z}{dy^4}\right)+\dots\right].$$

Now $$\frac{1}{z}\frac{dz}{dx} = -l-\frac{p/a}{1-x/a+y/b}+\frac{p'/a'}{1+x/a'-y/b'}.$$

$$\therefore\quad \frac{1}{z}\frac{d^2z}{dx^2}-\left(\frac{1}{z}\frac{dz}{dx}\right)^2 = -\frac{p/a^2}{(1-x/a+y/b)^2}-\frac{p'/a'^2}{(1+x/a'-y/b')^2}.$$

Let us call
$$1 - x/a + y/b = L,$$
$$1 + x/a' - y/b' = M.$$

Then
$$\frac{1}{z}\frac{d^2z}{dx^2} + \frac{1}{z}\frac{d^2z}{dy^2} = \left(-l - \frac{p/a}{L} + \frac{p'/a'}{M}\right)^2 + \left(-m + \frac{p/b}{L} - \frac{p'/b'}{M}\right)^2 - \frac{p/a^2}{L^2} - \frac{p'/a'^2}{M^2} - \frac{p/b^2}{L^2} - \frac{p'/b'^2}{M^2}$$
$$= l^2 + m^2 + 2l\left(\frac{p/a}{L} - \frac{p'/a'}{M}\right) - 2m\left(\frac{p/b}{L} - \frac{p'/b'}{M}\right) + \frac{p^2/a^2 + p^2/b^2 - p/a^2 - p/b^2}{L^2}$$
$$+ \frac{p'^2/a'^2 + p'^2/b'^2 - p'/a'^2 - p'/b'^2}{M^2} - \frac{2pp'}{LM}\left(\frac{1}{aa'} + \frac{1}{bb'}\right).$$

But
$$l^2 + m^2 = p^2/a^2 + p^2/b^2 + p'^2/a'^2 + p'^2/b'^2 - 2pp'\,(1/aa' + 1/bb')$$
$$-mp/b + lp/a = -p^2/a^2 + pp'/aa' - p^2/b^2 + pp'/bb'$$
$$-lp'/a' + mp'/b' = +pp'/aa' - p'^2/a'^2 + pp'/bb' - p'^2/b'^2.$$

$$\therefore\quad \frac{1}{z}\frac{d^2z}{dx^2} + \frac{1}{z}\frac{d^2z}{dy^2}$$
$$= \left(\frac{p^2}{a^2} + \frac{p^2}{b^2}\right)\left\{\frac{1}{L^2} - \frac{2}{L} + 1\right\} + \left(\frac{p'^2}{a'^2} + \frac{p'^2}{b'^2}\right)\left\{\frac{1}{M^2} - \frac{2}{M} + 1\right\}$$
$$- 2pp'\left(\frac{1}{aa'} + \frac{1}{bb'}\right)\left\{\frac{1}{LM} - \frac{1}{L} - \frac{1}{M} + 1\right\} - \left(\frac{1}{a^2} + \frac{1}{b^2}\right)\frac{p}{L^2} - \left(\frac{1}{a'^2} + \frac{1}{b'^2}\right)\frac{p'}{M^2}$$
$$= \left(\frac{1}{a^2} + \frac{1}{b^2}\right)\left\{p^2\left(1 - \frac{1}{L}\right)^2 - \frac{p}{L^2}\right\} + \left(\frac{1}{a'^2} + \frac{1}{b'^2}\right)\left\{p'^2\left(1 - \frac{1}{M}\right)^2 - \frac{p'}{M^2}\right\}$$
$$- 2pp'\left(\frac{1}{aa'} + \frac{1}{bb'}\right)\left(1 - \frac{1}{L}\right)\left(1 - \frac{1}{M}\right).$$

Thus the frequency on a square, when $h = k = 1$, is
$$z\left[1 + \frac{1}{24}\left\{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\left(p^2\left(1 - \frac{1}{L}\right)^2 - \frac{p}{L^2}\right) + \left(\frac{1}{a'^2} + \frac{1}{b'^2}\right)\left(p'^2\left(1 - \frac{1}{M}\right)^2 - \frac{p'}{M^2}\right)\right.\right.$$
$$\left.\left. - 2pp'\left(\frac{1}{aa'} + \frac{1}{bb'}\right)\left(1 - \frac{1}{L}\right)\left(1 - \frac{1}{M}\right)\right\}\right],$$
to the first approximation, where z is the middle ordinate, and L, M are the values of $1 - x/a + y/b$, $1 + x/a' - y/b'$ for x, y, the centres of the square, sides $h = k = 1$.

We have here a formula for obtaining a frequency from the middle ordinate in terms of the mid-ordinate with a correction, which will be small, so long as the base (hk) for which we are obtaining the frequency is small compared with the standard deviations of the variables.

6. *Practical work.*

The data I had before me with which to illustrate the results of the preceding theory were the data referred to before, contained in the results of 2922 daily observations of the barometric heights at Southampton and Laudale. The moments of this correlation table had been worked out previously by Professor Pearson in connection with some other work, and I have to thank him for his kindness in permitting me to use these results, which have saved me much labour.

Referring to the Southampton variate as x and the Laudale as y the results are as follows:

$$\sigma_x = 3\cdot250067, \quad \sigma_y = 3\cdot932290, \quad r = \cdot780225,$$
$$\beta_{10} = \cdot171140, \quad \beta_{01} = \cdot224536,$$
$$q_{21} = \cdot286962, \quad q_{12} = \cdot310386.$$

The unit of working is $0''\cdot1$.

The preliminary work is the finding of θ, ϕ, λ, s from equations (35), (36), (37), and as the elimination of all but one of these constants from the equations would mean a stupendous piece of algebra, it was thought better to find them by a process of approximation. As a help to locating the values of θ and ϕ we observe that $l = -\frac{p}{a} + \frac{p'}{a'} = \frac{p}{a}(\theta - 1)$ and $m = \frac{p}{b} - \frac{p'}{b'} = \frac{p}{b}(1-\phi)$; as for large values of x and y the frequency is zero, we shall have l and m positive, and therefore we shall have $\theta > 1 > \phi$. Further, in order to get the four constants λ, θ, ϕ, s we really only need four of the momental constants, r, β_{10}, β_{01}, q_{21}, q_{12}; but since if we use q_{21} and not q_{12} we may be giving greater weight to one part of the table than to another, it is better to form one equation from the two equations (37), and obtain in this way four equations for our constants. We obtained these results:

$$\lambda = 1\cdot155645, \quad s = 11\cdot88185, \quad \theta = 2\cdot831825, \quad \phi = \cdot680439.$$

From these we had

$$p = 10\cdot88185\ , \quad p' = 12\cdot73120,$$
$$a = 17\cdot34984\ , \quad a' = 7\cdot167960,$$
$$b = 8\cdot817063, \quad b' = 15\cdot160064,$$
$$l = 1\cdot148924, \quad m = \cdot394396,$$

and finally from (38) $\qquad z_0 = 64\cdot87113,$

and from [21 (i)] $\qquad \bar{x} = 1\cdot24537, \quad \bar{y} = 1\cdot44314.$

The mean for the Southampton (x) distribution being at $29''\cdot9839$, and that for the Laudale (y) distribution being at $29''\cdot8488$, the mode and origin of our surface is at $30''\cdot10845$ (Southampton), $29''\cdot99315$ (Laudale).

We are now in a position to calculate the mid-ordinates of our theoretical surface corresponding to the observed frequencies, and by using the approximate formula developed above, we can obtain the theoretical frequencies with reasonable accuracy. These are shewn in Table I with the observed frequencies. The fact that the total frequency of the theoretical surface agrees within $0\cdot1\ ^\circ/_\circ$ of the observed frequency is to a certain extent a measure of the reasonableness of this approximation to the individual theoretical frequencies. A comparison of the observed and theoretical frequencies in Table I shews that the surface fits reasonably well the observed data. To shew this fit further the theoretical surface has been constructed in model form from the mid-ordinates. I owe this model to Miss H. G. Jones, whom I would like to thank here for her kindness and patience

TABLE I. *Frequency of Barometric Heights at Southampton and Laudale* (2922 *days observations*). *Actual and Theoretical Frequencies.*

Height of Barometer at Southampton in Inches

Laudale in Inches.

					30·6	30·5	30·4	30·3	30·2	30·1	30·0	29·9	29·8	29·7	29·6	29·5	29·4	29·3	29·2	29·1	29·0	28·9	28·8	28·7	28·6	28·5	
31·0			—		—					—		—							—					—	—	—	
		0·1	—		—					—		—							—					—	—	—	0·1
30·9		—	—		—					—		—							—					—	—	—	—
		0·3	0·4	0·3	0·1					—		—							—					—	—	—	1·2
30·8		1	—	—	—	—	—			—		—							—				—	—	—	—	2
		0·4	·2	1·6	1	0·4	0·1			—		—							—				—	—	—	—	4·8
30·7		1	—	4·5	4	3	1·5		—	—		—							—			—	—	—	—	—	14
		0·3	·7	3·8	4·5	3·1	1·3		0·1	—		—							—			—	—	—	—	—	15·2
30·6		1		7·75	9·75	8·5	3·5		0·25	—		—							—			—	—	—	—	—	36
		0·1	·7	5·2	9·5	9·9	6·6		1·0	0·2		—							—				—	—	—	—	36·8
30·5	—	1	0·5	6	9	17	15·75	6	5·5	1·75	0·5	—							—				—	—	—	—	64
	—	—	0·2	4	2	18·9	17·7	11	3·3	1·8	0·5	0·1							—				—	—	—	—	71·6
30·4	—	—	0·5	5	1	22	24·75	39	19·75	2·25	4·5	—							—				—	—	—	—	41
	—	—	—	2	1	24·1	30·8	26	16·7	7·9	2·8	0·8							—				—	—	—	—	24
30·3	—	—	—	5	1	15·5	24·75	55	50	5·5	6								—				—	—	—	—	00
	—	—	—	1	7	22·0	38·1	42·8	34·4	9·9	9·9								—				—	—	—	—	81·3
30·2	—	—	—		4·25		24	38	·5	66	12	·5	4	2·75			—	—	—	—	—	—	—	—	—	—	63
	—	—	—		3·2		35	51	·8	39	23·2	·0	4	1·4	0·4		—	—	—	—	—	—	—	—	—	—	236·4
30·1	—	—	—		2		21	23	·75	55	46·75		15	3·25			—	—	—	—	—	—	—	—	—	—	60·5
	—	—	—		1·0		25	48	·4	55	39·9	·0	10	4·3			0·1	—	—	—	—	—	—	—	—	—	78·6
30·0	—	—	—		—		11	26	·25	63	58	·5	24	0			1	—	—	—	—	—	—	—	—	—	277·5
	—	—	—		0·3		14	36	·8	63	4·1	·9	20	9·8			0·4	0·1	—	—	—	—	—	—	—	—	300·8
29·9							4	19	31	54·25		57	2 ·5	11		·	0·5	1	—				—	—	—	—	283·5
	—	—	—	—			6·4	21·6	43	59·5	·2	48·	3 ·3	17		3·3	1·2	0·4	0·1				—	—	—	—	302
29·8	—	—	—	—			7	13	24	41·75	·75	48	3 ·75	18	1	6·5	3·5	0·5	—	1	—		—	—	—	—	277·5
	—	—	—	—			2·2	10·8	28	47·	·3	52·	3 ·2	24	1	6·2	2·6	0·9	0·3	0·1	—		—	—	—	—	284
29·7	—	—	—	—			2	8	12	36	·2	47	4 ·25	19	1	7·5	5	5	2·5	0·5			—	—	—	—	245
	—	—	—	—			0·6	4·3	15	31	·8	48·	4 ·8	29	1	·6	4·4	1·8	0·7	0·2	0·1		—	—	—	—	251·6
29·6	—	—	—	—			0·5	4	8·5	6		7	32·75	25	23	2·5	8	3	1	3	—		—	—	—	—	212
	—	—	—	—			0·1	1·4	6·8	8·2		9·6	38·9	31	21	2·6	6·5	3·0	1·2	0·5	0·2		—	—	—	—	212·8
29·5	—	—	—	—				1	3	3		1	45	39	20	0·5	8	2·5	—	0·5	—		—	—	—	—	192
	—	—	—	—				0·3	2·5	8·9		8·2	31·8	28	22	4·3	8·2	4·2	1·9	0·8	0·3	0·1	—	—	—	—	171·6
29·4	—	—	—	—					0·5	5		7·2	24·5	23		3·5	7·2	5·25	3	0·5	1	1	1	—	—	—	135
									·8	3·7		·3	23	23		4·5	9·1	5·1	2·5	1·1	0·5	0·2	0·1	—	—	—	132

Height of Barometer at																											
29·3	—	—	—	—	—	—	—	—	—	2	6	13·25	17·25	18·5	11·5	10·5	7·75	5·75	1	2	1	1	—	—	—	—	97·5
	—	—	—	—	—	—	—	—	*0·2*	*1·3*	*4·6*	*9·9*	*17·1*	*17·6*	*16·7*	*13·2*	*9·1*	*5·5*	*3·0*	*1·5*	*0·7*	*0·3*	*0·1*	—	—	—	*98·8*
29·2	—	—	—	—	—	—	—	—	—	2	5·25	5·25	6	14	13	8	7	3	2	—	1	—	—	—	1	—	67·5
	—	—	—	—	—	—	—	—	—	*0·4*	*1·8*	*4·9*	*8·8*	*11·8*	*12·5*	*11·0*	*8·3*	*5·5*	*3·2*	*1·6*	*0·8*	*0·4*	*0·2*	*0·1*	—	—	*71·3*
29·1	—	—	—	—	—	—	—	—	—	1	0·25	2·75	5·5	10·5	10	12·5	6	9·5	3	—	1	1	—	—	—	—	63
	—	—	—	—	—	—	—	—	—	*0·1*	*0·6*	*2·1*	*4·6*	*7·2*	*8·6*	*8·4*	*6·9*	*4·9*	*3·1*	*1·8*	*0·9*	*0·5*	*0·2*	*0·1*	—	—	*50*
29·0	—	—	—	—	—	—	—	—	—	—	1	—	4	3	5·75	3·75	6·5	6	4·5	2·5	1·5	—	—	—	—	—	38·5
	—	—	—	—	—	—	—	—	—	—	*0·3*	*0·8*	*2·2*	*3·9*	*5·4*	*5·8*	*5·3*	*4·1*	*2·8*	*1·7*	*1·0*	*0·5*	*0·2*	*0·1*	—	—	*34*
28·9	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2·75	5·75	2·5	4·5	4	—	2	1	1	—	—	—	24·5
	—	—	—	—	—	—	—	—	—	—	—	*0·3*	*0·9*	*2·0*	*3·1*	*3·7*	*3·7*	*3·2*	*2·3*	*1·5*	*0·9*	*0·5*	*0·3*	*0·1*	*0·1*	—	*22·6*
28·8	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2·5	2	1·5	0·5	0·5	—	3	—	—	—	—	11
	—	—	—	—	—	—	—	—	—	—	—	—	*0·3*	*0·9*	*1·6*	*2·2*	*2·5*	*2·3*	*1·8*	*1·3*	*0·8*	*0·5*	*0·3*	*0·1*	*0·1*	—	*14·7*
28·7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5	1·5	—	1·5	1	3	—	—	—	—	—	—	7·5
	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	*0·3*	*0·8*	*1·2*	*1·5*	*1·5*	*1·3*	*1·0*	*0·7*	*0·4*	*0·2*	*0·1*	*0·1*	—	*9·2*
28·6	—	—	—	—	—	—	—	—	—	—	—	—	1	—	0·5	1	—	1	—	1	—	—	—	—	—	—	4·5
	—	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	*0·3*	*0·6*	*0·9*	*1·0*	*0·9*	*0·7*	*0·5*	*0·3*	*0·2*	*0·1*	*0·1*	—	*5·7*
28·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	*0·3*	*0·4*	*0·6*	*0·6*	*0·5*	*0·4*	*0·3*	*0·2*	*0·1*	*0·1*	—	*3·6*
28·4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5	—	—	—	1	—	—	1	2·5
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	*0·2*	*0·3*	*0·3*	*0·3*	*0·3*	*0·2*	*0·1*	*0·1*	—	—	*1·9*
28·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5	—	—	—	—	—	—	—	0·5
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	*0·2*	*0·2*	*0·2*	*0·2*	*0·1*	*0·1*	*0·1*	—	—	*1·2*
28·2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	*0·1*	*0·1*	*0·1*	*0·1*	*0·1*	—	—	—	*0·6*
28·1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	*0·1*	*0·1*	*0·1*	—	—	—	—	*0·4*
28·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
27·9	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Totals	1	4	4	30·5	52·5	107·5	140·5	237	315	395·5	382·5	339·5	288	201	150·5	98·5	65	50	23·5	15·5	7·5	7	3	—	1	2	2922
	0·1	*1·2*	*5·6*	*19·3*	*50·7*	*105·1*	*178·8*	*257·9*	*322*	*359·9*	*359·4*	*327·5*	*275·3*	*215·1*	*157·7*	*108·9*	*71·4*	*44·7*	*26·4*	*15·0*	*8·5*	*4·5*	*2·3*	*1·0*	*0·5*	—	*2918·8*

Theoretical frequencies in italics.

and good-will in undertaking such a hard task. As the observed data had been shown in model form many years ago by Professor Pearson*, we are enabled to compare the two, as we can compare data involving only one variable. The accompanying plate shews the two models and a "composite" photograph of them superposed. Finally we can obtain a further comparison of the theoretical with the observed by means of the Goodness of Fit Test. Table II shews the marginal arrays compared, with fairly good results.

When the Laudale and Southampton marginal totals are fitted by the usual four moment method, the theoretical curves when compared with the observations by means of the Goodness of Fit Test give P's of ·62 and ·59 respectively. I am indebted for these figures to Mr E. S. Pearson.

TABLE II.

Comparison of Actual and Theoretical Frequencies of Barometric Heights at Southampton and Laudale, from the Marginal Totals of Table I.

Laudale.

Barometric Height	Observed	Theoretical
Above 30·6	16	21·3
30·6	36	36·8
30·5	64	71·6
30·4	141	124·0
30·3	200	181·3
30·2	263	236·4
30·1	260·5	278·6
30·0	277·5	300·8
29·9	283·5	302·0
29·8	277·5	284·0
29·7	245	251·6
29·6	212	212·8
29·5	192	171·6
29·4	135	132·0
29·3	97·5	98·8
29·2	67·5	71·3
29·1	63	50·0
29·0	38·5	34·0
28·9	24·5	22·6
28·8	11	14·7
Below 28·8	17	22·6
Totals	2922	2918·8

$P = ·4$.

Southampton.

Barometric Height	Observed	Theoretical
Above 30·7	9	6·9
30·7	30·5	19·3
30·6	52·5	50·7
30·5	107·5	105·1
30·4	140·5	178·8
30·3	237	257·9
30·2	315	322·0
30·1	395·5	359·9
30·0	382·5	359·4
29·9	339·5	327·5
29·8	288	275·3
29·7	201	215·1
29·6	150·5	157·7
29·5	98·5	108·9
29·4	65	71·4
29·3	50	44·7
29·2	23·5	26·4
29·1	15·5	15·0
Below 29·1	20·5	16·8
Totals	2922	2918·8

$P = ·2$.

To get the comparison between the theoretical and observed data for the whole of the surface we had to group some of the cell-contents together, for if we had used the χ^2 test in the material as shewn in Table I we should have about 170 groups. In this connection we may recall an observation by Professor Pearson in

* [1897. It is a "pseudo-frequency surface," in which frequencies have been used as ordinates, instead of shewing them by the more recent and reasonable method of briquettes. Ed.]

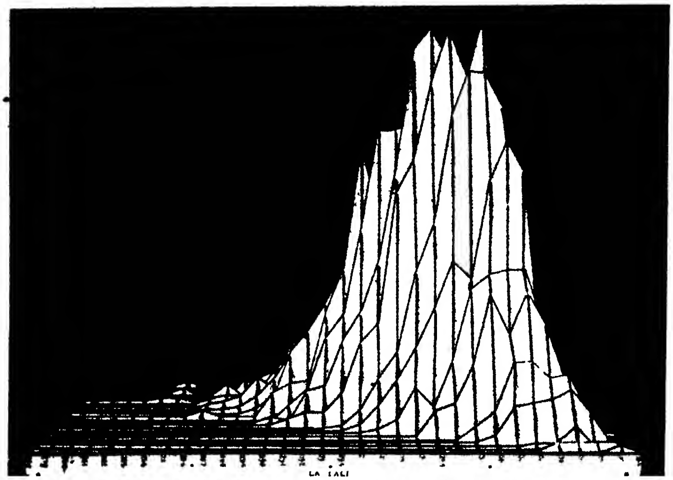


Fig 1.

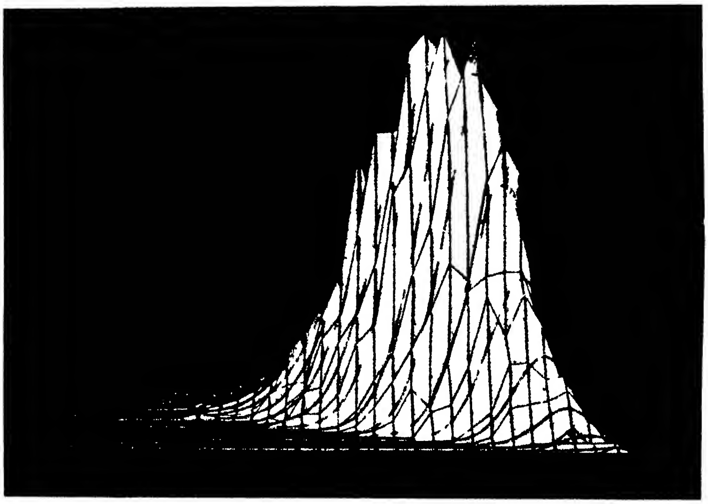


Fig 2.

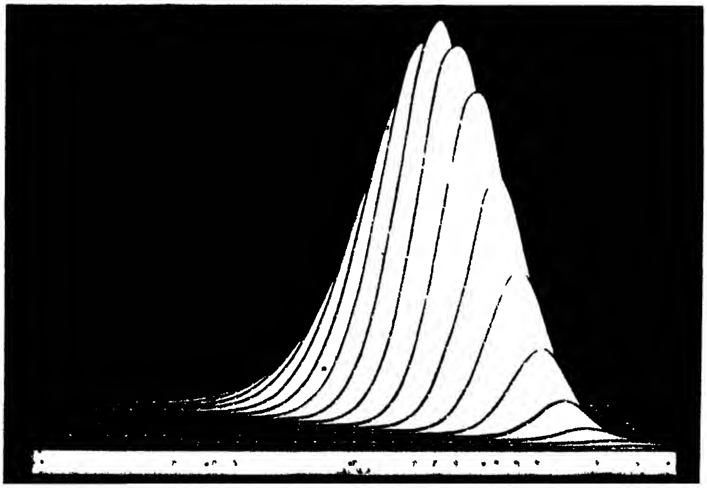


Fig. 3.

his original paper on the Goodness of Fit Test (*Phil. Mag.* Vol. L. (1900), p. 157*) to the effect that one should take a moderate number of groups in using the Test, otherwise when the grouping is too fine the value of χ^2 becomes imponderable and cannot be used to give any result, while when the grouping is too broad, until finally there is only one group, the value of χ^2 becomes zero and the fit is perfect. I have endeavoured to perform the necessary grouping in as systematic manner as possible as shewn by the thick lines in Table I. The results of the grouping are shewn in Table III where there are 24 groups. The resulting P is ·04, which is

TABLE III.

(*Obtained from Table I.*) *Shewing Actual and Theoretical Frequencies with Wider Grouping.*

Southampton.

	30·7—	30·6—30·4	30·3—30·1	30·0—29·8	29·7—29·5	29·4—29·2	29·1—
30·6—	19·25 *10·9*	30·25 *36·6*					
30·5—30·3		171·5 *191·7*	218·5 *170·4*	17·25 *19 1*			
30·2—30·0		118·5 *115·5*	432·5 *462·8*	247·0 *228·9*	20·25 *24·1*		
29·9—29 7			240·5 *262·0*	422·75 *422·5*	106·25 *130·1*		
29·6—29·4			51·5 *42·7*	255·5 *239·5*	187·0 *188·7*	64·5 *58·8*	
29·3- 29·1				72·5 *59·2*	108·5 *107 0*	45·0 *49·5*	8·0 *9·2*
29·0 —28·8					29·0 *32·4*	36·5 *38·8*	11·5 *10·0*
28·7—							8·0 *8·4*

(24 groups.) $P = {\cdot}04$. Theoretical figures in italics.

not as good a result as had been looked for, but which shews that the theoretical surface, even from the point of view of the χ^2 test, is a fair fit to the observed table of frequency.

7. An effort has been made here to find a skew surface which would fit reasonably well certain data, and to a certain extent the form of the surface was guessed at from the given distribution. For purposes of calculation it was needful that

* [The reference appears to be to p. 160, but surely the warning there refers to grouping in such fine intervals that the distribution of the cell-content follows a Poisson's series and not a normal curve, i.e. to cells with a few units in them? Otherwise the number of groups need not be small. ED.]

the required theoretical surface should have as simple an equation as possible. The equation to the surface we have considered is reasonably simple, and although it might be asserted that it is empirical, it appears to fit fairly well the observed data. The form assumed for the surface is not however entirely empirical, for it can be obtained as a special case from the general differential equations obtained from a fundamental theorem in probability, concerned with the problem of the drawing of two samples from a certain population, where correlation arises as a result of making these two samples have a part which is common to both. This method of trying to explain correlation in terms of chance problems, by expanding Bayes' Theorem into two dimensions, is itself purely arbitrary, but it seems the logical step and leads to differential equations which appear to be the type of differential equations we require for the two-dimensional surface. Having obtained the type of differential equation by this method, it seems preferable, at any rate in the beginning of such work, to experiment with different kinds of differential equations which can be immediately obtained from the general differential equations, but which do not necessarily correspond directly to a particular chance problem.

This appears to be the most useful course to pursue, as a general solution to our equations seems impossible at first sight, and as a treatment of more simple equations leads to results which are of interest and involves methods which may be used later in the more general treatment. For instance at first I had in view, as the method of obtaining the moments of the surface, the consideration of such an integral as $\iint \frac{d^2z}{dx\,dy} x^m y^n\, dx\,dy$, but in the work in connection with this paper I saw that I could get the results more easily by considering such integrals as $\iint \frac{dz}{dx} x^m y^n\, dx\,dy$ and $\iint \frac{dz}{dy} x^m y^n\, dx\,dy$; further I saw the possibilities which lay before me in treating such integrals as $\int \frac{dz}{dx} x\,dx$, $\int \frac{dz}{dy} y\,dx$ etc., when considering the array totals and the means of the arrays. From these points of view I think that the foregoing treatment of the skew surface justifies itself, even though we cannot find any physical counterpart to the two straight lines which I have taken as the bounds to the surface.

I have discussed elsewhere the problem of finding a skew surface which has Pearson Curves as its arrays, and have shewn that such surfaces, except for particular forms, are symmetrical. The papers dealing with this problem have not yet been published.

In the course of the work on this subject of skew-correlation surfaces, I have come to the following conclusions:

(1) That we must consider the problem of fitting to observational data surfaces which are bimodal, or multimodal, in form, but unimodal in fact. It will be seen that the array curves of the surface discussed in this paper are bimodal.

(2) That the type of regression line which appears in skew correlation is only linear in very particular cases, and generally is of the form given by the equation

$$\frac{\bar{x}_y - \bar{x}}{\sigma_x} = \text{infinite series in } \left(\frac{\bar{y}_x - \bar{y}}{\sigma_y}\right),$$

which may approximate, with given data, to a straight line, a parabola, a cubic etc.; but which, with simpler equations, may actually be a hyperbola of the form

$$\left(\frac{\bar{x}_y - \bar{x}}{\sigma_x} - \text{constant}\right)\left(\frac{\bar{y}_x - \bar{y}}{\sigma_y} - \text{constant}\right) = \text{constant},$$

or a cubic of the form

$$\left(\frac{\bar{x}_y - \bar{x}}{\sigma_x}\right)\left[\left(\frac{\bar{y}_x - \bar{y}}{\sigma_y}\right)^2 + \text{const.}\left(\frac{\bar{y}_x - \bar{y}}{\sigma_y}\right) + \text{const.}\right] = \text{const.} + \text{const.}\left(\frac{\bar{y}_x - \bar{y}}{\sigma_y}\right).$$

This "cubical regression" in the usual sense of the term is merely a short way of saying "the best fitting cubic to the regression line," which is really of the form $\frac{\bar{x}_y - \bar{x}}{\sigma_x} = \text{infinite series in } \left(\frac{\bar{y}_x - \bar{y}}{\sigma_y}\right)$, but we may have real cubical regression, the cubic being of the form $x(ay^2 + by + c) = A + By$, or some such equation.

(3) The only homoscedastic system with linear regression is the Gaussian surface. I do not know if this has been shewn to be the case, but I feel sure that this is so.

A SUMMARY OF THE PRESENT POSITION WITH REGARD TO THE INHERITANCE OF INTELLIGENCE*.

By ETHEL M. ELDERTON, Galton Fellow, University of London.

INTELLIGENCE is defined in the Oxford Dictionary as meaning primarily "the faculty of understanding" and it is in that sense that I propose to use the word. Secondary meanings for the word intelligence have grown up, meanings that have rather the sense of extent of knowledge, and it is that idea that we must banish from our minds. People with good intelligence will be capable of acquiring information readily and will assimilate it; but it is possible to be a regular repository of information and yet to be singularly lacking in the power to use it; such power lies in intelligence, that faculty of using one's existing information to arrive at a right decision in some new field of thought or at a reasoned course of action under unusual circumstances, and it is this power which will make the difference between a well informed intelligent person and a well informed unintelligent one. We shall expect a correlation between intelligence and acquired knowledge but not by any means necessarily a very high one. Is this faculty of understanding inherited or is it largely, if not wholly, the product of opportunity, of environment and of education? No one would suggest that all the children in one school are equally intelligent but are the variations we see due to differences in natural ability or to a different home environment, or to an unsuitable educational environment, or are the variations due chiefly to one of such causes, while the others are of some but of much less importance?

Of recent years education has received much attention, new systems have been tried, conference has followed conference and the child normal and abnormal is constantly discussed, and one of the questions before us is whether the ideal system of education can create or even increase intelligence, or whether it can only give to child or man the material for his intelligence to play upon. The advocates of any new system of education or of any thing else new (even of Eugenics) are sometimes inclined to attribute possibilities to their new theories which are unlikely to be realized. No new scheme can be started without a great deal of enthusiasm on the part of the starters, enthusiasts must be whole hearted supporters of their theories and naturally claims will sometimes be advanced beyond their legitimate boundaries. If we find that this faculty of understanding is innate it does not for a moment follow that education is useless or that the form it takes matters not at all. The sharper the tool is ground the greater the advantage to those who can make use of it.

* Being a lecture delivered at the Galton Laboratory of National Eugenics.

The study of the inheritance of intelligence appealed very strongly to Francis Galton; it was the cheerful side of Eugenics, the inheritance of marked ability and of the best characteristics of the race rather than of the worse aspects of humanity, that chiefly attracted his attention. Galton has stated that it was the publication of the *Origin of Species* in 1859 that turned his thoughts to the question of heredity in man. Current views of inheritance at that time were, he said, very vague and contradictory; generally speaking most authors agreed that all bodily and some mental qualities were inherited by brutes but refused to believe the same of man. Even the word heredity in those days was considered fanciful and unusual and Galton was chaffed by a friend for adopting it from the French.

If we imagine individuals arranged in order of their intelligence as they can be in order of their height or any other characteristic that can be measured numerically, then we may say that it was the inheritance of giant intellect, of the intellect of the foremost man in 4000 as Galton says in *Hereditary Genius*, that chiefly appealed to him. He was not content that a man should be a Fellow of the Royal Society only, but he must be something more even than that, for he must have some additional title to fame before Galton put him in the first rank of English Men of Science. To show how much the cheerful side of Eugenics appealed to Francis Galton I need only remind the reader of his chief works, *Hereditary Genius*, 1869, *English Men of Science*, 1874, and to a lesser extent *Human Faculty*, 1883, and *Natural Inheritance*, 1889. The first work he suggested to the first Galton Fellow was a book on Noteworthy Families followed almost at once by a study of the Inheritance of Ability from the Oxford Class Lists It may be urged that Galton proved that the highest form of intelligence was inherited, but I feel sure that he thought that further investigations of a more definite and numerical nature were required, that the study must be extended to every grade of intelligence and that even if we showed to his and our satisfaction that inheritance of intelligence was of vital importance, we still had to investigate whether such inheritance could be largely modified by environment, either physical or educational.

Before the publication of the first papers of the first Galton Fellow another attempt had been made to measure the inheritance of intelligence in what we can regard as ordinary individuals. This was organized by Professor Pearson and the data were collected from pairs of brothers and sisters in schools; the results were first given in the 4th Huxley Lecture and were re-published in 1904 in *Biometrika*, Vol. III. p. 131 et seq. The material was obtained from very many schools and provided information, not only as to the intelligence of the children but also as to their health; further various physical measurements were included.

I will explain the terms then used with regard to intelligence in some detail as the scale with one modification is the one we still use in the Galton Laboratory

A. Very Dull. Capable of holding in their minds only the simplest facts and incapable of perceiving or reasoning about the relationship between facts.

B. Slow Dull. Capable of perceiving relationship between facts in some few fields with long and continuous effort, but not generally without much effort or without much external assistance.

C. Slow. Very slow in thought generally, but with time understanding is reached.

D. Slow Intelligent. Slow generally although possibly more rapid in certain fields.

E. Intelligent. Ready to grasp and capable of perceiving facts in most fields. Capable of understanding without much effort.

F. Quick Intelligent. Very bright and quick both in perception and acquirement and this not only of customary but of novel facts. Ready to reason rightly on purely self initiative. It will be noticed that in all these definitions it is the idea of understanding and of reason that is uppermost and not the idea of information acquired.

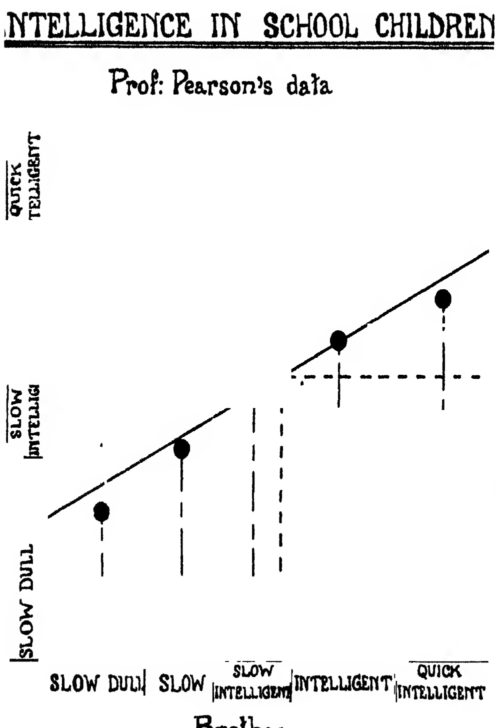


Diagram I.

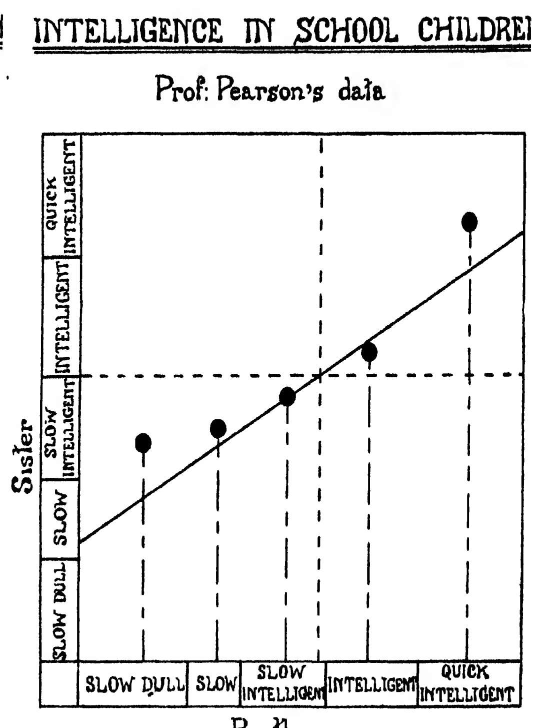


Diagram II.

Diagrams I and II show for brothers and for brothers and sisters how steadily the intelligence of the second member of a pair of "siblings" increases with the intelligence of the first. If one considers a group of boys with slow dull intelligence it will be seen that the average intelligence of the brothers of these boys is in the slow group and that the brothers do not even reach the middle of that group, but that when boys have been classed as intelligent their brothers also are found just in the intelligent group. Pairs of brothers and sisters show the same thing; the slow intelligent boys have sisters whose average intelligence is also slow intelligent

and quick intelligent boys have sisters who, on the average, have a high grade of intelligence.

We must now consider the coefficients of correlation in siblings for the physical as well as the mental characters which were obtained from Pearson's data; some of them are given in the following table:

TABLE I.

	Brothers	Sisters	Brother—Sister
Head Length ..	·50	·43	·46
Head Height ...	·55	·52	·49
Eye Colour .	·54	·52	·53
Intelligence ...	·52	·50	·49

It will be seen from this table that all the coefficients of correlation are very similar in value; as we should lay no stress on differences in percentages of five or six so we lay no stress on differences in correlation coefficients of 05 and ·06—they are insignificant. In considering these sibships it has been argued that we were dealing with widely different environments and that the brothers were alike because of the similarity of environment. The agreement between the results for intelligence and for physical characters seems to me to render the argument invalid but we will now see from the Oxford Class Lists, and from Charterhouse school where the environment is more uniform, how far intelligence runs in families*. We are assuming that intelligence and success in the final schools at Oxford are highly correlated and that intelligence and the form reached in a public school are also correlated; this I think is on the whole true, though naturally there will be many exceptions, but for the moment let us consider that we are discussing not intelligence but success. There are certain other characteristics that are required for success as well as intelligence, such as health and perseverance, but I doubt whether success in Oxford Finals could be achieved without intelligence. Here we will consider only fathers and sons for by so doing we obtain I think a very uniform environment. The sons took their degree before 1892; they all had

TABLE II.

Degree at Oxford taken by the Father	Percentage of Sons who took I and II Class Honours
I and II	27
III	15
Pass	12
None	9

* "The Inheritance of Ability," by E. Schuster and Ethel M. Elderton, *Eugenics Laboratory Memoirs*, I.

fathers at Oxford and though the environment would doubtless vary the variations would be comparatively slight. There would be no lack of the necessities of life and all the sons would come from cultured homes.

It will be seen that the number of sons who obtained first and second class honours falls steadily according to the degree taken by the father.

It is possible that two groups of men are included in this table. The first consisted of men who were placed in the position they deserved, men who tried for honours whether they obtained them or not and those who did not enter for honours because it was recognized that they were not up to the required standard, and the second group of men who did not try for honours though intellectually they may have been up to the standard. Family tradition and outside interests influence a man in deciding whether he will enter for honours or not and this applied even more thirty years ago than it does now.

At Charterhouse the boys are living under similar environments and though home conditions may vary, such variations will only be slight. The following table gives the percentage of brothers who were in groups I and II at ages 15, 16, and 17 according to the place in the school attained by a brother at that age. Group I includes the sixths, group II the fifths and so on down to group VII which includes

TABLE III.

Percentage of Brothers in Groups I and II.

	15 years	16 years	17 years
I	32·1	79·3	84·6
II		49·7	67·8
III	21·2	35 1	45·9
IV	15·0	21·9	42·8
V	9·7	19·0	28·6
VI	6·0	12·0	
VII	1·3		
Average	15·0	34 0	60·8

all boys in the shell. Of all boys of 15 years 15 per cent. are found in groups I and II but of the boys who were in groups I and II, 32 per cent. of their brothers reached that position at 15 years while of the boys who were in shell at 15 years only 4 per cent. of their brothers reached this highest grade. Pairs of brothers at ages 16 and 17 show the same thing, 34 per cent. of boys of 16 are in groups I and II and 61 per cent. of boys of 17 but the percentage of boys in the two highest groups varies from 79 to 12 at age 16 and from 85 to 29 at age 17 according to the position reached by the brother. The mean correlation coefficient for the three ages is ·48 which agrees very well with that found for brothers in the school data where the teachers estimated the intelligence.

It must be remembered that up to this time no-one had talked about tests for intelligence or about intelligence quotients. Binet did not publish his work on the subject till 1908, the year after the paper on the Oxford Class Lists was published, and the only way to judge intelligence was by estimating the character, using such a scale as I have described on p. 379. These estimates had to be made by those who might be supposed to know the children well, generally by the masters and mistresses. In order to test how far such an estimate could be relied upon Professor Pearson when dealing with intelligence in school children had asked two or three different teachers in several schools to apply the classification to between 30 and 50 pupils known to each of them. Classifications were to be made absolutely independently, often by teachers of different subjects. Just under 85 per cent. of the children were put into the same classes by the different teachers while another 10 per cent. differed by only one class. Further investigations as to the adequacy of the teacher's estimate were made later. Teachers in four different Aberdeen schools judged the mental capacity of 249 boys in four groups, excellent, good, moderate, and dull. Each boy's examination place was given and Miss H. G. Jones in 1909* compared the teacher's estimate of intelligence with the place attained in the examination. It may be argued that an examination is not a perfect test of intelligence and I quite agree; an examination is supposed to test information as well as intelligence but unless it is very badly conducted the more intelligent people will be found near the top and the least intelligent at the bottom of the list. The place in the examination was divided by the number of boys in the class but perhaps the easiest way in which we can see the result is by giving the average position that would be attained out of 100 boys by those of excellent, good, moderate, and dull intelligence. The average position attained by boys of excellent intelligence in 100 boys is fifteenth, by boys of good intelligence thirty-fourth, by boys of moderate intelligence sixty-second and by boys of bad intelligence eighty-eighth, which shows that the teacher's estimate and the result of an examination test agree very well.

In 1911 Dr Waite worked on the report sheets for one term from two London secondary schools for boys†. Each master was asked to judge the mental capacity of every boy he taught in five grades from very able to slow. Each boy was judged by four masters and the marks he might gain could vary between four and twenty, the higher numbers representing the better intelligence; these estimates

TABLE IV.

Grades of Intelligence.

	4 to 9	10 and 11	12 and 13	14 and 15	16 and over
Place in Examination	82	60	39	30	15
Place in Form (first school) ...	82	57	41	32	15
,, ,, (second school) .	70	59	44	37	18

* *Biometrika*, Vol. VII. pp. 542—548. † *Biometrika*, Vol. VIII. pp. 79—93.

were then compared with the results of an examination held in all subjects in the one school and with the place in form gained for work done in school in both the secondary schools. We will give again the average place gained if there were 100 boys being considered.

This table shows again that when the master's estimate of the boy's intelligence is a low one the boy does badly in examination and in his class work, only reaching on the average to about eightieth in 100 boys, but that when the masters estimate the boy's intelligence as good, the place in examination and term work is about fifteenth in 100.

Mr W. H. Gilby considering boys in eight primary schools who were judged by 36 teachers all of whom had been in charge of their class for nearly twelve months found very similar results*. In this case the scale described before on p. 379 was used with one alteration, "*E*" the intelligent group being subdivided into *E* and *F* while *F* became *G*. The boys' intelligence as judged by their masters was compared with their place in an examination which was held by another man and with the percentage of marks gained by the boys in the head master's examination held the previous term. These are given in Table V below:

TABLE V.

Master's Estimate.

	B	*C*	*D*	*E*	*F*	*G*
Place in examination out of 100 boys	84	74	62	46	25	12
Percentage of marks gained in the head master's examination	43	54	61	70	79	85

The correlation coefficients given in Table VI show that there is a close association between the teacher's estimate and the school record.

TABLE VI.

	Teacher's estimate of Intelligence and Place in Examination		Teacher's estimate of Intelligence and Place in Form	
	Crude	Corrected for Age and Form	Crude	Corrected for Age and Form
Aberdeen	·72	·70	—	—
London Secondary Schools (1)	·69	·84	·62	·78
" " " (2)	—	—	·48	·61
Primary Schools †	·68	·68	—	—

Some masters were slightly influenced in their judgment of intelligence by the age and position in the school of the boy they were considering and allowance had to be made. It is interesting to notice that there is less agreement between

* *Biometrika*, Vol. VIII. pp. 94—108. † Mean of two values.

the teacher's estimate and the place in form in the second London Secondary school than in the first, but this can be accounted for, I think, by the fact that in the second school about a third of the boys had only been in the school for about three months and in the first school all boys had been in the school for at least six months and were consequently better known by their masters.

These investigations strengthened our faith in the estimates of intelligence and we decided to work out the correlation of intelligence in parents and adult children from data collected in the form of family schedules by Professor Pearson. In 1905 these forms had been issued asking for information about the health, intelligence, temperament, success in life, age at death and cause of death in families. Such schedules were formidable in size and the filled in forms came in so slowly that we are still asking for more. The data required were for a father and mother, their children, their parents, brothers and sisters, uncles and aunts, grandparents and cousins. It is I think very selected data with regard to intelligence; we should hesitate to give such a form to any one of say "slow" intelligence nor would such a person as a rule accept one! The result of this selection is that we have rather a large number of "very able" individuals, especially among the recorders, from the very fact that it was people of this type who were willing and able to give the required information. When we have more forms I think it will be as well to separate the person who filled up the form and his children from the grandparents and their children. Further there was a difficulty in the definition of intelligence. Only six classes were given and we found a tremendous heaping up in the intelligent class, the top but one, and Professor Pearson began to think that "slow intelligent" which to us meant average, meant to most people rather slow and that in the intelligent group we had many people included who were very little, if at all, above the average, so he divided the intelligent group into two parts "distinctly capable" and "fairly intelligent," but until we have more schedules in which this division is used there is some difficulty about grouping for working purposes. Tables VII and VIII give the results in rather a different form. We suppose that the intelligence of the community can be calculated in units called mentaces*, the larger the number of mentaces the more intelligent is the individual.

TABLE VII.

Father's Intelligence	Average number of Mentaces	Average Number of Mentaces in Son	Average Number of Mentaces in Daughter
Slow ...	269	348	260
Slow Intelligent ...	360	408	403
Fairly Intelligent ...	454	479	458
Distinctly Capable ..	554	497	472
Very Able ...	672	515	487
Mean ...	510	483	458

* By reduction to a normal scale of frequency: see *Biometrika*, Vol. III, p. 147.

The average number of mentaces of a father varies from 269 when he is "slow" to 672 when he is "very able." It will be seen that on the whole the "slow" and "slow intelligent" fathers do not have children as "slow" as themselves, but that their children have decidedly fewer mentaces than the average. The "distinctly capable" and "very able" fathers have children not so capable as themselves but they have a greater number of mentaces than the average mothers and children show the same thing but with even greater uniformity. The daughters of

TABLE VIII.

Mother's Intelligence	Average Number of Mentaces	Average Number of Mentaces in	
		Son	Daughter
Slow	266	385	351
Slow Intelligent	363	461	411
Fairly Intelligent ..	453	485	468
Distinctly Capable ..	566	501	482
Very Able	705	527	516
Mean . ..	497	486	467

"slow" fathers had fewer mentaces than we should have expected but the daughters of "slow" mothers have done better than their mothers which is what we should expect from Galton's law of regression. The mean of the correlation coefficients for parent and child is ·44 which does not agree badly with what we find when we compare physical measurements in parent and child.

While these schedules were being collected the Binet-Simon tests had been extended for all age groups of normal children and the Intelligence Quotient began to be used as a measure of intelligence both abroad and in America. We had satisfied ourselves that the teacher's estimate of intelligence was on the average a sound one but there were two drawbacks to this method of judging intelligence which we fully recognized; the one was that it is very difficult to get a uniform standard over the whole country when you are dependent on definitions such as we have given of the grades of intelligence, and secondly you must have an intelligent and interested teacher. It is no good pretending that all teachers are equally intelligent, those that know the facts of their particular subject may be bad judges of other matters, but on the whole such teachers will not be sufficiently interested to undertake work of the nature I have described, which is unpaid and needs time and energy outside school hours, although as long as such estimates are made voluntarily there is very little danger under this second head. But the first difficulty is, I think, a real one, for though people in one town or district will use words with a certain meaning, travel 200 miles north and words such as "fair" begin to mean something different. Also if a man or woman has been teaching for long in any one school the average for that school may become his or her idea of "average" generally, and "fairly intelligent" in one school may

be only equal to "slow intelligent" in another. With these facts in mind the appearance of the Intelligence Quotient may be hailed with joy.

The use of mental tests in schools is still not very common in England though it seems to be gaining ground and we have to turn to America for the most extensive discussion of and work on the subject. For those who are ignorant of the meaning of the "Intelligence Quotient" I may explain that certain tests have been designed for each physical age of a child, which the average child of that age should be able to pass. The average child of five years' physical age would have a mental age of five years and its intelligence quotient would be the mental age divided by the physical age multiplied by 100 that is to say 100. If a child of five had a mental age of six years its intelligence quotient would be $(6 \div 5) \times 100$, i.e. 120, but if it failed at the five year old test and passed the four year old test only the intelligence quotient would be $(4 \div 5) \times 100$, i.e. 80, but a child can of course reach any mental age such as 5·2 or 5·8 years if it passes the five year old tests and some of the six year old ones. To the original Binet-Simon tests others have been added; the Stanford Revision is now more generally used in America and it has obvious advantages over the original Binet-Simon tests. It has to be remembered that Binet's work was pioneer work and after some experience it was found that his tests at the older ages were too difficult and there was a distinct fall in the intelligence quotient with the age of the subject. Many and varying tests have been used and I am not competent to express an opinion as to which test is the best to use or what method of scoring is best. On statistical grounds any test that has to make allowance for age is at a disadvantage. The actual tests employed under different systems seem very similar, but there are certain differences in the way in which the tests are given, the amount of help that may be contributed by the examiner and what percentage of tests for any year must be solved to attain the mental age of that year. In any mental tests the most important thing is to insure that one is using not a test of existing information, but a real test of intelligence, and those who have used these methods in America consider that they do test intelligence, not knowledge, and that children who have received no formal instruction may have a high mental age when competing with children who have been two or more years at school. Examples are given which individually are convincing but from a statistical standpoint one would like more extended experiments in this direction. Different people might consider that some tests were not entirely free from the objection I have named, that even if they did not test information they tested experience, nevertheless I believe a study of the tests shows that those who claim that they are testing intelligence and not knowledge are in the main justified in their assumption though I think myself there are some exceptions. On the whole one feels that if the instructions to the examiners are carried out faithfully the shy and nervous child will not be at a disadvantage, for the essential point is sympathy with the child and there are frequent warnings against confusing a shy child and instructions to the examiner to leave a test alone if the child seems bewildered and return to it later when confidence is restored. We all know what queer things will alarm certain small

children and examiners are warned to beware of this. In the test when three orders are given the subject is asked to place a key on a chair and it is suggested that some children will not touch a key and that if a child fails to start to execute the orders another test of the same sort not using a key shall be tried later. I cannot think that everyone would be an efficient examiner and I hold that the intelligence and sympathy that are necessary for the teacher who is going to estimate a child's intelligence adequately will be nearly if not quite as necessary in the examiner who is going to test a child's intelligence especially when dealing with young children. When the tests are standardized it seems to me the great advantage will be uniformity of definition, but I am not sure that the use of tests will displace entirely the teacher's judgment or that it would be wise that it should. These tests, as Binet says, have been specially designed to test general intelligence rather than some special faculty and such a faculty may not come to light if a Binet test is used exclusively. I should never suppose that any man or woman who had been with a child for a few hours or even for a few weeks would be a judge of that child's ability but a teacher who has seen a child in school and at play with its companions for a year ought, as a rule, if he or she is at all observant and sympathetic, to have a sound idea of that child's intelligence. There is not perfect agreement between the intelligence of the child as estimated by the teacher and as judged by the intelligence test, but both might be equally good estimates of intelligence judged from a rather different standpoint.

In the diagrams and tables so far given we have measured the strength of the inheritance of intelligence when that intelligence has been judged by some one who knows the children or adults concerned, now let us see whether the intelligence as judged by the intelligence quotient will yield like results.

In the *Journal of Delinquency*, Vol. IV. Dr Kate Gordon in testing the intelligence of children in three orphanages in California gave the intelligence of 91 pairs of siblings. Here we have very uniform environment and Professor Pearson found that the coefficient of correlation was ·51 which agreed very well with that found between brothers and sisters when the teachers estimated the intelligence; he discussed the question in *Biometrika*, Vol. XII. p. 367, and an examination of the diagram in that paper shows how closely the results for orphanage siblings resemble those for the English school children of Professor Pearson's own data. Dr Kate Gordon continued her investigations and in a Report to the State Board of Control of California on *The Influence of Heredity on Mental Ability* she gave the results of estimating the intelligence of 216 pairs of siblings. The Stanford Revision of the Binet-Simon scale was used, and the intelligence quotient was calculated. The examinations were carried out between July 1918 and September 1920, mainly on children in orphanages. The original intention of Dr Gordon had been to keep the pairs of siblings in each orphanage separate and to calculate the coefficients between siblings living under identical environment; she obtained results that varied from $\cdot 27 \pm \cdot 126$ in one orphanage to $\cdot 69 \pm \cdot 086$ in another, the latter result being obtained from 20 pairs of siblings and the former

from 26 pairs. The probable error of the difference is not three times that difference (·42 ± ·156) and we cannot therefore say that there is a significant difference between the pairs of siblings in these two orphanages but one feels very strongly that correlation coefficients obtained from such small numbers are essentially unsatisfactory.

The point that was very noticeable at the first reading was the correlation of ·61 which Dr Gordon found when considering 216 pairs of siblings, a distinctly higher correlation than we should have expected from the earlier examination of orphanage children which gave a value of ·51. As we were anxious to investigate the point a little further Dr Gordon most kindly sent us a copy of her original observations and a first examination led us to think that this high value of ·61 for the intelligence between siblings might be due to the diversity in race of the children considered which included Jews, Mexicans, and judging by the names, Italians, and other races. The expectation that we should find greater variability among these children than among the other orphanage children was not realized however as will be seen by an examination of the various constants for the 478 children who were examined.

TABLE IX.

	Later data	Earlier data
Mean Intelligence Quotient ...	90·48 ± ·47	92·86 ± ·84
Standard Deviation	15·275 ± ·333	16·727 ± ·591
Coefficient of Variation ...	16·882 ± ·379	18·014 ± ·657
Correlation Coefficient... ...	·61 ± ·03	·51 ± ·05

If anything Dr Gordon's earlier series discussed by Professor Pearson in *Biometrika* shows a greater variability than this later series and a lower correlation and the source of the difference in the coefficient must be sought elsewhere; in itself the difference is not significant but ·61 is a higher value for the resemblance between brothers than we should expect to find.

Dr Gordon in considering families of which more than two members were tested used no child more than once, combining any one sibling with the next younger one. In this way families of three would yield only one pair of siblings, families of four and five two pairs and of six and seven three pairs and so on. As a check on Dr Gordon's result of ·61 we took all possible pairs, made the table symmetrical and then found a correlation coefficient for intelligence between siblings of ·467 ± ·026! The difference here is ·14 ± ·039 and is almost certainly significant. Whichever way you choose to pair the siblings both are samples of the same population and the results should not differ by more than three times the probable error, and if they do there is some anomaly somewhere. From the original data, using Dr Gordon's method of pairing, we found 219 pairs of siblings and a coefficient of ·593 ± ·030 which confirms generally Dr Gordon's result. There

was however a distinct difference in the mean intelligence quotient of the older and younger siblings which only showed itself when using this method of pairing; the mean intelligence quotient of the elder sibling was found to be 87·8 and of the younger 92·1. It is known that when the Binet-Simon test is used there is a lower average intelligence quotient for older children, but this difference in the quotient with age is not usually found when the Stanford Revision is employed; any such fact if it existed would account for the different coefficients of correlation found by the two methods of pairing. We then worked out the correlation coefficient between age and the intelligence quotient and found that it was $-{\cdot}310 \pm {\cdot}028$, that is to say the intelligence quotient decreased with age and the regression was $-1{\cdot}59$, which means that for every increase in one year of physical age the intelligence quotient was reduced roughly by $1\frac{1}{2}$. Diagram III, p. 391, for which I am much indebted to Miss I. McLearn*, shows very clearly a gradual decrease in intelligence with an increase in age. This seems to explain the variation in the coefficients found. When all pairs of siblings are taken and not only the pairs close to one another in age we may be decreasing the correlation automatically, and we must therefore correct for age before we can discuss the real correlation found for intelligence between siblings from the data. There are not enough pairs to make a separation of the sexes practicable, and such a division is not necessary here, for in orphanages where children of both sexes occur they do not differ significantly in the mean intelligence quotient. There are two possible methods of correcting for age. We can find the regression equation for the intelligence quotient on age, correct the quotient of each individual child and thus eliminate age. If we use this method and then proceed in the ordinary way with a symmetrical table we get a correlation coefficient of $\cdot540 \pm \cdot024$ which is a reasonable value in very good agreement with previous results found for the correlation in intelligence between brothers. Diagram IV, p. 392, shows that the regression is reasonably linear and that the points lie very evenly along the regression line. This method of correcting for age is not possible when broad categories are used for measuring intelligence; we must then use a partial correlation; we resolved to do this in the present instance as a check on the previous result; we further used Dr Gordon's method of pairing and then corrected for age to see how it compared. We decided first to use the method described by Professor Pearson in a paper *On Homotyposis in Homologous but Differentiated Organs*†, but to use this method the correlation between the growth in the elder brother's intelligence and the younger brother's age and between the younger brother's intelligence and the elder brother's age should be the same and this was only the case when all the tables were made symmetrical. From the figures given in Table X (p. 393) it will be seen that the differences are not really significant but in the two first columns we used partial correlation in the ordinary way.

* I owe the drawing of all the diagrams in this paper to Miss McLearn and would like to express here my appreciation of the value of her help. I have also cordially to thank Miss M. Moul for assistance in some of the laborious calculations.

† *Roy. Soc. Proc.* Vol. LXXI. p. 289.

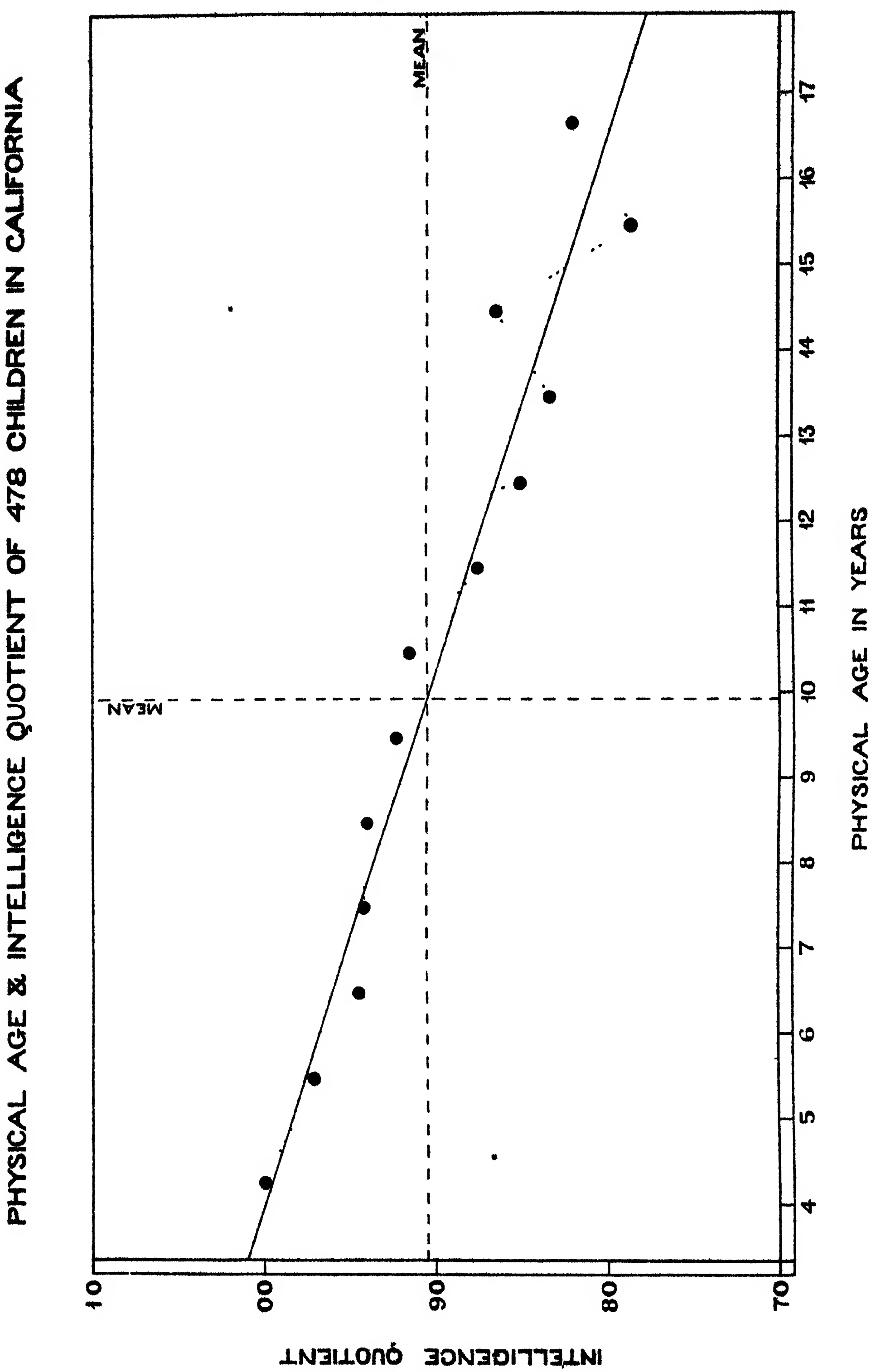
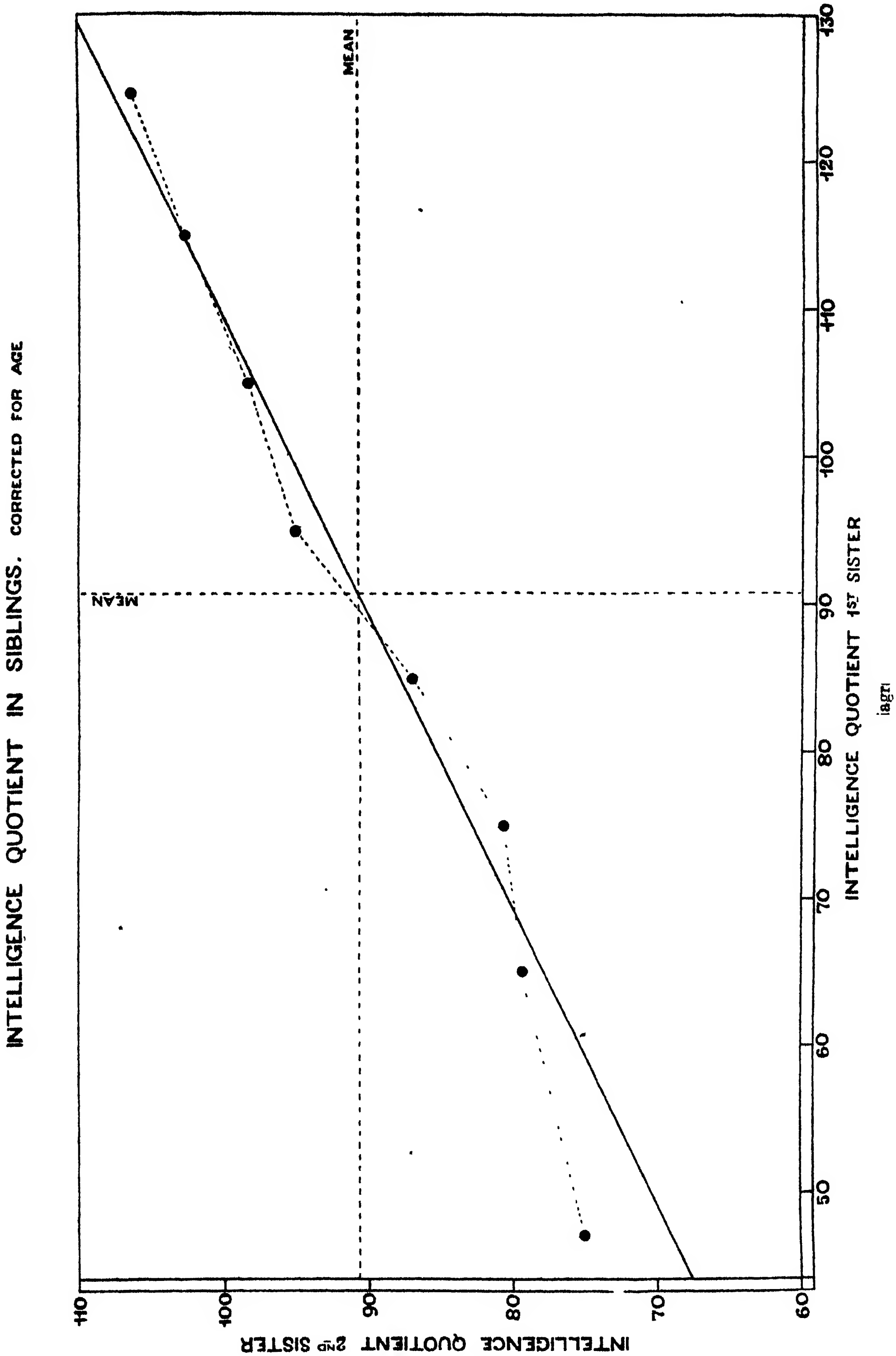

PHYSICAL AGE & INTELLIGENCE QUOTIENT OF 478 CHILDREN IN CALIFORNIA
MEAN
MEAN
INTELLIGENCE QUOTIENT
10
00
90
80
70
PHYSICAL AGE IN YEARS
4
5
6
7
8
9
10
11
12
13
14
15
16
17

Diagram III.


INTELLIGENCE QUOTIENT IN SIBLINGS. CORRECTED FOR AGE
MEAN
MEAN
INTELLIGENCE QUOTIENT 1ST SISTER
50
60
70
80
90
100
110
120
130
INTELLIGENCE QUOTIENT 2ND SISTER
60
70
80
90
100
110

It will be seen that when all pairs of siblings are considered the results agree very well with each other and with the correlation of ·540 obtained when we corrected each individual's intelligence quotient in accordance with age. Using Dr Gordon's method certain members of the family will constantly be omitted but even as it is there is no significant difference between ·58 and ·54 when the probable error in the first case is ·03 and in the second ·02. These values of ·54 agree very well with other results, namely those for adult brothers from the family records ·54 and for brothers at school ·52.

TABLE X.

	Dr Gordon's method of pairing	All pairs of siblings	All pairs of siblings and tables symmetrical
I. Q. of Elder and I. Q. of Younger...	·593 ± ·030	·534 ± ·024	·467 ± ·026
I. Q. of Elder and Age of Elder	− ·245 ± ·043	− ·280 ± ·031	− ·328 ± ·030
I. Q. of Elder and Age of Younger...	− ·189 ± ·044	− ·130 ± ·033	− ·022 ± ·034
I. Q. of Younger and Age of Elder...	− ·255 ± ·043	− ·183 ± ·033	− ·022 ± ·034
I. Q. of Younger and Age of Younger	− ·299 ± ·014	− ·266 ± ·031	− ·328 ± ·030
Age of Elder and Younger	·827 ± ·014	·702 ± ·017	·193 ± ·033
I. Q. of Elder and I. Q. of Younger for constant age of Elder and Younger	·578 ± ·030	·544 ± ·024	·532 ± ·024

The chief interest in this study of orphanage children lies I think in three directions. In the first place it shows that orphanage siblings resemble one another about as closely as children whose home conditions will vary in a more marked degree. In the second place it affords some evidence that whether the intelligence of children is measured by the intelligence quotient or by the estimate of teachers, the correlation between the intelligence of siblings is of the same order. It is obvious that in neither case have we the ideal conditions for making a dogmatic statement as the orphanage children were tested by the Stanford Revision and the school boys by the teachers, but I think the data collected add something to our knowledge of the hereditary factor in intelligence. Perhaps the chief interest really lies in the use of the intelligence quotient. One had hoped that the Stanford Revision had removed the necessity of allowing for age, which was one of the drawbacks to the Binet-Simon tests, but apparently it is not always the case. Here there is a well marked and steady fall in the quotient with age and we must either assume that there is something inadequate in the tests or that residence in an orphanage tends to decrease the intelligence of the inmates!

In any work dealing with an hereditary factor correction for age means a great deal of extra work and when dealing with environmental questions would add very much to one's labours especially if it were required to make some other factor constant. If this difficulty about age in connection with the use of tests cannot be conquered we shall begin to wonder whether a return to the results obtained by the now more old fashioned method of estimation has not much to recommend it. Professor Pearson showed in the paper in *Biometrika*, Vol. XII. before referred

to, how closely intelligence as obtained by estimate agreed with intelligence as measured by Stanford Binet-Simon tests and the labour is much reduced in dealing with any factor when age has been eliminated.

It may be suggested that since correction must be made for age, correlating the mental age of one sibling with another and making the physical age constant would be a better method of proceeding since it would save calculating the intelligence quotient, but here again there is a difficulty. It was found that there was greater overlapping in the scores of the older children than in the scores of the younger ones* and that one year in physical growth did not correspond to one year in mental growth. For instance, taking a standard test for five year old children, it was found that about twenty-three per cent. more six year old than five year old children passed the tests, but if a standard test for ten year old children was taken it was not twenty-three per cent. more of the eleven year old children but twenty-three per cent. more of the twelve year old who passed the test. At present these tests are still partly in the experimental stage and such an anomaly as that to which I have referred might be removed, but until it has been, we must continue to use the intelligence quotient. At present whatever tests are used an investigation into the relationship between age and the quotient must be undertaken before it will be safe to draw any conclusions of either an hereditary or environmental nature. In many instances I have seen the Stanford Revision used when there was no correlation with age, but certainly in the data here examined such a correlation existed and though correction for age did not make a very great difference it showed that the question of age cannot at present be neglected when considering intelligence as measured by the intelligence quotient.

From Dr Gordon's data collected in orphanages we found that when intelligence is measured by the Stanford Revision tests the resemblance between siblings is about ·54, while from data collected by Professor Pearson where the environment presumably differs more widely and when the intelligence is estimated by the teachers the resemblance between brothers is ·52 at school and ·54 for adult brothers when the judgment is that of a relative, and we feel that the inheritance of intelligence is a fact very little if at all affected by environment. It might be argued however that in making this statement we were comparing facts that were not strictly comparable, since we had no evidence that if the orphanage children had been judged by the teachers we might not have found a much higher association between the intelligence of siblings than we actually found in the school children, and that if the Charterhouse boys and the brothers in the family schedules had been tested by Binet-Simon tests there might have been less association found than there was when they were judged by place in school or by the estimate of their relatives. What we want are pairs of siblings who have been judged in both ways, by test and by the teacher's estimate, and if the correlation between intelligence in siblings comes out the same in both cases we shall feel that our conclusions are justified. Owing to the energy and kindness of Dr Drinkwater of Wrexham we have recently received some material in which both methods of

* *Journal of Educational Psychology*, Vol. XII. January 1921, p. 8 et seq.

ascertaining intelligence have been applied to the same children. The numbers of pairs of brothers and sisters are however very small and owing to a difference in the intelligence of boys and girls we cannot combine the sexes here, but the material is, as far as I know, the first of this kind collected for brothers and sisters and is therefore valuable*.

Our most grateful thanks are due to Dr Drinkwater and to the head teachers of the schools concerned who made the testing possible, and to all those teachers who estimated the intelligence of the children. The first examination was of pairs of brothers and sisters in a high class school in Wrexham, the best school in the district in which the parents formed the aristocracy of the working classes in the town. Dr Drinkwater first used the Binet-Simon tests but found that for children of ten years and over the tests were too difficult and he then used the Stanford Revision on all children over ten and on all children that he examined subsequently. Dr Drinkwater re-tested some of the younger children by the Stanford Revision and found that the intelligence quotient obtained by the Binet-Simon method was confirmed by the Stanford Revision, but one feels that it is not very satisfactory to have two different tests used in the same school and sometimes on members of the same family, and we shall see presently that the results in this first school are more irregular than in the second school where the method employed was uniform throughout. The teacher's estimate was also obtained for each child quite independently of the tests, which were carried out by Dr Drinkwater himself. The second school was of a very different type and the children were inferior in every respect to those who were found in the first school, home housing conditions were worse and the occupations of the fathers were less skilled. The average intelligence quotient differed in the two schools and owing to the different tests used in the better class school we preferred at first to keep the two schools separate, though we combined them later, such a separation of the schools has the disadvantage that we are dealing with comparatively few cases. Table XI gives the average intelligence quotient for the two schools for boys and girls, and for convenience we shall call the schools A and B, A being the better class school which was examined first.

TABLE XI.

	Average Intelligence Quotient	
	A	*B*
Boys	101·4	94·0
Girls ..	97·2	88·7

It will be seen that in both schools the average intelligence of the girls is lower than that of the boys and that the intelligence in the inferior type of school is lower

* Dr P. Stocks has collected similar data for individuals and gave some of his results in a paper read before the Society of Biometricians and Mathematical Statisticians, and the correlations he found between the intelligence quotient and the teacher's estimate of intelligence agree very well with the results obtained from Dr Drinkwater's data.

than in the better. These conclusions are confirmed by the teacher's estimate. Seven groups were used for grading the intelligence, *A*, *B*, *C*, *D*, *E*, *F* and *G*, but in all cases we grouped *A* and *B* together as "very dull" and *F* and *G* together as "distinctly capable," *C* can be defined as "dull," *D* as "rather slow" and *E* as "intelligent." Diagrams V—VIII on p. 397 will show that the average intelligence of boys in school *A* is in the "intelligent" group, though fairly close to the boundary of "slow," but that the average for the girls is in the "slow" group. In school *B*, the average for the boys, according to the teacher's estimate, is almost on the boundary line of "slow" and "intelligent," and for girls on the boundary between "slow" and "dull." We shall return to this difference in intelligence in the sexes later; we did not find it in the orphanage data collected by Dr Gordon, nor has it generally been found in America when boys and girls under 15 years are considered.

We first found in both schools the correlation between the teacher's estimate of intelligence and the intelligence quotient. The diagrams to which we have already referred on p. 397 will show that there is a well-marked association and that the regression may be considered linear, though among "very dull" boys we find a lower intelligence quotient than we should expect, but the mean is based on only ten cases. We found the following values for the correlation ratio between the teacher's estimate of intelligence and the intelligence quotient:

	School *A*	School *B*
Boys ...	$\cdot 662 \pm \cdot 029$	$\cdot 581 \pm \cdot 030$
Girls ...	$\cdot 617 \pm \cdot 026$	$\cdot 789 \pm \cdot 021$

This does not show anything like perfect correlation but only a well-marked association. In one or two cases where Dr Drinkwater noticed a decided difference between the teacher's estimate and the intelligence quotient he re-tested but without finding much change. One would hardly expect perfect correlation between the two and it is quite possible that both are equally good measures of intelligence from a different standpoint.

The next point to be discussed is the relationship between age and intelligence; it will be remembered that Dr Drinkwater used the Stanford Revision test after experimenting with the Binet-Simon in an effort to obtain a measure of intelligence independent of age. As far as the boys are concerned he was successful and there is no association between the intelligence quotient and age in either school, but this is not the case with the girls, for we find a correlation coefficient of $-\cdot 225 \pm \cdot 040$ between age and the intelligence quotient in school *A*, and of $-\cdot 297 \pm \cdot 049$ in school *B*. Both these coefficients are significant, but since no decrease in the intelligence quotient is found with age among the boys we were at once inclined to attribute it, not to the tests adopted, but to some other factor and this idea receives some support from the fact that there is a correlation between age and the teacher's estimate of intelligence for girls. For girls the correlation coefficient is $-\cdot 146 \pm \cdot 041$ (assuming a normal distribution for intelligence) in school *A* and $-\cdot 241 \pm \cdot 051$ in school *B*, but there is no decrease in intelligence as measured

INTELLIGENCE QUOTIENT & TEACHER'S ESTIMATE

Dr. Drinkwater's data, earlier Series

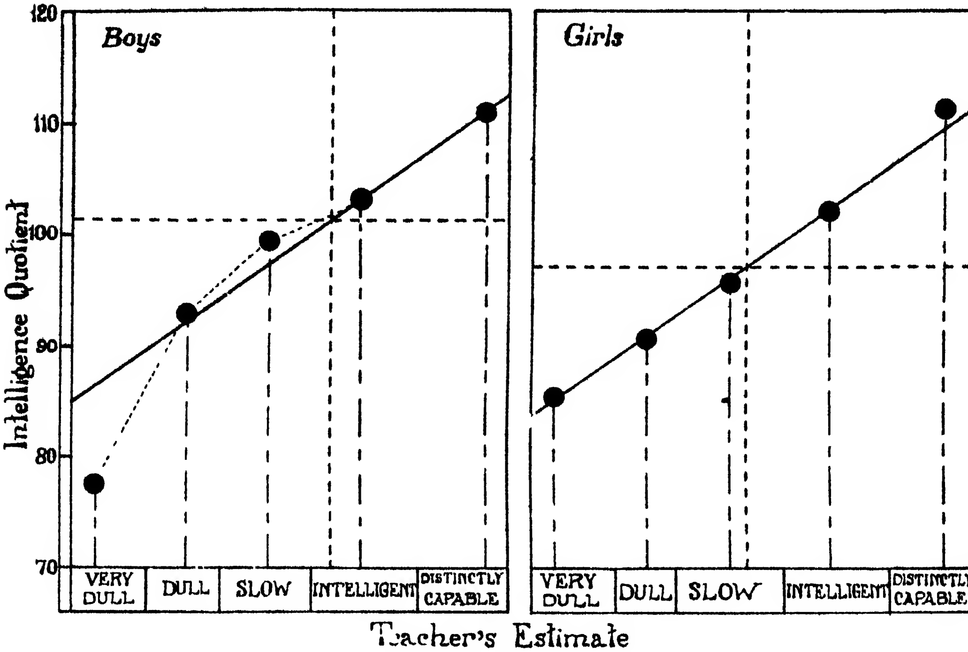


Diagram V School A. Diagram VI. School A.

INTELLIGENCE QUOTIENT & TEACHER'S ESTIMATE

Dr. Drinkwater's data, later Series

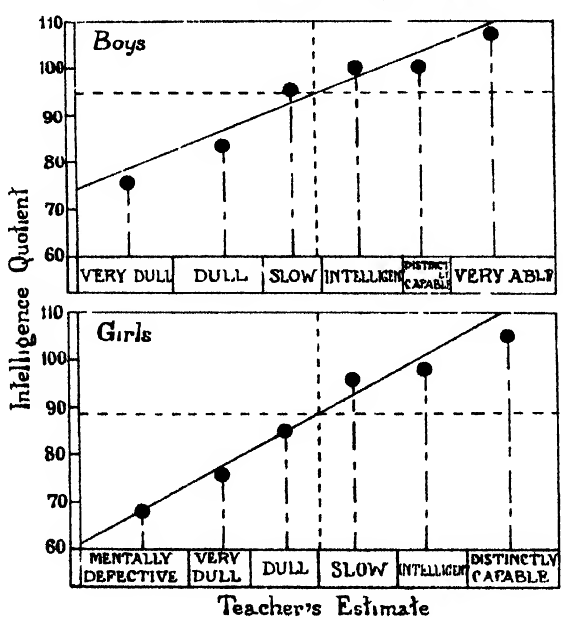


Diagram VII. School B.

Diagram VIII. School B.

by the teacher or by the intelligence quotient with the age of the boys, all the coefficients found being positive and none significant. This seems to indicate that for some reason the older girls in these schools are less intelligent. If it occurred in boys as well as in girls and began suddenly at age 11 (in school *B* there is a tendency for this to happen among the girls, but the numbers are too few to be sure on this point) we might attribute it to the transference of the brighter children to secondary schools. Under our present educational system this takes place at about the age of 11 years, and there is evidence in the notes received from the school that the older children in some families are at the county school, but we should expect this transference to apply to boys as well as girls. There seemed only two possibilities which would account for the difference between boys and girls and neither is really confirmed by further information received from Dr Drinkwater. First that for some reason the brighter boys in Wrexham do not enter secondary schools but go straight to work, or that there were so many more places for boys than girls at secondary schools that many boys who were only just "above the average" were transferred, while only the brightest girls went on, but this second suggestion will not really account for the entire absence of any decrease of intelligence with age in the boys, because the brightest boys would certainly be transferred first. The suggestion that the brightest boys go straight to work is not confirmed by further enquiries carried out by Dr Drinkwater as the following extract from the letter of the head master of school *A* will show "With regard to the falling off of the I. Q. in children of 11 and upwards* a very decided factor in the case of my own school is the fact that such a big number of my brightest boys go to the secondary school at 11 and 12. The last two years I lost over 30 each year and this out of about 120 which is a large percentage. Another cause might be the employment morning and evening of a number of my bigger boys." With reference to the girls in school *B* the head mistress reports that 15 girls were transferred to a county school and 13 to a convent school and adds, "Some of our girls from 11 years upwards have to work very hard at home, especially when the mothers are ill, etc. I cannot help thinking that the result of the war stress is being shown now as I have a number of retarded children."

The head mistress of school *A* writes, "During the three years prior to your inspection 48 girls, who would normally have been in the senior classes at the time of your inspection, passed on to the secondary school through Scholarship or Free Place. Free Places are, as you know, awarded to all scholars who attain a certain standard of work. In addition another 6 left for various secondary schools as paying pupils. You will see therefore that our brightest material is passed out of school between 11 and 12 years, consequently only inferior material is left. As a rule this material comes from socially inferior homes, and the scholars are kept away a good deal for work in the homes. There is another factor which I think helps to account for the low quality of our senior material. These girls have grown up during the upset of war time—throughout the 7 years of their attendance national

* As we have said this "falling off" does not occur among boys but it would appear from this letter that it should do!

conditions have been abnormal, and in many instances homes have reflected this abnormality—there has been less discipline, and this fact may have reacted on the children*. The aspect of the senior classes of primary schools has changed very considerably of late years—we have practically only the residue of children who are not able to benefit by higher education—entrance to secondary schools has become increasingly easy, and there are so many free places, that a much higher proportion of pupils passes on now than formerly."

It is quite clear from these letters that the more intelligent girls do go on to secondary schools, which accounts for the lower intelligence quotient found among the older girls left at school, but we are just as much in the dark as to why there is no trace of this among the boys. A much larger proportion of boys than girls go on to secondary schools. "Over 30 boys" in one year in school *A* as compared with only 16 girls from the same school†, and apparently there are more girls in that school than boys. We have only before us the cases in which more than one of a family is considered, but from these data we find 188 girls under 11 years as compared with 132 boys, so that we may infer I think that the boys more frequently move on to secondary schools than the girls, but even if this be so we have not accounted for the absence of any decrease in intelligence among the older boys in these schools. It has been suggested that the oncoming of puberty which is earlier in girls than in boys means a diminution in intelligence between the ages of 11 and 14, but such an explanation has received no confirmation from work done in America, where retardation occurs after 14 and not before when Stanford Revision tests are used‡.

Table XII gives the crude correlation coefficients, that is to say, uncorrected for age, between the different types of sibship in the two schools, using in each case two measures of intelligence, the intelligence quotient and the teacher's estimate.

TABLE XII.

	School *A*		School *B*	
	Intelligence Quotient	Teacher's Estimate	Intelligence Quotient	Teacher's Estimate
Brother—Brother ...	·669 ± ·053	·768 ± ·040	·394 ± ·055	·351 ± ·057
Sister—Sister ..	·272 ± ·060	·450 ± ·052	·461 ± ·065	·460 ± ·065
Brother—Sister ...	·326 ± ·044	·543 ± ·036	·149 ± ·036	·466 ± ·035

Correction for age makes only slight changes and from this table it is obvious that the results found in school *A* are very irregular. If we consider only school *B*

* A good deal of this letter seems to show that the school mistress reckoned knowledge as a factor of intelligence! This would weaken the correlation of the teacher's estimate with the Binet-Simon tests record.

† It would be 18 per year if we included those who went on as paying pupils.

[‡ There are distinct traces of it in the change of psychical characters with age in my school data and this remark applies much more to girls than boys. K. P.]

we find that the resemblance in intelligence among siblings is practically the same whether we judge that intelligence by the tests or by the teacher's estimate and such a result gives one great confidence in the inheritance coefficient as worked out solely on the teacher's estimate in the past. For all pairs of siblings it is lower than we should expect but it must be remembered that we are dealing with selected data, with a school in which the pupils rarely come from better class homes, that therefore the variation will not be very marked and consequently the correlation coefficients will not be as high as if we were dealing with more variable material. The coefficients of variation in this school do not differ significantly from those found for orphanage children when the whole series was taken; here they are 16·3 for boys and 16·9 for girls and among orphanage children the coefficient of variation was 16·9 for both sexes taken together. In school *A* on the other hand the resemblance in intelligence between sisters and between brothers and sisters seems to differ according to the measure of intelligence used. The differences with the probable errors are:

Sister-Sister	...	·178 ± ·067,
Brother-Sister	...	·217 ± ·057.

In the first case the difference is not three times the probable error and may or may not be significant, but in the second case the difference does exceed three times the probable error and must, I think, be regarded as significant. The value found for the resemblance between sisters when the intelligence quotient is used is very low absolutely and relatively as compared with the other school and though the coefficient of variation has sunk to 13·48 it seems a hardly sufficient diminution in variability to account for the low value. For brothers the correlation is much higher than in school *B* but as the number of pairs of brothers is only 49 we cannot expect results that are likely to be of much value. Comparing the two measures of intelligence there is fair agreement. The resemblance between pairs of sisters and pairs of brothers and sisters when the teacher's estimate is used is much the same in both schools but we certainly cannot say the same for the intelligence quotient and correction for age makes very little difference. Can these irregularities be due to the use of two different tests on some of the younger children in this particular school?

Diagrams IX–XIII show the regression lines for each type of sibship and for each school, the unbroken line giving the regression line when the intelligence is measured by the intelligence quotient and the broken line when it is estimated by the teacher. It must be remembered that in every case means are based on a very few cases which causes many irregularities. The general trend of the two regression lines in school *B* is very similar but as we should expect the agreement is not so good in school *A*. Possibly a brother-brother diagram in school *A* would show a better result but when only 49 pairs of brothers are available the drawing of a diagram seems waste of time. We must now correct the girls' intelligence for age since there is a quite significant correlation between age and intelligence. There is no need to apply this correction to the boys since the correlations are

·050 ± ·051 in school *A* and ·069 ± ·046 in school *B* when intelligence is estimated by the intelligence quotient and ·031 ± ·049 and ·103 ± 045 when the intelligence is estimated by the teachers. It will be obvious that none of these are significant correlations and all are positive which emphasizes the difference from the girls where in all four cases we found negative significant correlations between their age and intelligence. The first method used for correcting for age was applied to the

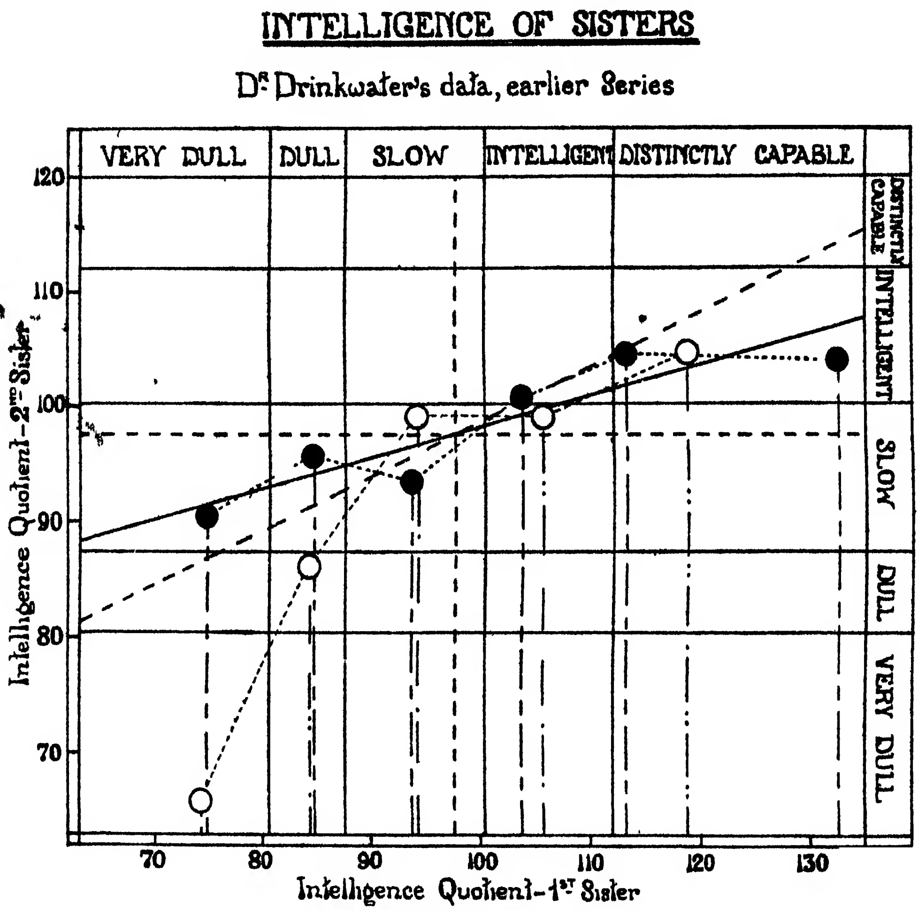


Diagram IX. School *A*.

intelligence quotient only, the regression of intelligence on age was found and the required correction made in each individual case with the following results.

		School *A*	School *B*
Sister-Sister	...	174 ± ·062	·507 ± ·061
Brother-Sister	...	·296 ± ·045	·445 ± ·036

It will be seen that only slight changes have been made, but in both cases in school *A* the coefficients of correlation are even lower than they were before and

that between sisters occurring alone would hardly be significant. We will now use partial correlation and find for both teacher's estimate and intelligence quotient the correlations when age is constant. Working with broad categories we can find the contingency coefficient (C_2) or, assuming a normal distribution for intelligence, we can find "*r*." We have worked out the tables in both ways and give the two results; correcting for age makes very little difference, in fact, taking into consideration the probable error, we have made no significant change in our coefficients and it may be asked why we took the trouble to make the corrections, but until the whole thing is worked out it is not possible to foresee the result. When any

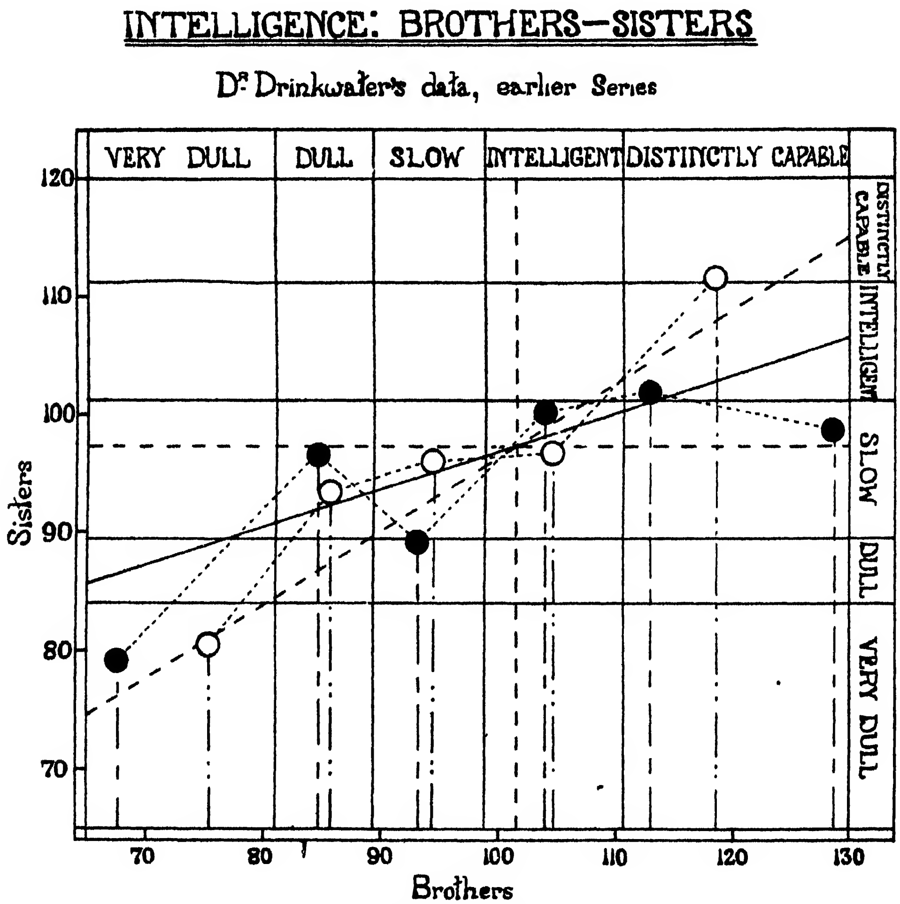


Diagram X. School *A*.

significant correlation exists between age and some character the correction for age must be made. When the changes are so slight it seems needless to repeat the comments made on the crude table. The probable errors are large and, taking this fact into account, School *B* seems to give results which we might anticipate and it makes very little difference whether the estimate of intelligence is by a test or by the teacher, but in School *A* the anomalies remain as marked as before.

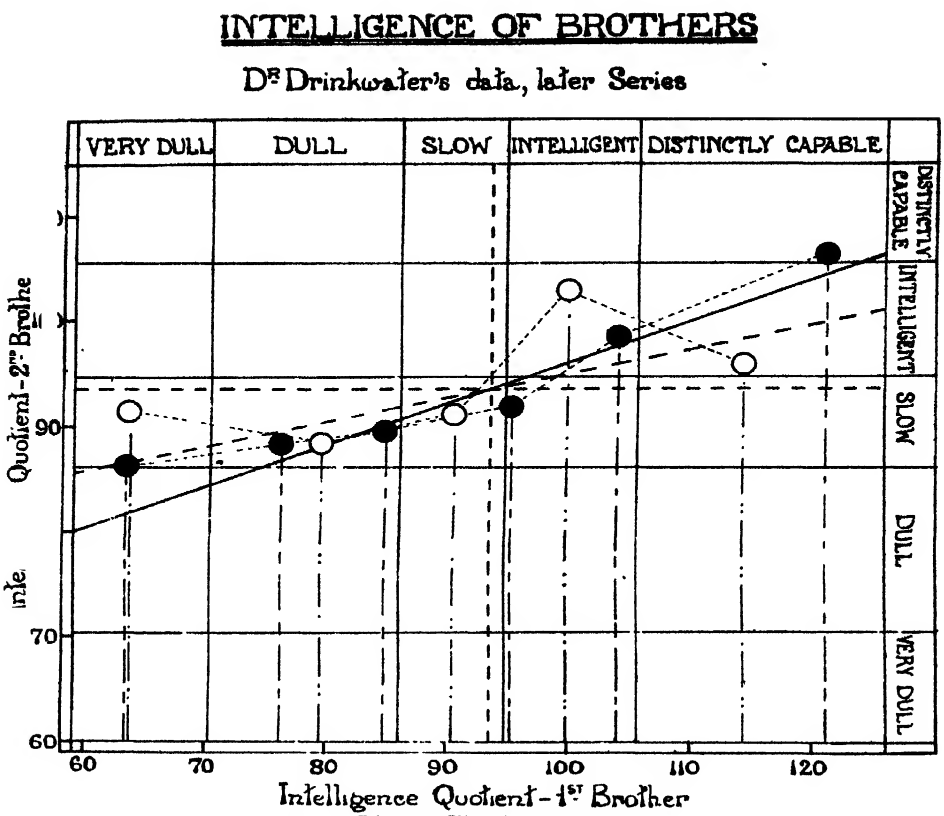


Diagram XI. School *B*.

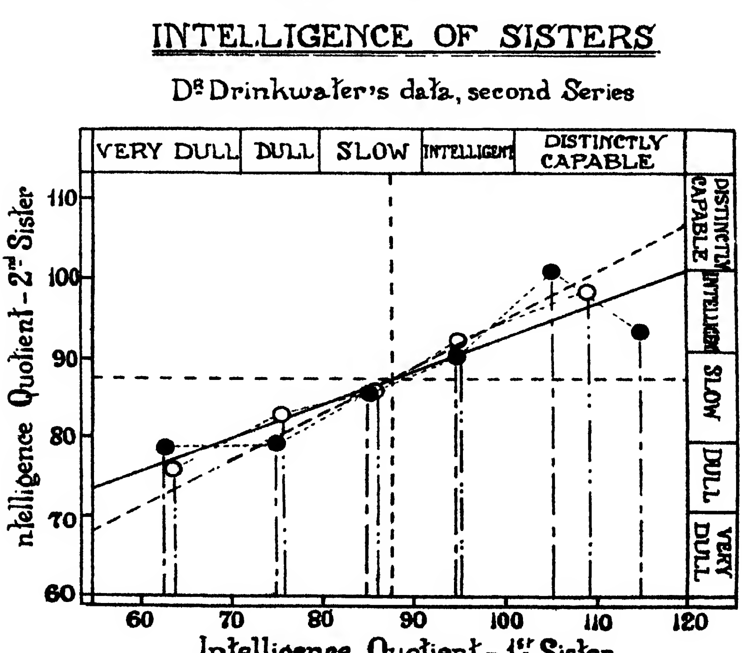


Diagram XII. School *B*.

There seems a definite reason why the sexes should not be combined, but no really cogent reason* for not combining the schools except the fact of the change of test in School *A* among the younger children, so we decided to put together the pairs of siblings from the two schools and see whether larger numbers would give

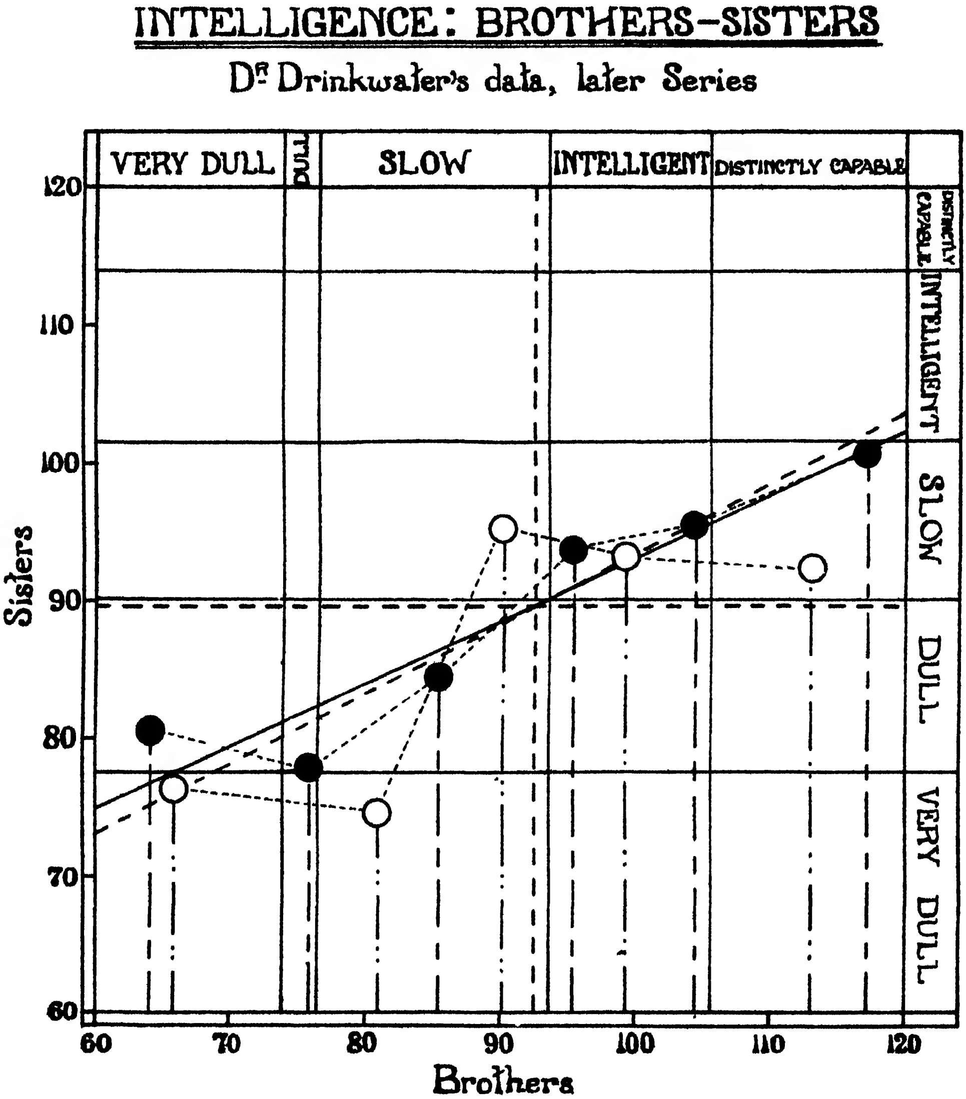


Diagram XIII. School *B*.

us greater uniformity of result. It is not getting over the differences between the schools but is perhaps worth while considering. First we correlated age with teacher's estimate of intelligence and then with the intelligence quotient and found

* There is a possible reason for combining them in that the two schools have really selected each a portion of the general population.

TABLE XIII.

Inheritance of Intelligence.

		Estimated by the Intelligence Quotient		Estimated by the Teacher	
		School A	School B	School A	School B
Pairs of Sisters	Intelligence of Elder and of Younger	·272 ± ·060	·461 ± ·065	·450 ± ·052	·460 ± ·065
	" " " Age of Elder ...	− ·221 ± ·040	− ·297 ± ·049	− ·146 ± ·041	− ·241 ± ·051
	" " " Age of Younger ...	− ·254 ± ·060	− ·226 ± ·079	− ·198 ± ·062	− ·254 ± ·078
	Intelligence of Younger and Age of Elder ...	− ·270 ± ·060	* − ·060 ± ·083	− ·182 ± ·062	− ·204 ± ·080
	" " " Age of Younger	− ·221 ± ·040	− ·297 ± ·049	− ·146 ± ·041	− ·241 ± ·051
	Age of Elder and Younger	·610 ± ·040	·509 ± ·062	·610 ± ·040	·509 ± ·062
	Intelligence of Sisters for constant age } using r	·221 ± ·061	·462 ± ·065	·470 ± ·050	·411 ± ·067
	} using C_2	—	--	·603 ± ·042	·451 ± ·066
Pairs of Brother-Sister	Intelligence of Brother and of Sister	·326 ± ·044	·449 ± ·036	·543 ± ·036	·466 ± ·035
	" " " Age of Brother ...	·050 ± ·049	·069 ± ·046	·031 ± ·049	·103 ± ·045
	" " " Age of Sister ...	− ·032 ± ·049	− ·102 ± ·045	·041 ± ·049	− ·152 ± ·040
	Intelligence of Sister and Age of Brother ...	− ·042 ± ·049	·207 ± ·043	− ·148 ± ·048	·096 ± ·045
	" " " Age of Sister ..	− ·221 ± ·040	− ·297 ± ·049	− ·146 ± ·048	− ·297 ± ·049
	Age of Brother and Sister	·151 ± ·048	·009 ± ·045	·151 ± ·048	·009 ± ·045
	Intelligence of Brother-Sister for constant age } using r	·329 ± ·044	·438 ± ·036	·564 ± ·035	·440 ± ·037
	} using C_2	—	—	·520 ± ·036	·429 ± ·037
	Pairs of Brothers	·669 ± ·053	·394 ± ·055	† ·768 ± ·040	† ·351 ± ·057

again a negative result for girls − ·266 ± ·036, when the estimate was the intelligence quotient and − ·309 ± ·035, when the estimate was that of the teacher, while for boys it was ·008 ± ·034 and ·060 ± ·034 respectively for the two estimates of intelligence. Again in this case we must correct the tables in which the girls appear for age and we get the following.

TABLE XIV.

	Intelligence in Siblings judged by	
	Intelligence Quotient	Teacher's Estimate
Brother-Brother	·489 ± ·041	·486 ± ·041
Sister-Sister	·425 ± ·041	·580 ± ·034
Sister-Sister corrected for age ...	·384 ± ·042	·540 ± ·036
Brother-Sister	·529 ± ·024	·486 ± ·025
Brother-Sister corrected for age ...	·532 ± ·024	·499 ± ·025

* Re-checked with the same result, but note the large P.E.

† Using C_2.

Here the results are more uniform, we have nothing so strange as the value found for the resemblance in the intelligence quotient between sisters in School *A*, and except for sisters the two estimates of intelligence agree very well. It is as well to notice that on the whole the resemblance between siblings when the teacher's estimate is taken is lower than when the Stanford Revision test is used, from which we may fairly argue, I think, that the results already obtained for inheritance of intelligence by biometricians are not over-estimated even though use has been made of the teacher's judgment rather than of some numerical test *. The resemblance between brothers and sisters and between parent and child is very similar in value to the resemblance found for physical characters and we must now see whether there is anything equal to heredity in determining the grade of intelligence. The Americans have done some work on this point to which I shall refer in a moment but there is room for much more. In our own laboratory Dr Heron took the intelligence of children in the London elementary schools and compared it with the physical condition of the children, with stunted growth and with bad home condition as measured by cleanliness. Since the estimates varied in every school all were kept separate and the results given in Table XV represent the means of many observations, negative in some schools, positive in others.

TABLE XV.

Intelligence and	Boys	Girls
Height ...	·10	·07
Weight ...	·06	·03
Cleanliness...	·14	·07
Nutrition ...	·01	·08
Glands ...	·09	·08
Tonsils ...	− ·01	·11
Teeth ...	·08	·09

These coefficients are all small, many not significant, the highest is between cleanliness and intelligence in boys, which is ·14, which holds the same position in the scale of intelligence that 14 per cent. holds in the scale of percentages. Such a coefficient though significant in itself may be only a secondary effect of intelligence; if the more intelligent parents and the better type generally keep their children cleaner than the less intelligent parents there should be a correlation between intelligence and cleanliness in the child, but ignoring this possibility we can see that the correlation coefficients given in Table XV are all small and not comparable in size to those found for the hereditary influence.

Other data that we have are from Edinburgh where the results found are given in Table XVI. Here the numbers dealt with are much smaller and none of the results found are significant except those between the intelligence of the children

* It is impossible to consider the data here dealt with as final, not only are the numbers inadequate, but there is a want of that smoothness, which almost invariably will be found when judgments are given by specially trained examiners dealing with adequate numbers.

and the economic condition of the home, and unless one is to suppose no connection between good homes and a rather better type of intelligence in the parents such a relationship may be a secondary effect of the hereditary nature of intelligence. Some of the coefficients are negative and some positive and none of them mean anything—there is no association between the moral and physical condition of the parents and the intelligence of their children, and only a very slight one between the economic condition of the home and the intelligence of the children, which may be an indirect result of heredity.

TABLE XVI.

	Boys	Girls
Intelligence and drinking of the father...	−·07	·05
" " " mother.	·05	−·09
Morality of the parents	−·07	·03
Physical condition of the parents ...	−·04	·06
Economic condition of the home ...	·10	·16
Overcrowding	·02	·04

As far as we can see from all the material we have examined the environment matters little compared with the hereditary factor. Here we are still working on the teacher's estimate of intelligence, but there seems no reason to suppose that the intelligence quotient will yield different results for environment since it did not give us very different results for heredity, but before long we ought to be able to test children in Great Britain and work out the influence of environment on the intelligence quotient. Some work in this direction has been done in America already and Terman* considers that the culture of the home, apart from heredity, has little influence and that of the dozen or more children who have been removed from an unsatisfactory to a satisfactory home environment not one showed an improvement in his intelligence quotient. "A dozen or more" cases are not enough but if every case of this kind were noted, I mean wherever there was a change of environment, we should in time get some real evidence on the point.

In America also much work has been done showing the persistence of the intelligence quotient from year to year in the same child. Re-tests were given to 315 children and the correlation between one test and a later one was ·93. Some of these children were re-tested more than once and all the cases were used. There were 33 examiners and in 72 per cent. of cases the earlier and later tests were given by different examiners. In 86 cases the re-test was at less than a years interval, in 138 cases between 1 and 3 years, in 85 between 3 and 5 years and in 127 there was more than 5 years between the tests and Terman, on this basis, considers that, apart from minor fluctuations due to temporary factors, the feeble minded remain feeble minded in spite of special schools, the dull remain

* Lewis M. Terman, *The Intelligence of School Children*, London, 1921.

dull, the average remain average and the superior remain superior. This does not by any means imply that education is unnecessary, it sharpens the tools and provides the material for the brain to use, and the keener the instrument the more the brain can achieve, but the efficiency of the output is I think limited by the power of the brain that produces it. In the midst of many new experiments in education the use of re-tests would quickly prove whether intelligence can be increased by any or all of these new systems. Even if it were proved that no real increase in intelligence followed a change in educational environment it does not mean that we should remain in our old ways. New methods may improve the edge of the razor without changing its metal, but in these days it is important, I think, to distinguish between the possibility of increasing intelligence and the possibility of giving an existing intelligence help in achieving success. Education in even the ideal form may fail in the former case and succeed in the latter but they are not one and the same thing and it is no use to have a perfect system of education if the brains that should profit by it are decreasing in power, if men and women of intelligence are leaving fewer and fewer children to inherit their ability.

All men are not born equal. I must quote Francis Galton, for after all he still remains one of our chief authorities on inheritance—"I acknowledge freely the great power of education and social influences in developing the active power of the mind, just as I acknowledge the effect of use in developing the muscles of a blacksmith's arm, and no further. Let the blacksmith labour as he will, he will find there are certain feats beyond his power that are well within the strength of a man of herculean make, even although the latter may have led a sedentary life.... In running, in rowing, in walking, and in every other form of physical exertion, there is a definite limit to the muscular powers of every man, which he cannot by education or exertion pass. This is precisely analogous to the experience that every student has had of the working of his mental powers." To each of us, a limit is set, a limit as far as one can see at present due to heredity rather than to opportunity and to the intelligence of our parents and ancestors rather than to the educational system under which we were reared.

1 Ulna of Eagle, left wing, ventral aspect

2 a } White tailed Eagle left wing, Cartilaginous Sesamoid of Elbow Joint in two aspects 2a anterior
2 b. } and 2 b posterior

3 a }
3 b } House Sparrow Elbow Joint Posterior and Ventral views (× 8) to show Ulnar Sesamoid

Addendum to a "Memoir on the Sesamoids of the Knee-Joint."

Biometrika, Vol. XIII. pp. 133–175 and pp. 350–400.

On p. 387 of the above memoir reference was made to Owen's statement that in certain birds there is an ossiculum belonging to the ulna which Owen states is "essentially the separated olecranon of that bone." This detached sesamoid, which has been termed the "ulnar patella," is stated by Owen to occur in many of the *Raptores* and in the Swifts, while in the Penguins it is double. The importance of the matter for the above paper lay in the association as homologous of the olecranon with the cnemial crest of the tibia, and the possibility of the sesamoids of the elbow and knee-joints arising from a partial break up of these appendages

I have examined the elbow-joint of a number of *Raptores* without much success in the discovery of ulnar sesamoids Owls, Vultures and Hawks do not seem generally to possess them. The ulnar joint of the American Sea Eagle (that bird Franklin called it "of bad moral character," which for an inexplicable reason the United States has chosen for its emblem) has an ulnar joint as shown in Plate I, Fig. 1 without sesamoid In the White-tailed Eagle I have, indeed, found a *cartilaginous* sesamoid of the elbow-joint: see Figs. 2 *a* and 2 *b*. In the House Sparrow there is, however, a well-marked ulnar sesamoid: see Figs. 3 *a* and 3 *b*. The drawings seem to suggest that the olecranon is at any rate very much reduced where the ulnar sesamoid exists I should prefer not to insist on any too sweeping statement till far more elbow-joints in birds have been studied. But it seemed worth while adding these pictures of the elbow-joint and of the ulnar sesamoid as an addendum to the above mentioned paper.

I have to thank very heartily Mr S. Steward of the Royal College of Surgeons for the drawing of the American Sea Eagle, Miss Ida McLearn for those of the White-tailed Eagle, and my colleague, Dr Katherine Watson, for the dissection and drawings of the elbow-joint of the House Sparrow. K. P.

ON AN UNUSUAL CASE OF DIGITAL ANOMALY.

My attention was drawn to this case by Miss L. Eckenstein, the subject being described as a thumbless, but five fingered woman. The subject herself said that she had originally seven toes on each foot, those on the left foot being removed when her right foot was amputated as a child. Dr Julia Bell kindly had a preliminary interview with and accompanied the subject to the radiography, while Mr E. S. Pearson photographed the hands. The following description of the skiagrams and photographs was drawn up by my colleague Dr Percy Stocks. K. P.

The case provides a somewhat unusual combination of congenital anomalies of the extremities. The subject, a female, presented from birth the following abnormalities which are illustrated in the accompanying photographs and skiagrams.

(1) Both hands showed an identical malformation of the thumb, which was represented by a digit with three joints having the characteristics of a finger rather than a thumb. In length and circumference this 1st digit closely resembled the forefinger, from which it was slightly abducted (see Plates I. Fig. 1, IV and V). The metacarpophalangeal joint was almost on a level with that of the neighbouring forefinger, being only about 7 mm. proximal to it instead of about 30 mm. as in a normal thumb. As the photographs indicate, this digit was slightly rotated inwards towards the palm and adductor and opponens muscles were apparently present which enabled the tips of the 1st and 2nd digits to be opposed The nail was equal in size to that of the 2nd, 3rd and 4th digits.

Examination of the skiagrams (Plate III) shows that, as regards the carpus, the os magnum and trapezium are unusually large and articulate with each other at their proximal ends, separating the small trapezoid from the scaphoid which normally articulate with each other. The proximal end of the 1st metacarpal fits into a V-shaped concavity formed by the distal ends of the trapezium and trapezoid, and articulates equally with both these bones, instead of with the trapezium alone. The proximal end of the 2nd metacarpal fails to articulate with the trapezium, as is usual, but obliquely articulates with the base of the 1st metacarpal. The 1st metacarpal is approximately equal in length to the 3rd, and about 5 mm. shorter than the 2nd. The phalanges are almost exactly similar in size and shape to those of the index finger, but are somewhat curved with concavity towards the central axis of the hand.

(2) A photograph (Plate I, Fig. 2) taken when the patient was a girl shows that there was apparently congenital absence of the right tibia, whilst the fibula, presumably present, was only about a quarter of its normal length. This was accompanied by extreme equino-varus of the right foot, as is usual in such cases. The great toe of this foot was abnormally broad, and has the appearance of being grooved longitudinally, whilst on its pre-axial side is a complete supernumerary digit with an appendage of skin along its inner border which probably represents a rudimentary 7th digit. This deformed limb was amputated at the knee in youth, and further details are not available.

Ferguson and Stocks

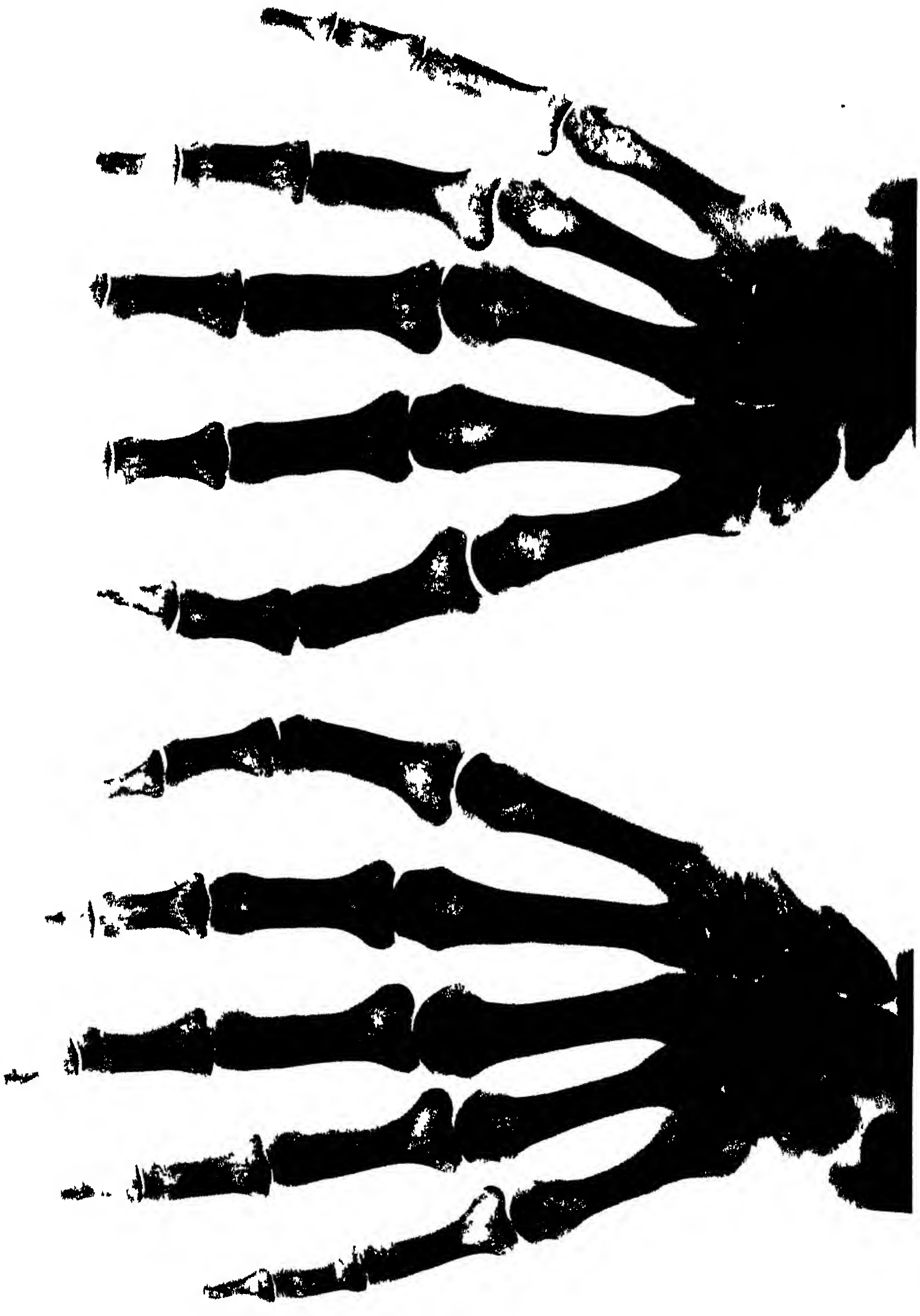
Fig. 1

Fig. 2

Subject as a child, before amputation of right foot or supernumerary toes of left.
From Photographs by Dr J. Dill Russell, 1897.

Pearson and Stocks

Skiagram of Left Foot in its present state.
Radiographed by Dr R. W. A. Salmond, 1922.



Skiagrams of Left and Right Hands   Radiographed by Dr R. W. A. Salmond 1922

Fig. 1. Palms of Left and Right Hands. Photographed by E. S. Pearson, 1922. Fig. 2.

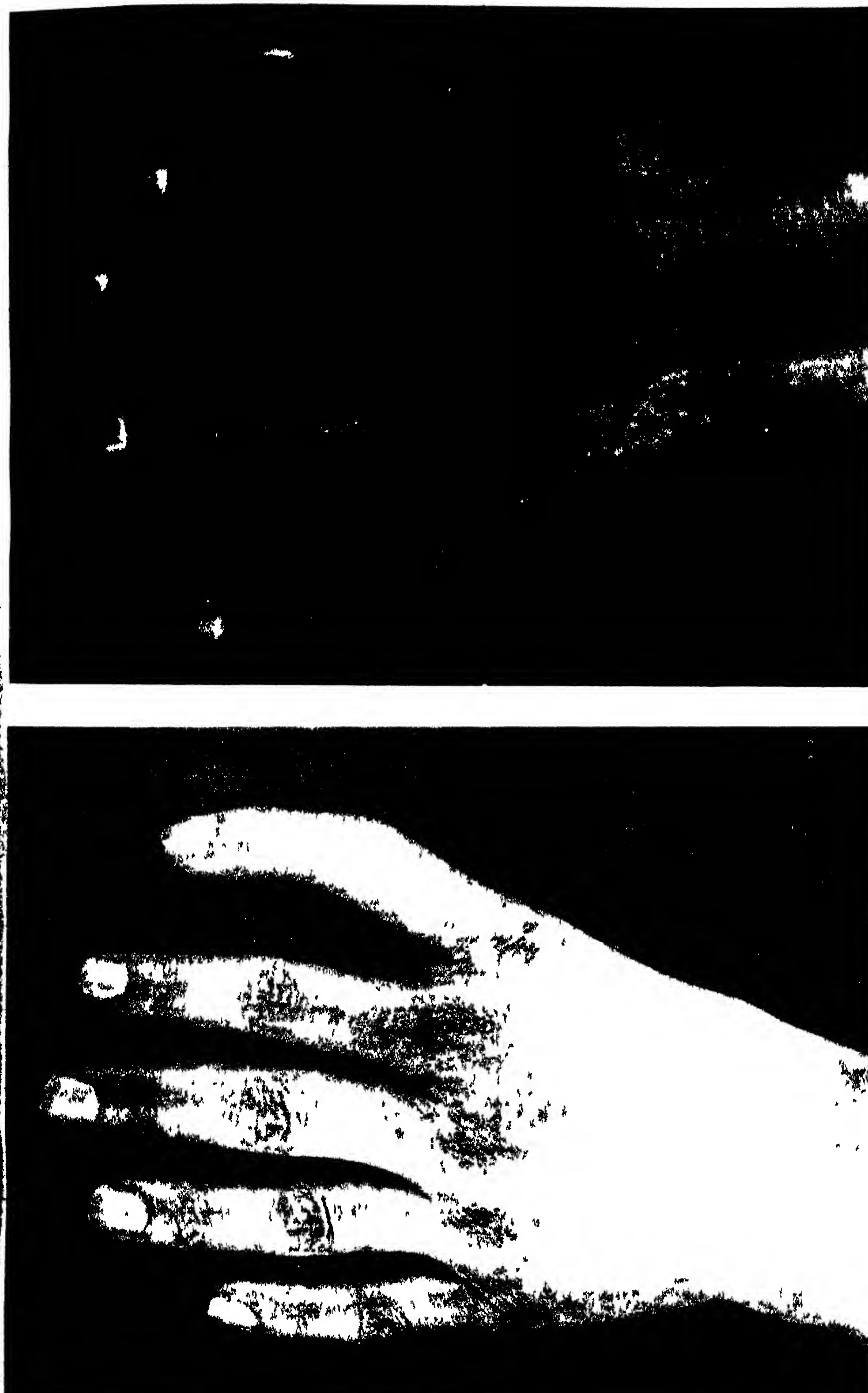


Fig.

Backs of Left and Right Hands   Photographed by L. S. Pearson, 1922

(3) The left foot showed the presence of a rudimentary 6th digit on the pre-axial side of the great toe, which was evidenced by a bony projection on the inner side of the last phalanx, and another near the base of the metatarsal. There was also a skin appendage on the phalangeal projection which possibly represents the rudiments of a 7th digit. A skiagram (Plate II) of this foot shows that the internal cuneiform bone is enlarged on its inner side, having an additional articular surface for another metatarsal bone on inner side of the metatarsal of great toe. There is also an appearance of a longitudinal groove separating these portions of the bone. The metatarsal and 1st phalanx of the great toe are normal in outline, and two sesamoids are present near the proximal end of the former. Articulating with the internal cuneiform and base of 1st metatarsal is an additional metatarsal bone, ending after about 25 mm in an attenuated stump. Attached by a bridge of bone to the inner side of the last phalanx of the great toe is an additional phalanx, which articulates with the 1st phalanx of the great toe There appear to be one or more sesamoids near the base of this extra phalanx.

All the anomalies in this case were pre-axial, and with exception of the absence of tibia, symmetrical. Deficiency of the tibia is frequently associated with defective development of the great toe, its association with supernumerary toes on this side of the foot is exceptional As regards the hand and foot abnormalities, two cases have been reported which resemble this case to some extent, both of which had hereditary history of polydactylism. McKellar(1) records the case of a female (VI. 4 in Lewis's pedigree of this family(3)), in whom the 1st digit of each hand had a metacarpal and three phalanges, whilst in each foot the great toe was bifid, and webbed from the metatarsal onwards. Struthers(2) mentions a case in one of his families (II. 3 in Lewis's pedigree(3)) who had six toes on each foot and "very long thumbs."

In the present case the subject states that to her knowledge there is no history of any hereditary anomaly of hand or foot. It is worth noting that she finds no difficulty in carrying on her profession of dressmaker, she opposes forefinger and "finger thumb" as easily and successfully as most persons do the forefinger and thumb.

We have to thank the subject for the fulness of the information she has given us with regard to her relatives, and their freedom from anomaly. We are deeply indebted to Dr R. W. A. Salmond for his skiagrams of hands and foot, and to Dr J Dill Russell for photographs he took of the subject as a child before amputation of the right foot, and removal of the supernumerary toes on the side of the big toe of the left foot by the late Professor Christopher Heath. No publication with regard to the matter appears then to have been made.

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MISCELLANEA.

I. On the Correction necessary for the Correlation Ratio η*.

BY KARL PEARSON, F.R.S.

(1) In a paper published in *Biometrika*, Vol. VIII. p. 254, I have dealt with the mean value of η, the correlation ratio in samples from material in which the actual value of η is zero, and shewn that if κ be the number of arrays, then when we sample uncorrelated material the mean value of η is $(\kappa-1)/N$, where N is the size of the sample. That paper, however, did not deal more than superficially, and incorrectly, with the case of the mean value of η when η is not zero in the sampled population. I propose to discuss this problem at greater length in the present paper, noting that the chief difficulty lies in being quite certain that all the terms necessary to our degree of approximation have been really included. The general rule I think is that in proceeding to terms of the order $1/N$ we must keep not only first powers of statistical differentials but squares and products of statistical differentials; while if we wish to go to the next higher approximation we must retain not only cubes but fourth powers of statistical differentials. Our purpose at present is to deal only with large samples, and it will be adequate to proceed only to squares and products of such differentials.

(2) Let there be two variables x, y, means $\bar{x}$, $\bar{y}$, standard-deviations σ_x, σ_y, correlation coefficient r; we will take x_s for any particular value of x, and y_t for a particular value of y; the corresponding marginal frequencies being n_{x_s}, n_{y_t}, and the mean and standard-deviation of the array of y's corresponding to x_s being $\bar{y}_{x_s}$ and $\sigma_{y_{x_s}}$. We shall use η for the correlation ratio $\eta_{y,x}$ of y on x. Then by definition:

$$\eta^2=\frac{S\{n_{x_s}(\bar{y}_{x_s}-\bar{y})^2\}}{N\sigma_y^2} \quad\text{...(i)},$$

where for brevity we write

$$\sigma_M^2=S\{n_{x_s}(\bar{y}_{x_s}-\bar{y})^2\}/N \quad\text{...(ii)},$$

while

$$\sigma_y^2=S\{n_{y_t}(y_t-\bar{y})^2\}/N \quad\text{...(iii)}.$$

All statistical differentials shall denote the excess the given quantity has in a sample over its value for the sampled population, and when a short quantity is placed above a symbol, that symbol will be supposed to refer to the sampled population. Starting from $\eta^2=\sigma_M^2/\sigma_y^2$, we have

$$\breve{\eta}^2+d\eta^2=\frac{\breve{\sigma}_M^2+\delta\sigma_M^2}{\breve{\sigma}_y^2+\delta\sigma_y^2}=\frac{\breve{\sigma}_y^2\,\delta\sigma_M^2-\breve{\sigma}_M^2\,\delta\sigma_y^2}{\breve{\sigma}_y^2(\breve{\sigma}_y^2+\delta\sigma_y^2)}+\breve{\eta}^2.$$

Thus:

$$d\eta^2=\left(\frac{\delta\sigma_M^2}{\breve{\sigma}_y^2}-\breve{\eta}^2\frac{\delta\sigma_y^2}{\breve{\sigma}_y^2}\right)\left(1-\frac{\delta\sigma_y^2}{\breve{\sigma}_y^2}+\frac{(\delta\sigma_y^2)^2}{\breve{\sigma}_y^4}\right)$$

$$=\frac{\delta\sigma_M^2}{\breve{\sigma}_y^2}-\breve{\eta}^2\frac{\delta\sigma_y^2}{\breve{\sigma}_y^2}+\breve{\eta}^2\frac{(\delta\sigma_y^2)^2}{\breve{\sigma}_y^4}-\frac{\delta\sigma_M^2\,\delta\sigma_y^2}{\breve{\sigma}_y^4}$$

$$+\text{third order terms} \quad\text{..............................(iv)}.$$

Denoting mean values by square brackets we have to our order of approximation:

$$[\delta\eta^2]=\frac{[\delta\sigma_M^2]}{\breve{\sigma}_y^2}-\breve{\eta}^2\frac{[\delta\sigma_y^2]}{\breve{\sigma}_y^2}+\breve{\eta}^2\frac{[(\delta\sigma_y^2)^2]}{\breve{\sigma}_y^4}-\frac{[\delta\sigma_M^2\,\delta\sigma_y^2]}{\breve{\sigma}_y^4} \quad\text{..................(iv)}^{bis}.$$

We have accordingly to find four mean quantities, $[\delta\sigma_M^2]$, $[\delta\sigma_y^2]$, $[(\delta\sigma_y^2)^2]$ and $[\delta\sigma_M^2\,\delta\sigma_y^2]$. Consider first

$$\bar{y}_{x_s}=\frac{S(n_{x_s y_t}\,y_t)}{\breve{n}_{x_s}}.$$

* Reproduced from lecture notes.

Accordingly $$\delta\bar{y}_{x_s}=\frac{S\{\delta n_{x_s v_t} y_t\}}{\bar{n}_{x_s}}\left(1-\frac{\delta n_{x_s}}{\bar{n}_{x_s}}\right)-\bar{y}_{x_s}\frac{\delta n_{x_s}}{\bar{n}_{x_s}}\left(1-\frac{\delta n_{x_s}}{\bar{n}_{x_s}}\right) \quad \text{(v)},$$
to our order of approximation.

From this it follows, remembering that*

$$[\delta n_{x_s v_t}]=0, \qquad [\delta n_{x_s v_t}\,\delta n_{x_s}]=\bar{n}_{x_s v_t}\left(1-\frac{\bar{n}_{x_s}}{N}\right), \qquad [(\delta n_{x_s})^2]=\bar{n}_{x_s}\left(1-\frac{\bar{n}_{x_s}}{N}\right),$$

that $$[\delta\bar{y}_{x_s}]=0 \quad \text{(vi)}.$$

Again from (v):

$$\delta\bar{y}_{x_s}=\frac{S(\delta n_{x_s v_t} y_t)}{\bar{n}_{x_s}}-\frac{\bar{y}_{x_s}}{\bar{n}_{x_s}}\delta n_{x_s}+\text{square and higher terms}$$
$$=\frac{S\{(\delta n_{x_s v_t})(y_t-\bar{y}_{x_s})\}}{\bar{n}_{x_s}}+\dots,$$

$$\delta\bar{y}_{x_s}^2=\frac{S\{(\delta n_{x_s v_t})^2(y_t-\bar{y}_{x_s})^2\}}{\bar{n}_{x_s}^2}+\frac{2S'\{(\delta n_{x_s v_t}\,\delta n_{x_s v_{t'}})(y_t-\bar{y}_{x_s})(y_{t'}-\bar{y}_{x_s})\}}{\bar{n}_{x_s}^2}+\text{third order terms},$$

where S' is a summation for all different values of t and t'.

Or: $$[(\delta\bar{y}_{x_s})^2]=\frac{S\left\{\bar{n}_{x_s v_t}\left(1-\frac{\bar{n}_{x_s v_t}}{N}\right)(y_t-\bar{y}_{x_s})^2\right\}}{\bar{n}_{x_s}^2}-\frac{2S'\{\bar{n}_{x_s v_t}\bar{n}_{x_s v_{t'}}(y_t-\bar{y}_{x_s})(y_{t'}-\bar{y}_{x_s})\}}{N\bar{n}_{x_s}^2}+\text{etc.},$$

remembering $$[\delta n_{x_s v_t}\,\delta n_{x_s v_{t'}}]=-\frac{\bar{n}_{x_s v_t}\bar{n}_{x_s v_{t'}}}{N},$$

$$=\frac{\sigma_{y_{x_s}}^2}{\bar{n}_{x_s}}-\left\{\frac{S\{\bar{n}_{x_s v_t}(y_t-\bar{y}_{x_s})\}}{\bar{n}_{x_s}}\right\}^2\frac{1}{N}+\text{etc.}$$

But $$S\{n_{x_s v_t}(y_t-\bar{y}_{x_s})\}=0.$$

Hence $$[(\delta\bar{y}_{x_s})^2]=\sigma_{y_{x_s}}^2/\bar{n}_{x_s} \quad \text{(vii)},$$

i.e. the same value as if n_{x_s} were constant in all the samples.

Again $[(\delta\bar{y})^2]$ is the standard-deviation squared of means of samples $=\sigma_y^2/N$. Further we shall require $[\delta\bar{y}_{x_s}\delta\bar{y}]$. Now:

$$\delta\bar{y}_{x_s}=\frac{S\{\delta n_{x_s v_t}(y_t-\bar{y}_{x_s})\}}{\bar{n}_{x_s}}+\text{etc.},$$

$$\delta\bar{y}_{x_{s'}}=\frac{S\{\delta n_{x_{s'} v_t}(y_t-\bar{y}_{x_{s'}})\}}{\bar{n}_{x_{s'}}}+\text{etc.},$$

$$[\delta\bar{y}_{x_s}\delta\bar{y}_{x_{s'}}]=-\frac{SS\{\bar{n}_{x_s v_t}\bar{n}_{x_{s'} v_{t'}}(y_t-\bar{y}_{x_s})(y_{t'}-\bar{y}_{x_{s'}})\}}{N\bar{n}_{x_s}\bar{n}_{x_{s'}}}+\text{third order terms}$$
$$=-\frac{S\{\bar{n}_{x_s v_t}(y_t-\bar{y}_{x_s})\}}{N\bar{n}_{x_s}}\times\frac{S\{\bar{n}_{x_{s'} v_{t'}}(y_{t'}-\bar{y}_{x_{s'}})\}}{\bar{n}_{x_{s'}}N}+\text{etc.}$$

Or, $[\delta\bar{y}_{x_s}\delta\bar{y}_{x_{s'}}]=0$, because both these factors vanish ....(viii).

Now consider $$[\delta\bar{y}_{x_s}\delta n_{x_s}]=\frac{S\{[\delta n_{x_s v_t}\,\delta n_{x_s}](y_t-\bar{y}_{x_s})\}}{\bar{n}_{x_s}}$$
$$=S\left\{\frac{\bar{n}_{x_s v_t}}{\bar{n}_{x_s}}\left(1-\frac{\bar{n}_{x_s}}{N}\right)(y_t-\bar{y}_{x_s})\right\}$$
$$=\frac{1}{\bar{n}_{x_s}}\left(1-\frac{\bar{n}_{x_s}}{N}\right)S\{\bar{n}_{x_s v_t}(y_t-\bar{y}_{x_s})\}$$
$$=0 \quad \text{(ix)}$$

* *Biometrika*, Vol. III. p. 273 *et seq.*

Similarly

$$[\delta\tilde{y}_{x_s}\,\delta n_{x_{s'}}] = \frac{S\{[\delta n_{x_s y_t}\,\delta n_{x_{s'}}]\,(y_t - \tilde{y}_{x_s})\}}{\breve{n}_{x_s}} - \frac{\breve{n}_{x_{s'}}}{N}\,\frac{S\{\breve{n}_{x_s y_t}\,(y_t - \tilde{y}_{x_s})\}}{\breve{n}_{x_s}}$$

$$= 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{x})^*$$

Now

$$\delta\bar{y} = \frac{S(\delta n_{x_s}\,\tilde{y}_{x_s}) + S(\breve{n}_{x_s}\,\delta\tilde{y}_{x_s})}{N}.$$

Multiply by $\delta\tilde{y}_{x_s}$ and take the mean, then by (viii), (ix) and (x) above

$$[\delta\tilde{y}_{x_s}\,\delta\bar{y}] = \frac{\breve{n}_{x_s}}{N}[(\delta\tilde{y}_{x_s})^2] = \frac{\breve{\sigma}_{yx_s}{}^2}{N} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{xi}).$$

Combining these results we have

$$[(\delta\tilde{y}_{x_s} - \delta\bar{y})^2] = \frac{\breve{\sigma}_{yx_s}{}^2}{\breve{n}_{x_s}} - \frac{2}{N}\breve{\sigma}_{yx_s}{}^2 + \frac{\breve{\sigma}_y{}^2}{N} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{xii}).$$

We can now obtain the mean value of $\sigma_M{}^2$ in samples. Since

$$\sigma_M{}^2 = S\{n_{x_s}(\tilde{y}_{x_s} - \bar{y})^2\}/N,$$

$$\sigma_M{}^2 + \delta\sigma_M{}^2 = S\{(n_{x_s} + \delta n_{x_s})(\tilde{y}_{x_s} - \bar{y} + \delta\tilde{y}_{x_s} - \delta\bar{y})^2\}/N,$$

and

$$N\delta\sigma_M{}^2 = S\{\delta n_{x_s}(\tilde{y}_{x_s} - \bar{y})^2\} + 2S\{(\tilde{y}_{x_s} - \bar{y})\,n_{x_s}(\delta\tilde{y}_{x_s} - \delta\bar{y})\}$$
$$+ 2S\{(\tilde{y}_{x_s} - \bar{y})\,\delta n_{x_s}(\delta\tilde{y}_{x_s} - \delta\bar{y})\}$$
$$+ S\{n_{x_s}(\delta\tilde{y}_{x_s} - \delta\bar{y})^2\}$$
$$+ \text{terms of third order} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{xiii}).$$

The summations of the first line both vanish for they contain linear terms in δn_{x_s}, $\delta\tilde{y}_{x_s}$ and $\delta\bar{y}$ which vanish when we take mean values. We have shewn that $[\delta n_{x_s}\,\delta\tilde{y}_{x_s}]$ is zero. The mean value of $[\delta n_{x_s}\,\delta\bar{y}]$ is to be found from

$$[\delta n_{x_s}\,\delta\bar{y}] = [\delta n_{x_s}\,S(\delta n_{y_t}\,y_t)]/N$$
$$= S_t\left(\breve{n}_{x_s y_t} - \frac{\breve{n}_{x_s}\,\breve{n}_{y_t}}{N}\right) y_t/N$$
$$= \frac{\breve{n}_{x_s}}{N}(\tilde{y}_{x_s} - \bar{y}).$$

Accordingly the mean of the second line of the expression in (xiii) becomes

$$-2S\{\breve{n}_{x_s}(\tilde{y}_{x_s} - \bar{y})^2\}/N = -2\breve{\sigma}_M{}^2 = -2\breve{\eta}^2\breve{\sigma}_y{}^2,$$

and we have

$$N[\delta\sigma_M{}^2] = -2\breve{\eta}^2\breve{\sigma}_y{}^2 + S\left\{\breve{n}_{x_s}\left(\frac{\breve{\sigma}_{yx_s}{}^2}{\breve{n}_{x_s}} - \frac{2}{N}\breve{\sigma}_{yx_s}{}^2 + \frac{\sigma_y{}^2}{N}\right)\right\}.$$

Now $S(\breve{n}_{x_s}) = N$, $S(\breve{n}_{x_s}\breve{\sigma}_{yx_s}{}^2)/N =$ weighted mean square standard-deviation of the arrays $= (1 - \breve{\eta}^2)\,\breve{\sigma}_y{}^2$; and, if there be κ arrays, $S(\breve{\sigma}_{yx_s}{}^2)/\kappa =$ unweighted mean square standard-deviation of arrays $= \lambda_1(1 - \breve{\eta}^2)\,\breve{\sigma}_y{}^2$, let us say, where λ_1 will not differ much from unity and will equal unity if $\breve{\eta}$ be zero, or $\breve{\eta}$ be zero, or if there be homoscedasticity. Accordingly

$$\frac{[\delta\sigma_M{}^2]}{\breve{\sigma}_y{}^2} = \frac{(\kappa\lambda_1 - 2)(1 - \breve{\eta}^2) + 1 - 2\breve{\eta}^2}{N} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{xiv}).$$

We now take the second term of (iv)$^{\mathrm{bis}}$ and find very easily†:

$$\frac{[\delta\sigma_y{}^2]}{\breve{\sigma}_y{}^2} = -\frac{1}{N} \quad \ldots(\mathrm{xv}).$$

* See "Theory of Skew Correlation," *Drapers' Company Research Memoirs, Biometric Series*, No. XIV, p. 13, where (viii)—(x) have been previously demonstrated.

† This is really the familiar relation of text-books of theory of observations that mean $\sigma_y{}^2$ of samples $= \breve{\sigma}_y{}^2(N-1)/N$.

Hence $$\frac{[\delta\sigma_M{}^2]}{\breve{\sigma}_y{}^2}-\breve{\eta}^2\frac{[\delta\sigma_y{}^2]}{\breve{\sigma}_y{}^2}=\frac{(\kappa\lambda_1-1)(1-\breve{\eta}^2)}{N}\quad\text{(xvi)*}.$$

The third term of (iv)^{bis} involves $[(\delta\sigma_y{}^2)^2]$, or the standard-deviation of the second moment coefficient μ_2 of the y-variable. But this equals $(\mu_4-\mu_2{}^2)/N=\breve{\sigma}_y{}^4(\beta_2-1)/N$. Accordingly the third term is

$$\breve{\eta}^2[(\delta\sigma_y{}^2)^2]/\breve{\sigma}_y{}^4=\breve{\eta}^2(\beta_2-1)/N\quad\text{(xvii)}.$$

It now remains to find the last term depending on $[\delta\sigma_M{}^2\,\delta\sigma_y{}^2]$.

Remembering that $\sigma_y{}^2=S\{n_{y_t}(y_t-\bar{y})^2\}/N$, we easily find

$$\delta\sigma_y{}^2=\frac{S\{\delta n_{y_t}(y_t-\bar{y})^2\}}{N}-\frac{2\delta\bar{y}\,.\,S\{\breve{n}_{y_t}(y_t-\bar{y})\}}{N}+\text{terms of second order}$$

$$=\frac{S\{\delta n_{y_t}(y_t-\bar{y})^2\}}{N}+\text{terms of second order}\quad\text{(xviii)}.$$

Similarly from (ii):

$$\delta\sigma_M{}^2=\frac{S\{\delta n_{x_s}(\bar{y}_{x_s}-\bar{y})^2\}}{N}+\frac{2S\{\breve{n}_{x_s}(\bar{y}_{x_s}-\bar{y})\delta\bar{y}_{x_s}\}}{N}-\frac{2\delta\bar{y}\,S\{\breve{n}_{x_s}(\bar{y}_{x_s}-\bar{y})\}}{N}+\text{terms of second order}$$

$$=\frac{S\{\delta n_{x_s}(\bar{y}_{x_s}-\bar{y})^2\}}{N}+\frac{2S\{\breve{n}_{x_s}(\bar{y}_{x_s}-\bar{y})\,\delta\bar{y}_{x_s}\}}{N}+\text{terms of second order}\quad\text{(xix)}.$$

When we take the product it will only be needful to take terms of the first order in $\delta\sigma_y{}^2$ and $\delta\sigma_M{}^2$, if we neglect third order terms.

The sum in (xviii) refers to t and in (xix) to s.

We consider in the first place the mean $[\delta n_{x_s}\delta n_{y_t}]$. But this equals

$$\breve{n}_{x_s y_t}-\frac{\breve{n}_{x_s}\breve{n}_{y_t}}{N}\quad\text{(xx)}.$$

We next deal with $[\delta n_{y_t}\,\delta\bar{y}_{x_s}]$. But

$$\delta\bar{y}_{x_s}=S\{\delta n_{x_s y_{t'}}(y_{t'}-\bar{y}_{x_s})\}/\breve{n}_{x_s}+\text{etc.},$$

$$\therefore\ [\delta n_{y_t}\,\delta\bar{y}_{x_s}]=S\{[\delta n_{x_s y_{t'}}\,\delta n_{y_t}](y_{t'}-\bar{y}_{x_s})\}/\breve{n}_{x_s}$$

$$=\breve{n}_{x_s y_t}\left(1-\frac{\breve{n}_{y_t}}{N}\right)\left(\frac{y_t-\bar{y}_{x_s}}{\breve{n}_{x_s}}\right)-\frac{S\{\breve{n}_{x_s y_t}\ \breve{n}_{y_t}\}}{N}\ \frac{(y_t-\bar{y}_{x_s})}{\breve{n}_{x_s}}$$

$$=\frac{\breve{n}_{x_s y_t}(y_t-\bar{y}_{x_s})}{\breve{n}_{x_s}}-\frac{\breve{n}_{y_t}}{N}\frac{S\{\breve{n}_{x_s y_t}(y_t-\bar{y}_{x_s})\}}{\breve{n}_{x_s}}+\text{terms of third order.}$$

Here the second summation is clearly zero, or to our approximation

$$[\delta n_{y_t}\,\delta\bar{y}_{x_s}]=\frac{\breve{n}_{x_s y_t}(y_t-\bar{y}_{x_s})}{\breve{n}_{x_s}}\quad\text{(xxi)}.$$

Using (xx) we have:

$$[\delta n_{x_s}\delta\sigma_y{}^2]=S_t\left\{\left(\breve{n}_{x_s y_t}-\frac{\breve{n}_{x_s}\breve{n}_{y_t}}{N}\right)\frac{(y_t-\bar{y})^2}{N}\right\}$$

$$=\breve{n}_{x_s}\{\sigma_{y_{x_s}}{}^2+(\bar{y}_{x_s}-\bar{y})^2-\sigma_y{}^2\}/N\quad\text{(xxii)},$$

and from (xxi)
$$[\delta\bar{y}_{x_s}\,\delta\sigma_y{}^2]=S_t\{\breve{n}_{x_s y_t}(y_t-\bar{y}_{x_s})(y_t-\bar{y})^2\}/(N\breve{n}_{x_s})$$

$$=S_t(\breve{n}_{x_s y_t}\{(y_t-\bar{y}_{x_s})^3+2(\bar{y}_{x_s}-\bar{y})(y_t-\bar{y}_{x_s})^2$$
$$+(\bar{y}_{x_s}-\bar{y})^2(y_t-\bar{y}_{x_s})\})/(\breve{n}_{x_s}N).$$

Now if $(\breve{\mu}_3)_{y_{x_s}}$ be the third moment coefficient about its mean of the x_s array of y's for the sampled population, and if we note that $S_t\{\breve{n}_{x_s y_t}(y_t-\bar{y}_{x_s})\}=0$, we can write this result:

$$[\delta\bar{y}_{x_s}\,\delta\sigma_y{}^2]=\{(\breve{\mu}_3)_{y_{x_s}}+2(\bar{y}_{x_s}-\bar{y})\breve{\sigma}_{y_{x_s}}{}^2\}/N\quad\text{(xxiii)}.$$

* This agrees with the value given in *Biometrika*, *loc. cit.*, where the squares of the statistical differentials were neglected.

We have now to multiply (xix) by $\delta\sigma_y^2$, take the mean and use (xxii) and (xxiii). We find:

$$[\delta\sigma_M^2\,\delta\sigma_y^2]=\frac{1}{N^2}S_s\{\breve{n}_{x_s}\breve{\sigma}_{yx_s}^2(\bar{y}_{x_s}-\bar{y})^2+\breve{n}_{x_s}(\bar{y}_{x_s}-\bar{y})^4-\breve{\sigma}_y^2\breve{n}_{x_s}(\bar{y}_{x_s}-\bar{y})^2$$
$$+2\breve{n}_{x_s}(\breve{\mu}_3)_{yx_s}(\bar{y}_{x_s}-\bar{y})+4\breve{n}_{x_s}\breve{\sigma}_{yx_s}^2(\bar{y}_{x_s}-\bar{y})^2\}\text{.........(xxiv).}$$

It remains to consider these terms individually. First:

$$S\{\breve{n}_{x_s}\breve{\sigma}_{yx_s}^2(\bar{y}_{x_s}-\bar{y})^2\}.$$

If the system were homoscedastic this would equal

$$(1-\breve{\eta}^2)\breve{\sigma}_y^2\times S\{\breve{n}_{x_s}(\bar{y}_{x_s}-\bar{y})^2\}=(1-\breve{\eta}^2)\breve{\sigma}_y^2\sigma_M^2N=\breve{\eta}^2(1-\breve{\eta}^2)\breve{\sigma}_y^4N.$$

Accordingly let us take

$$\lambda_2=\frac{S\{\breve{n}_{x_s}\breve{\sigma}_{yx_s}^2(\bar{y}_{x_s}-\bar{y})^2\}}{N\breve{\eta}^2(1-\breve{\eta}^2)\breve{\sigma}_y^4},$$

and remember that λ_2 will not as a rule differ widely from unity.

Secondly: $$S\{\breve{n}_{x_s}(\bar{y}_{x_s}-\bar{y})^4\}.$$

If the system were linear this would equal

$$\frac{S\{\breve{n}_{x_s}\breve{r}^4(x-\bar{x})^4\breve{\sigma}_y^4\}}{\breve{\sigma}_x^4}=\frac{\breve{r}^4\breve{\sigma}_y^4}{\breve{\sigma}_x^4}N(\mu_4)_x=\breve{r}^4N\beta_2\breve{\sigma}_y^4,$$

where β_2 is the β_2 coefficient (i.e. μ_4/μ_2^2) of the x-variable and will not be very widely different from 3.

Accordingly let us take

$$\lambda_3=\frac{S\{\breve{n}_{x_s}(\bar{y}_{x_s}-\bar{y})^4\}}{3\breve{\eta}^4N\breve{\sigma}_y^4},$$

and remember that λ_3 may be put approximately unity.

Thirdly: $$S\{\breve{n}_{x_s}(\bar{y}_{x_s}-\bar{y})^2\}=N\breve{\sigma}_M^2=N\breve{\eta}^2\breve{\sigma}_y^2.$$

Lastly: $$S\{\breve{n}_{x_s}(\breve{\mu}_3)_{yx_s}(\bar{y}_{x_s}-\bar{y})\}$$

Now $(\breve{\mu}_3)_{yx_s}$ would be zero, if the arrays were symmetrical, and will generally be small. If the arrays have equal skewness the expression equals

$$(\breve{\mu}_3)_{yx_s}S\{\breve{n}_{x_s}(\bar{y}_{x_s}-\bar{y})\}=0.$$

Generally we may take it equal to $\lambda_4\sigma_y^4N$, where λ_4 will usually be very small.

Thus finally we may write:

$$\frac{[\delta\sigma_M^2\,\delta\sigma_y^2]}{\breve{\sigma}_y^4}=\frac{1}{N}\{5\lambda_2\breve{\eta}^2(1-\breve{\eta}^2)+3\lambda_3\breve{\eta}^4-\breve{\eta}^2+2\lambda_4\}\quad\text{......................(xxv).}$$

We can now marshal our results to obtain $[\delta\eta^2]$ from (xvi), (xvii) and (xxv). We have:

$$[\delta\eta^2]=\frac{1}{N}\{\kappa\lambda_1-1)(1-\breve{\eta}^2)+\breve{\eta}^2(\beta_2-1)-5\lambda_2\breve{\eta}^2(1-\breve{\eta}^2)-3\lambda_3\breve{\eta}^4+\breve{\eta}^2-2\lambda_4\}$$
$$=\frac{1}{N}\{(\kappa\lambda_1-1)(1-\breve{\eta}^2)+\breve{\eta}^2(\beta_2-5\lambda_2)+\breve{\eta}^4(5\lambda_2-3\lambda_3)-2\lambda_4\}\quad\text{..................(xxvi),}$$

where, collecting out values:

κ = number of arrays,

$$\lambda_1=\frac{S(\breve{\sigma}_{yx_s}^2)/\kappa}{(1-\breve{\eta}^2)\breve{\sigma}_y^2}=\text{unity approximately,}$$

$\beta_2=\beta_2$-coefficient of y-variable $=3$ approximately,

$$\lambda_2=\frac{S\{\breve{n}_{x_s}\breve{\sigma}_{yx_s}^2(\bar{y}_{x_s}-\bar{y})^2}{N\breve{\eta}^2(1-\breve{\eta}^2)\breve{\sigma}_y^4}=\text{unity approximately,}$$

$$\lambda_3=\frac{S\{\breve{n}_{x_s}(\bar{y}_{x_s}-\bar{y})^4\}}{3\breve{\eta}^4N\breve{\sigma}_y^4}=\text{unity approximately,}$$

$$\lambda_4=\frac{S\{\breve{n}_{x_s}(\breve{\mu}_3)_{yx_s}(\bar{y}_{x_s}-\bar{y})\}}{N\breve{\sigma}_y^4}=\text{zero approximately.}$$

Now it is clear that the actual values of these constants could be calculated, if with considerable labour, for the given sample and their values used instead of those of the sampled population as approximations, even as we calculate such expressions in probable errors, when the actual state of the sampled population is unknown. But for many purposes it will be quite adequate, since we are supposing the sample a large one, to insert the approximate values given on the right-hand side above, as they will be divided by N. In that case we have

$$[\delta\eta^2] = \frac{1-\breve{\eta}^2}{N}(\kappa - 1 - 2\breve{\eta}^2) \quad \text{.................................(xxvii).}$$

This result differs from my previous one, already referred to, by the presence of the $2\breve{\eta}^2$.

If $\tilde{\eta}^2$ be the mean η^2 of samples, we have

$$1-\tilde{\eta}^2 = 1-(\breve{\eta}^2+\delta\eta^2) = (1-\breve{\eta}^2)\left\{1-\frac{\kappa-3+2(1-\breve{\eta}^2)}{N}\right\}.$$

Or, if $\qquad \tilde{\epsilon} = 1-\tilde{\eta}^2$, and $\breve{\epsilon} = 1-\breve{\eta}^2$,

$$\tilde{\epsilon} = \breve{\epsilon}\left(1-\frac{\kappa-3}{N}\right) - \frac{2}{N}\breve{\epsilon}^2 \quad \text{......... ....(xxviii),}$$

which is a quadratic to determine $\breve{\epsilon}$ in terms of $\tilde{\epsilon}$; the required root since $\tilde{\epsilon}$ and $\breve{\epsilon}$ must vanish simultaneously is

$$= \frac{-\sqrt{\{1-(\kappa-3)/N\}^2-(8/N)\tilde{\epsilon}}+\{1-(\kappa-3/N)\}}{4/N}.$$

Or, approximately:

$$\breve{\eta}^2 = \frac{\tilde{\eta}^2-(\kappa-3)/N}{1-(\kappa-3)/N} - \frac{2(1-\tilde{\eta}^2)^2}{N\{1-(\kappa-3)/N\}^3} \quad \text{..................... ....(xxix).}$$

Now equation (xxix) is a relation between the mean $\tilde{\eta}^2$ for all samples and the $\breve{\eta}^2$ of the sampled population. In a practical example neither of these quantities is really known. We want to find $\breve{\eta}^2$, and all we are given is the value of η^2 in a single particular sample. It might be more or less reasonable to assume that the η^2 observed in the sample was the *modal* value of the η^2 for all samples. But so far there has been no discussion of the frequency curve for the distribution of η^2 in samples, and such discussion cannot be undertaken until we have greater knowledge of the form of skew bivariate frequency distributions. We do know that in the case of normal correlation the modal value of r^2 in samples differs from the value of r^2 in the sampled population by terms of the order $1/N$ and higher inverse powers. It may be taken for granted that modal η^2 in samples will differ from mean η^2 in samples—i.e. our $\tilde{\eta}^2$—by a term of order $1/N$. Accordingly if we replace $\tilde{\eta}^2$ by modal η^2 and eventually by the value in the single sample, we should not be justified in keeping the second or $(1-\tilde{\eta}^2)^2$ term in (xxix), and we might modify $\kappa-3$ in the first term by something of the order of a unit.

Accordingly it does not seem possible at present to go further than suggest that

$$\frac{\text{Observed } \eta^2-(\kappa-3)/N}{1-(\kappa-3)/N}$$

is a reasonable value for the η^2 of the sampled population, provided N is fairly large. Equation (xxix) would give as it stands a good result, if we could find the mean η^2 for a number of samples. Of course the first consideration in any investigation of η^2 is to determine whether it is comparable with $(\kappa-1)/N$. If it be less than this value we cannot assert significant association. If it be greater than this value we have to consider whether η as observed differs considerably from

$$\sqrt{\frac{\kappa-1}{N}+\cdot67449\,\frac{1}{\sqrt{N}}},$$

and for general purposes we must settle whether η differs from $\sqrt{(\kappa-1)/N}$ by, say, $1\cdot7/\sqrt{N}$.

II. Further Note on the χ^2 test of Goodness of Fit.

By KARL PEARSON, F.R.S.

I wish to make a correction to my former note on this subject on p. 189 of this volume. Mr Fisher on p. 90 of his paper* wrote:

"In 1915 Greenwood and Yule, using four-fold tables to test the effect of inoculation against typhoid and cholera, followed Pearson in applying Elderton's table with $n'=4$."

In my reply to Mr Fisher, I had not in mind what Greenwood and Yule had done. I only considered that they had applied χ^2 to a four fold table and it seemed to me that they ought accordingly to have taken $n'=4$, precisely as I personally should do for a four-fold table in general. But on examining Messrs Greenwood and Yule's paper I find that they are not dealing with what I should admit are true four-fold tables at all. *They are questioning whether two sets of alternative categories can be considered as samples of the same population.* In other words they are dealing with a special case of the problem which I have discussed in *Biometrika*, Vol. VIII. p. 250, and they have chosen, I think, somewhat arbitrarily to exhibit their data in a four-fold table form. I consider that their table is not a true four-fold table at all. In such a four-fold table the marginal totals are perfectly free, there are four frequency classes, and if the deviations of these classes from any assumed distribution be taken, then n' will equal 4, and this is the value which must be used to obtain P. This is equally true whether we put our four categories in four-fold order or in serial order.

But clearly as approached from the standpoint of *Biometrika*, Vol. VIII. the problem of Messrs Greenwood and Yule gives $n'=2$. This arises from the fact that they have arbitrarily fixed by the size of their inoculated and uninoculated groups two of the marginal totals Such marginal totals are not fixed in a true four-fold table, where these marginal totals are subject to the variations of random sampling. On this ground I venture to think it is not wise to speak of such tables as "four-fold tables" when applying the χ^2, P test. This use led the authors to apply χ^2 for $n'=4$ and not $n'=2$. In writing my note in the last number, I supposed the four-fold tables to which Mr Fisher referred as used by Messrs Greenwood and Yule were true four-fold tables with the natural freedom of margins.

It is such tables with which I deal in my Eighteenth Memoir on Evolution. I ask what is the probability that the distribution in four cells—which may be arranged serially or in quadrate form, i.e. with free margins—differs from a given null correlation theoretical distribution, and, as I pointed out in my note—not knowing the marginal distributions of the theoretical sampled population—we do what we always do when discussing probable error, put the unknown constants of the sampled population equal to the observed constants of the sample. This is not a fixing of the marginal totals and n' remains 4, and not the 2 of Messrs Greenwood and Yule's pseudo-four-fold tables.

III. Review: *Frequency Arrays.* By H. E. Soper, M.A.†

In his "Frequency Arrays," Mr Soper, by means of the symbolic expression of certain fundamental ideas, is enabled to demonstrate anew in a few pages many propositions which have been proved from time to time, the exposition of which has involved for the most part lengthy discussion. He hopes that familiarity with his method of treatment will lead to further discoveries in mathematical statistics and allied subjects. His book will appeal more to the mathematical statistician than to the practical statistician, and should appeal to the research worker, because a new world is opened to him where whole sentences are expressed quickly in symbolic forms, and where events may be said to move more rapidly than in the slow-moving

* *Journal of R. Statistical Society*, Vol. LXXXV. p. 90.

† Cambridge University Press.

algebraical world. The book will also appeal to the teacher because it should enable him to demonstrate the truth of many elementary propositions very quickly and easily.

The author starts out by representing the "frequency array" resulting from taking a sample of n individuals (with replacement) from a population containing individuals having characters A, B, C, ... in different degrees as $\Sigma p_{abc}\ldots A^a B^b C^c\ldots$, where the general term gives the frequency of occurrence (or the probability of occurrence) of individuals having the character A in degree a, the character B in degree b, the character C in degree c, and so on. The phrase "Frequency Array" he applies not only to the "array" in the ordinary sense of the term, in one variable; but also to describe a Table of Frequency in any number of variables.

When we wish to refer to the point-binomial we either write down a certain number of terms and the general term as follows:

$$p^n, np^{n-1}q, \frac{n \cdot n-1}{1 \cdot 2} p^{n-2} q^2, \ldots \frac{n!}{r!\,\overline{p-r}\,!} p^{n-r} q^r, \ldots$$

or we write down $(p+q)^n$, always understanding that this last expression is supposed expanded, and the connecting positive signs removed, the individual terms representing the frequency of occurrence of 0, 1, 2, ... r, ... successes, in a sample of n, when the chance of a success is q and the chance of a failure is $p=1-q$ for any one "draw" of a sample.

Mr Soper points out that the expression

$$\Sigma \frac{n!}{r!\,\overline{n-r}\,!} p^{n-r} q^r A^r = (p+qA)^n$$

represents in itself, without any mental restriction as to the removal of the conjunctive signs, and without any further explanations, the frequency distribution resulting from drawing a sample of n (with replacement) from a population, where the chance of any one draw being a success (A) is q and the chance of its being a failure is p. He points out that the expression $(p+q)^n$ in the ordinary notation is equal to unity and cannot therefore be used and submitted to any algebraic processes; but $(p+qA)^n$ can submit to such processes, with interesting results as the author shews in his book. He also extends this method of presentation to continuous variates and represents for instance the Gaussian frequency distribution of one variate as

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}}{\sqrt{2\pi}\,.\,\sigma} A^x \,.\, dx,$$

where $\frac{e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}}{\sqrt{2\pi}\,.\,\sigma}\,dx$ is the frequency of occurrence between x and $x+dx$ of the variate A, and the integral sign replaces the sign of summation in the previous expression. Thus this expression represents the frequency, when the distribution is Gaussian, for all values of x from $-\infty$ to $+\infty$. The author's applications of the ordinary laws of algebra to such expressions give the usual moment equations and so on very easily.

Mr Soper's notation and methods, being perfectly general, enable him easily to obtain the moments of the double hypergeometric series and the usual formulae which occur in problems of multiple regression. He also applies his methods to get the results which have been proved in other problems, such as that of random migration.

There is a great deal of concentrated work in Mr Soper's book, and it is to be hoped that work on the same lines may lead to new discoveries. He has only attempted to illustrate his methods on old problems to shew the uses to which this symbolic notation may be put. The author probably forgets that his readers are not so familiar with his mode of presenting a problem as himself, and contents himself with rather a short preface before plunging into the difficult part of the work. His book might therefore have been improved by a more detailed introduction. Finally we must thank Mr Soper for his short proof of the χ^2 formula, which he, perhaps characteristically, relegates to a footnote.

E. C. R.

IV. Review: *Some Formulas in the Theory of Interpolation of Many Independent Variables.* By Professor SEIMATSU NARUMI*.

This paper is mainly of theoretical interest. The author arrives at the extension of Newton's forward and backward interpolation formulae for the case of two variables and also at two-dimensional extensions of the so-called central difference formulae, which are generally attributed to Stirling, Bessel and Gauss, but are in reality due to Newton.

The reader is left to supply much of the algebra and the analysis seems to us somewhat involved, though admirably rigorous.

We refer to the formulae of Bessel, Stirling and Gauss as "so called" central difference formulae, because we wish to draw a distinction between them and Everett's central difference formula, which, we think, should be regarded as a true central difference formula. Everett's formula seems to possess two important properties:

(1) Only even order differences appear.

(2) Only $2n$ ordinates are needed to obtain correctness up to but not including $(2n+2)$th order differences.

The latter property is shared by the formulae of Bessel and Gauss though Stirling's requires $2n+1$ ordinates for the same accuracy; the former property however only Everett's formula possesses; the other three all containing, implicitly or explicitly, odd order differences.

When we extend Everett's formula to two dimensions property (1) still holds, and we find also that the interpolated value is based on the values of the function at the "nearest" points, a complete set of which always gives accuracy up to, but not including, a given even order of differences.

This property will presumably be shared by the two-dimensional forms of Gauss and Bessel, but not by the extended Stirling formula, and moreover Everett's form possesses the very great advantage (still greater in two dimensions than in one) over all three others, that odd order differences do not occur. This is indeed its distinguishing characteristic.

The extension of Newton's formula is of course not new and this is the only formula in the paper illustrated by a numerical example. Newton's formula is, of course, open to the objection that the interpolated value is not based on the values of the function at the "nearest" points, and we should like to have seen some numerical illustrations of Professor Narumi's other formulae and contrasted them with the two-dimensional form of Everett's central difference formula, given by Professor Pearson. For the above reasons we anticipate that the former will be found less convenient for numerical purposes. J. O. I.

* *Tôhoku Mathematical Journal*, Vol. XVIII. pp. 309–321.

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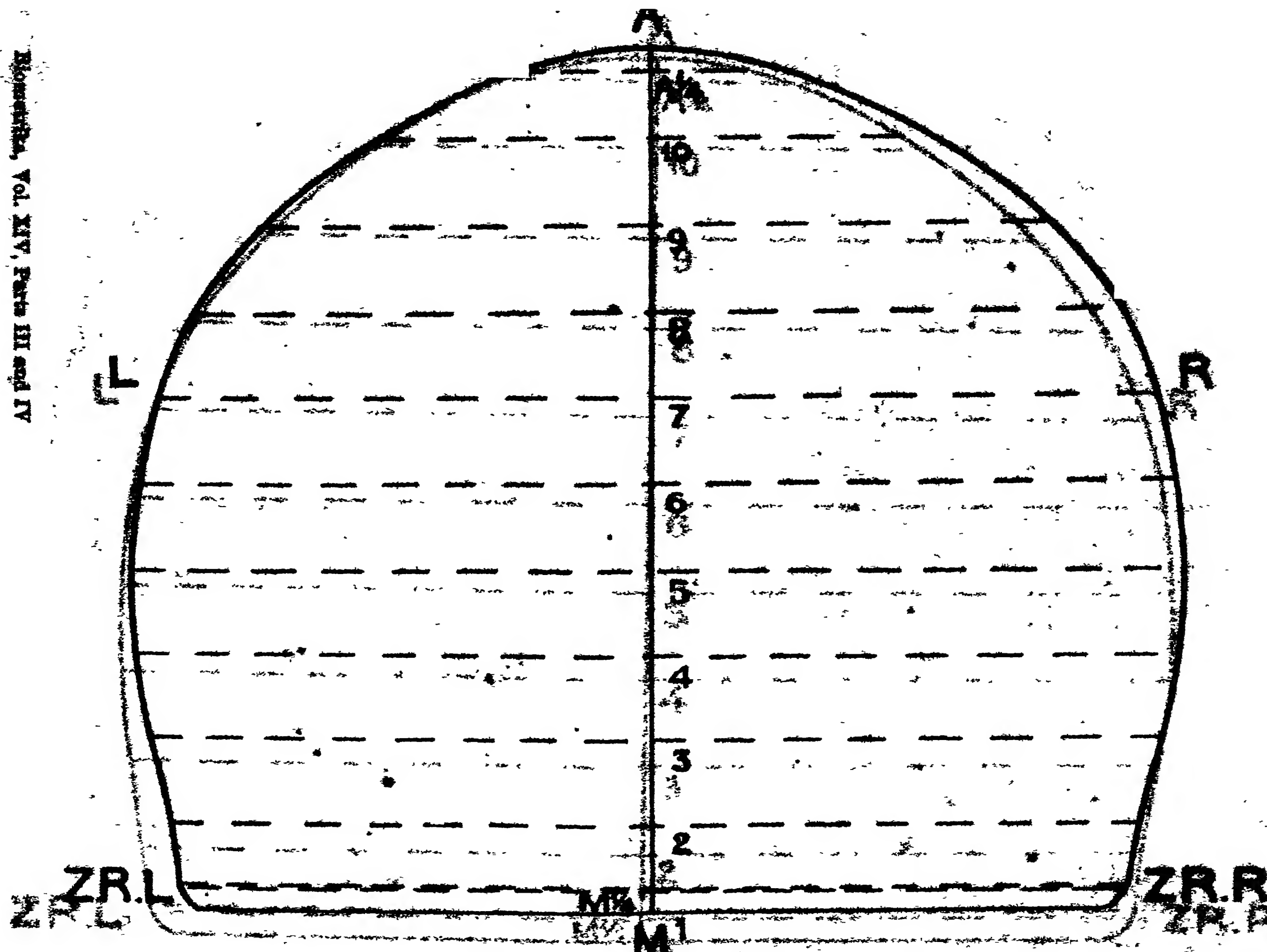

Fig. 1. Tibetan A. ♂. Vertical Type Contour. (17 Crania.)

Biometrika, Vol. XIV Parts III and IV

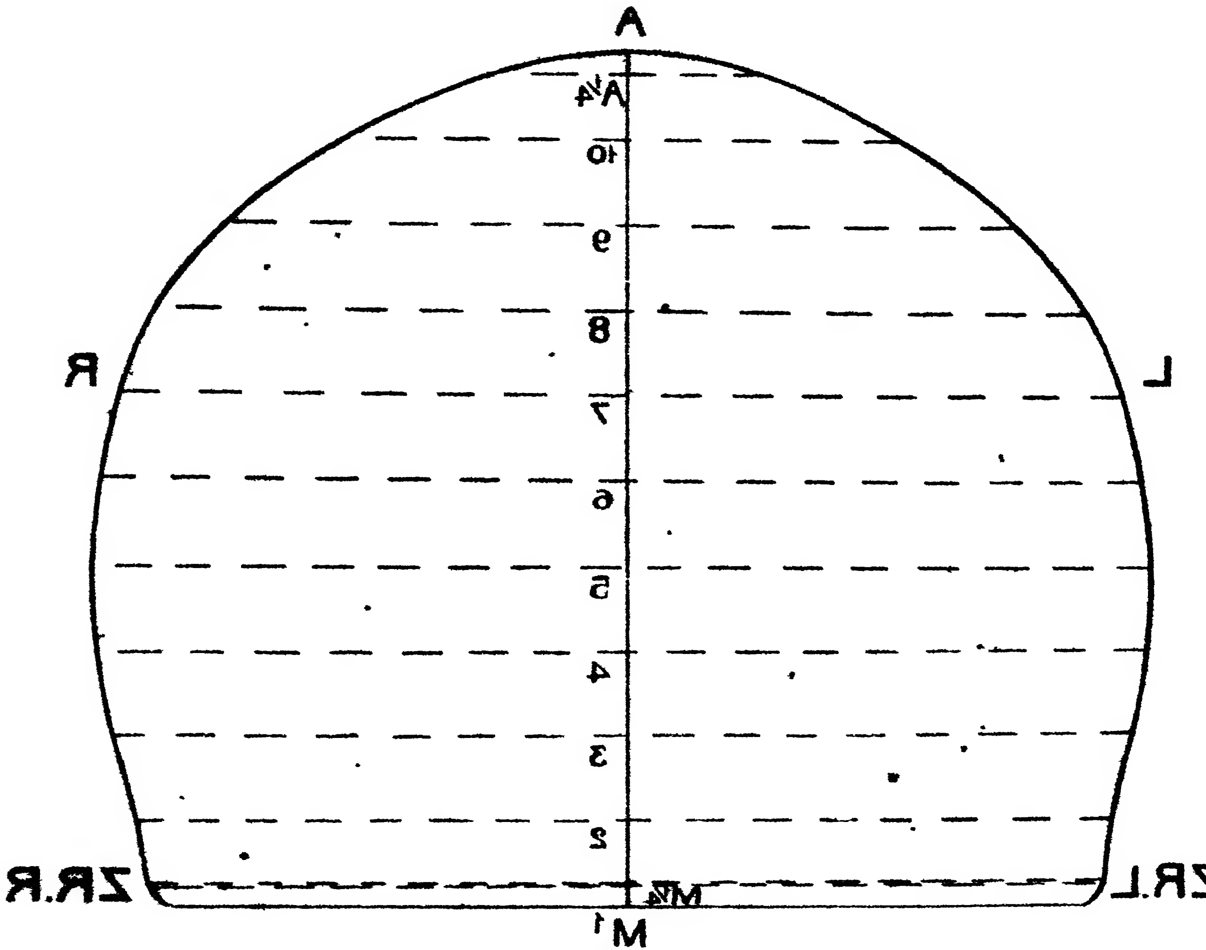

Fig. 1. Tibetan A. Vertical Type Contour (17 Crania)

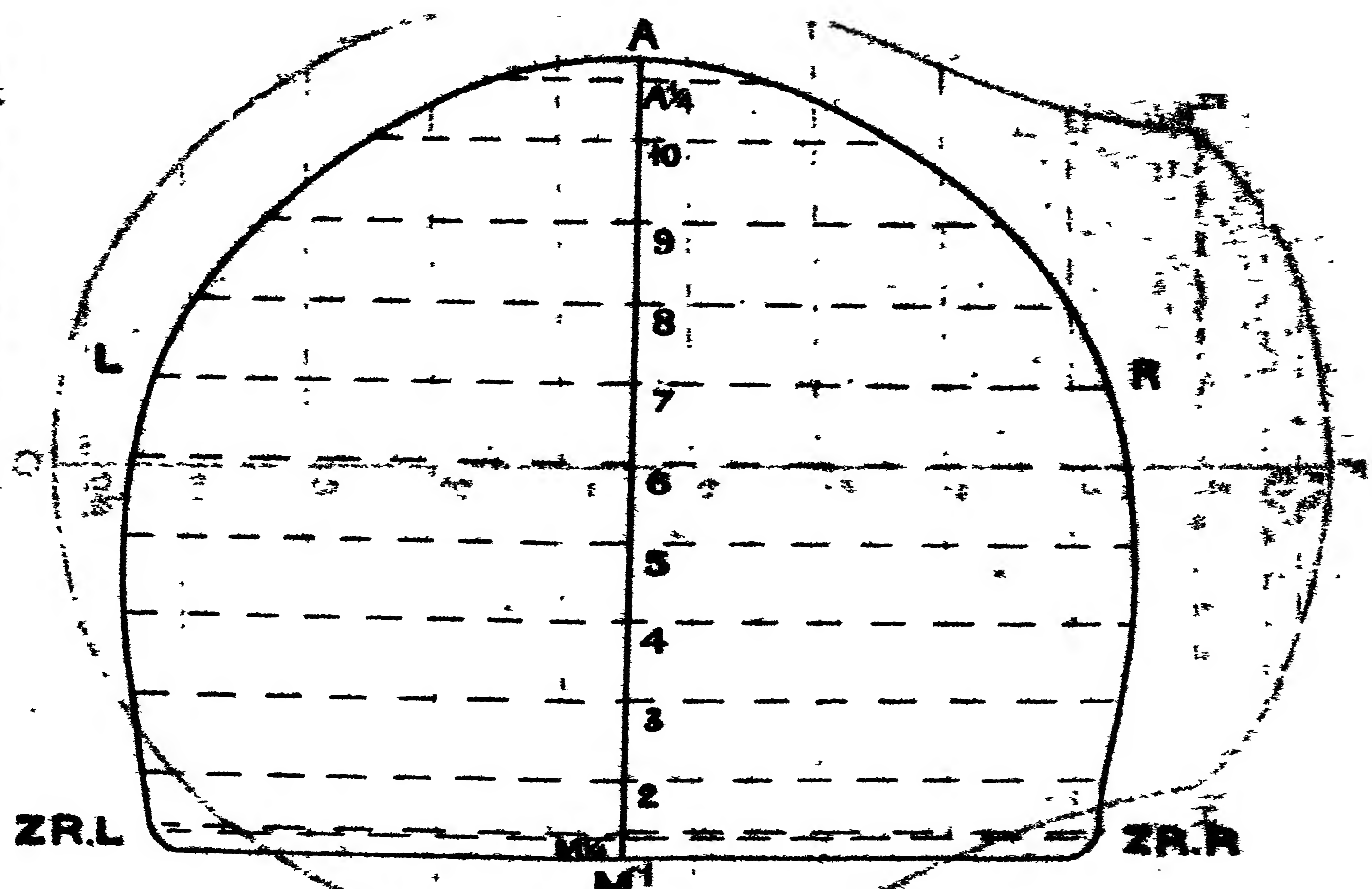

Fig. 2. Tibetan B. ♂. Vertical Type Contour. (15 Crania.)

Biometrika, Vol. XIV, Parts III and IV

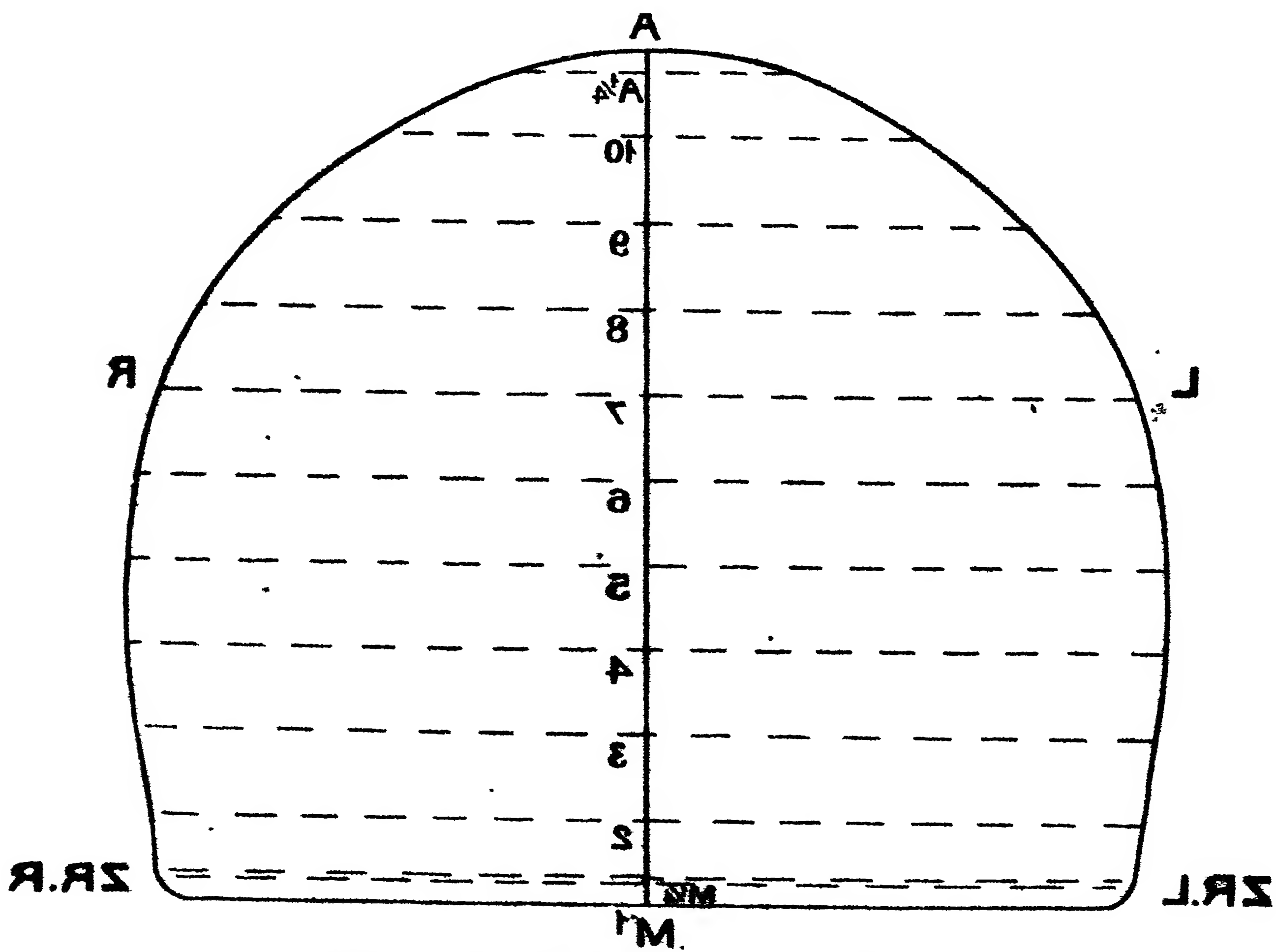

Fig. 2. Tibetan B 3. Vertical Type Contour (15 Crania.)

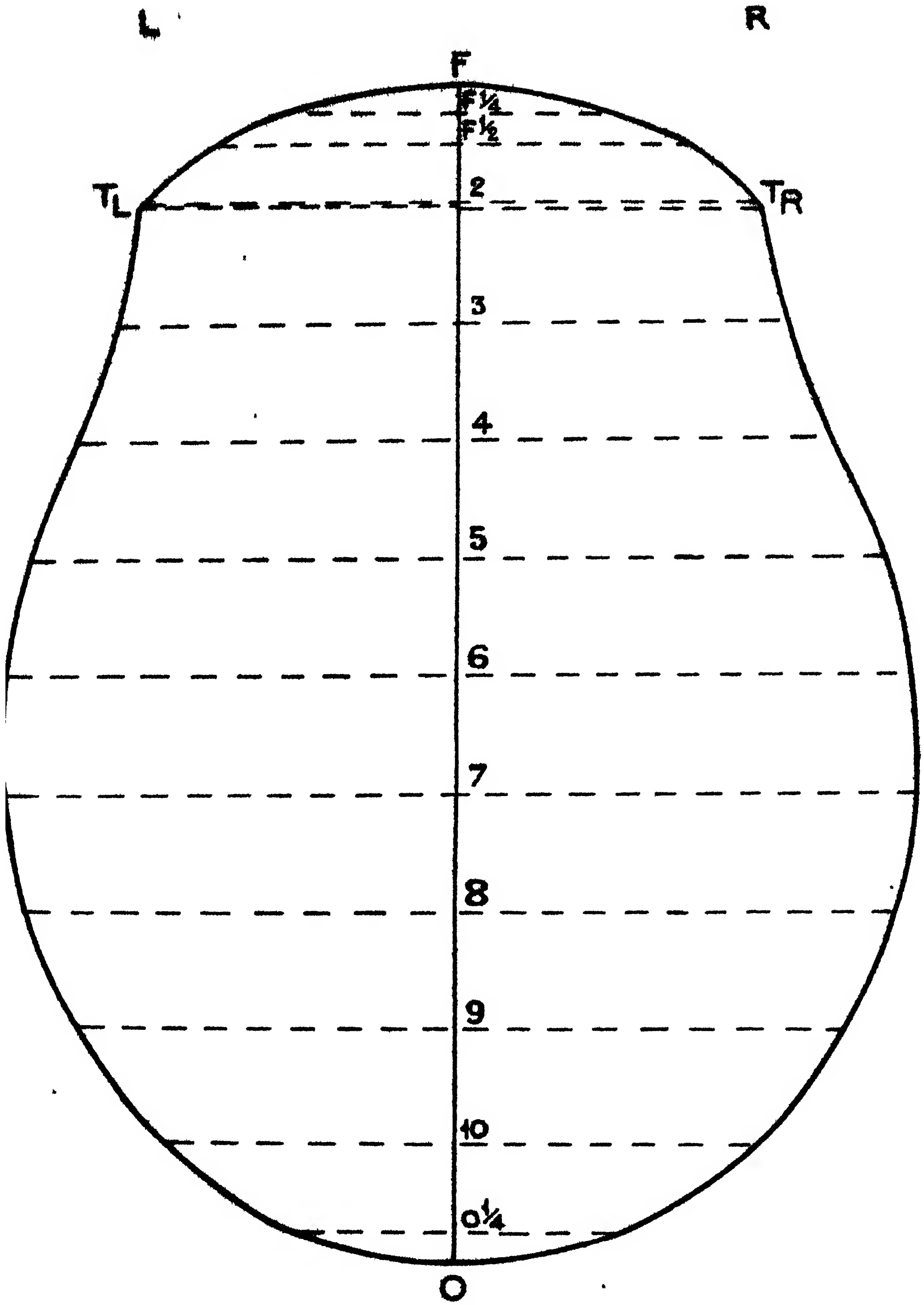

Fig. 8. Tibetan A. ♂. Horizontal Type Contour. (17 Crania.)

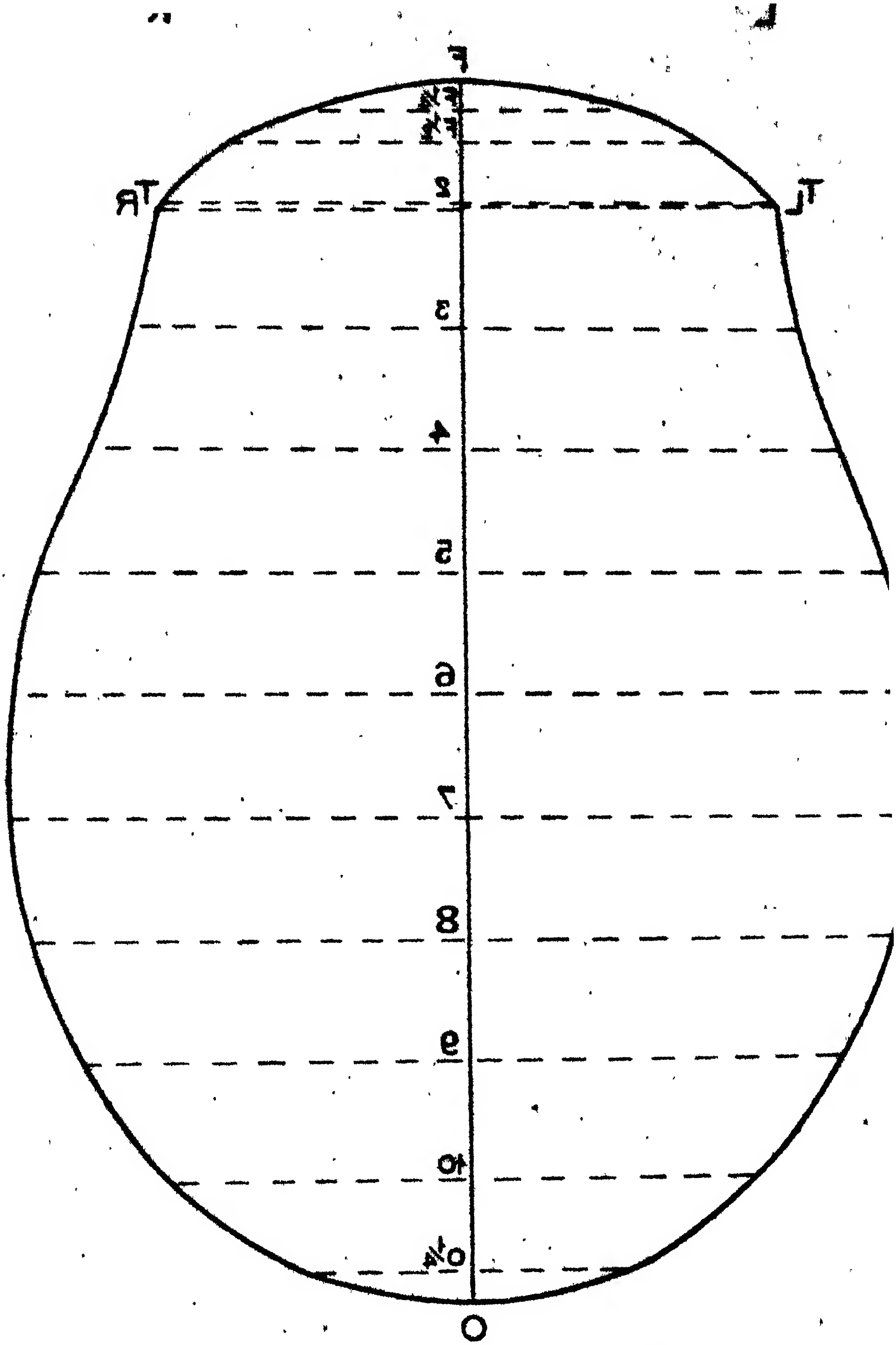

Fig. 8. Tibetan A. ♂. Horizontal Type Contour. (17 Crania.)

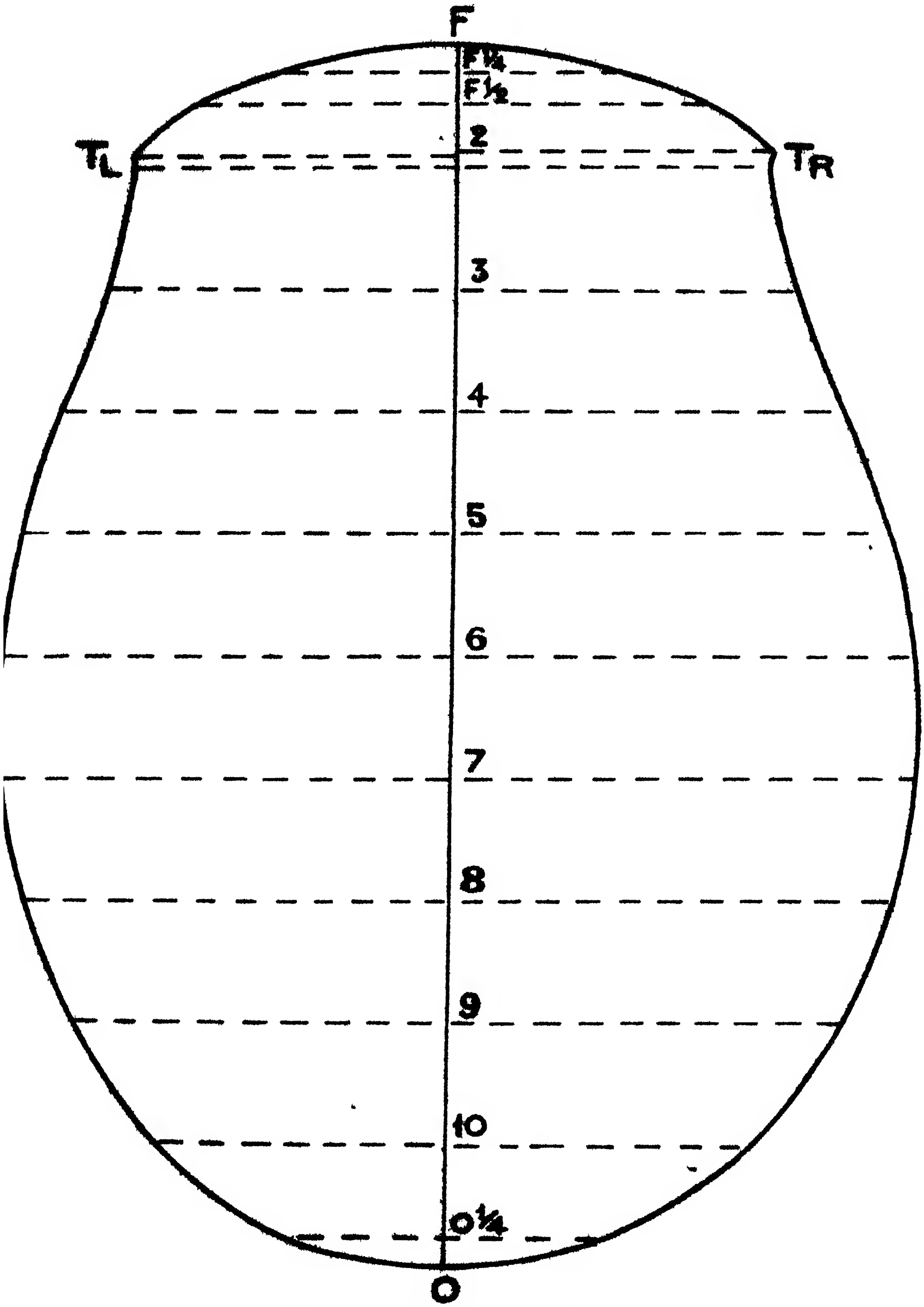

Fig. 4. Tibetan B. ♂. Horizontal Type Contour (15 Crania)

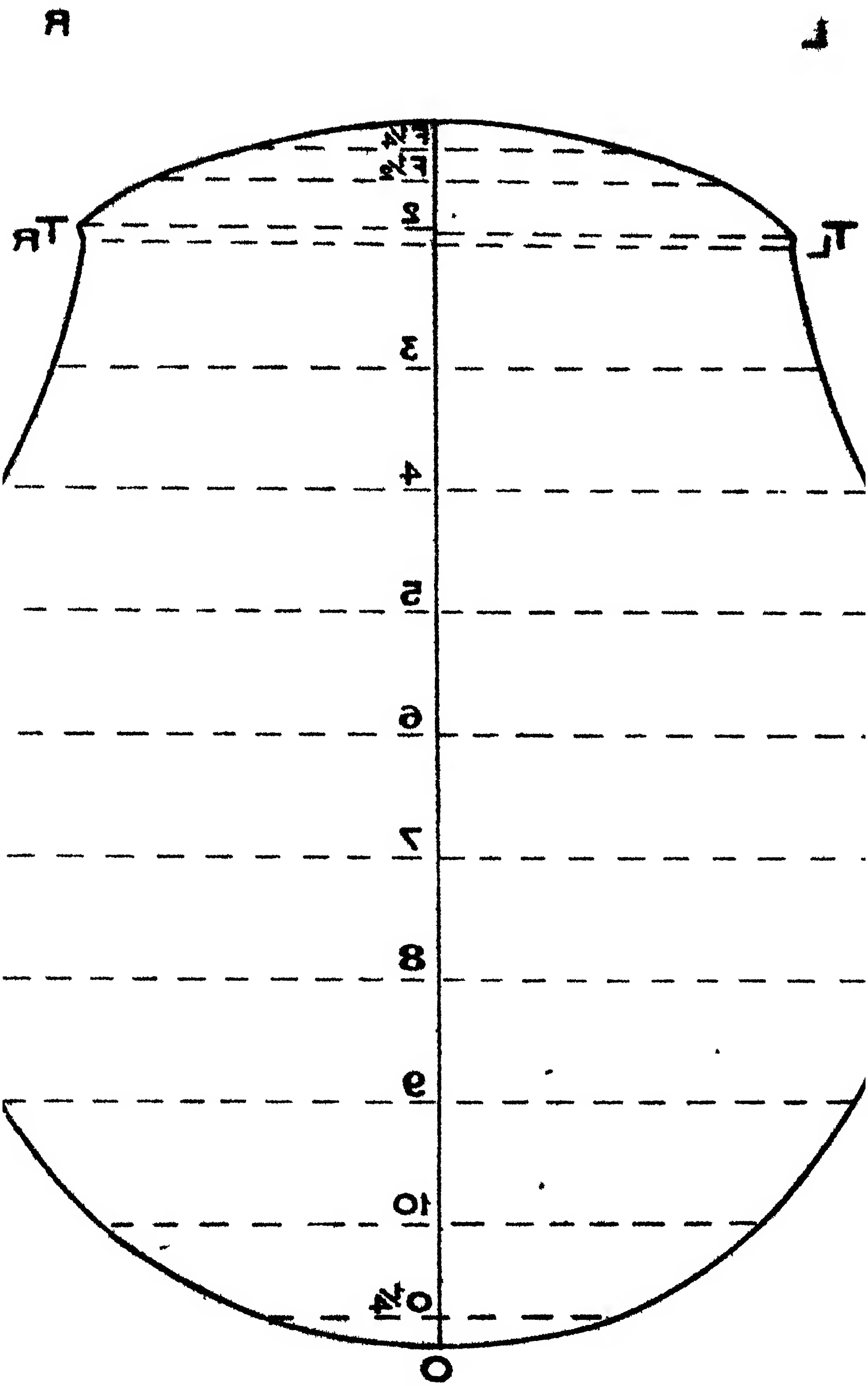

Fig 4. Tibetan B ♂ Horizontal Type Contour (15 Crania)

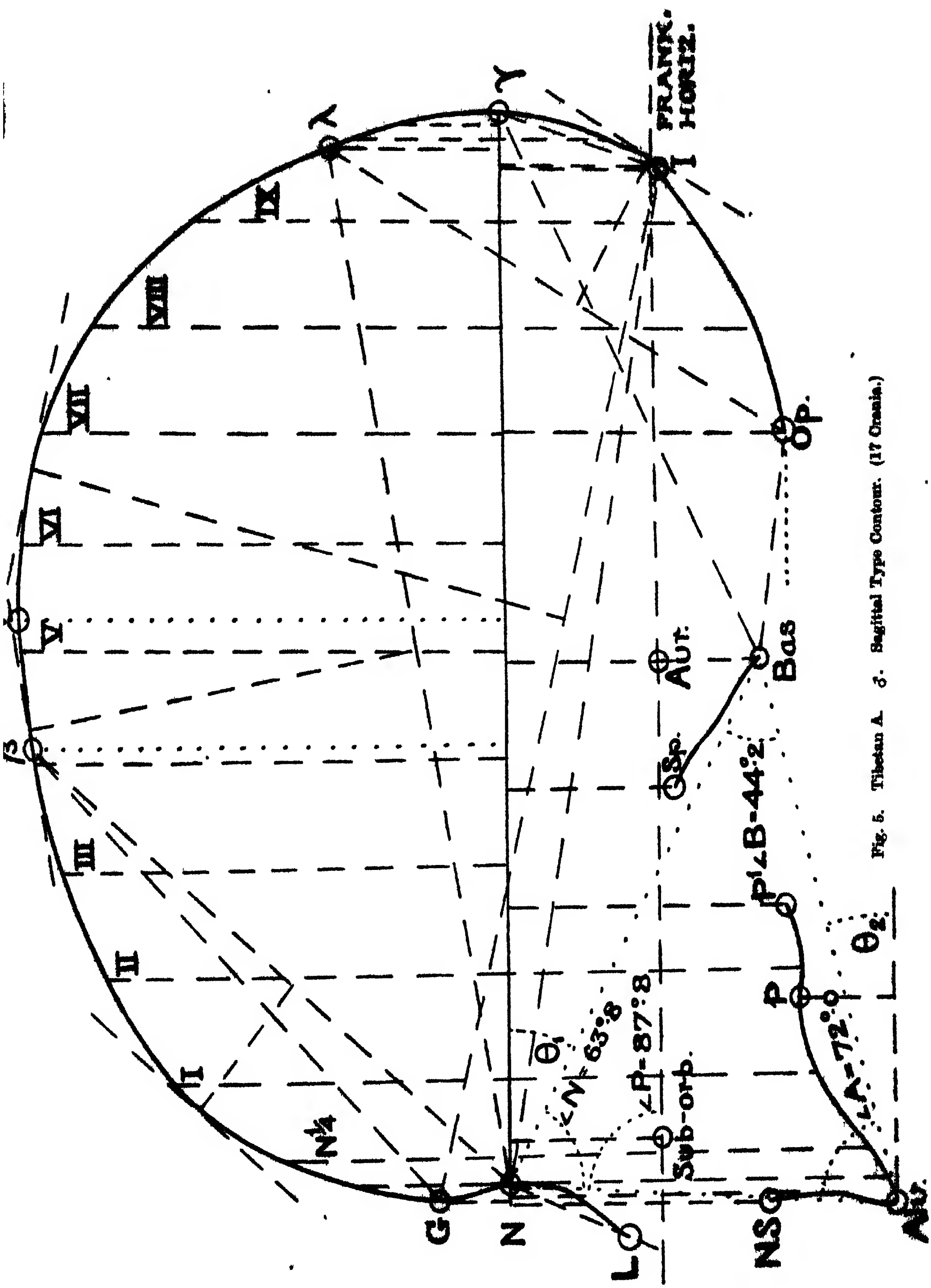

Fig. 5. Tibetan A ♂. Sagittal Type Contour. (17 Crania.)

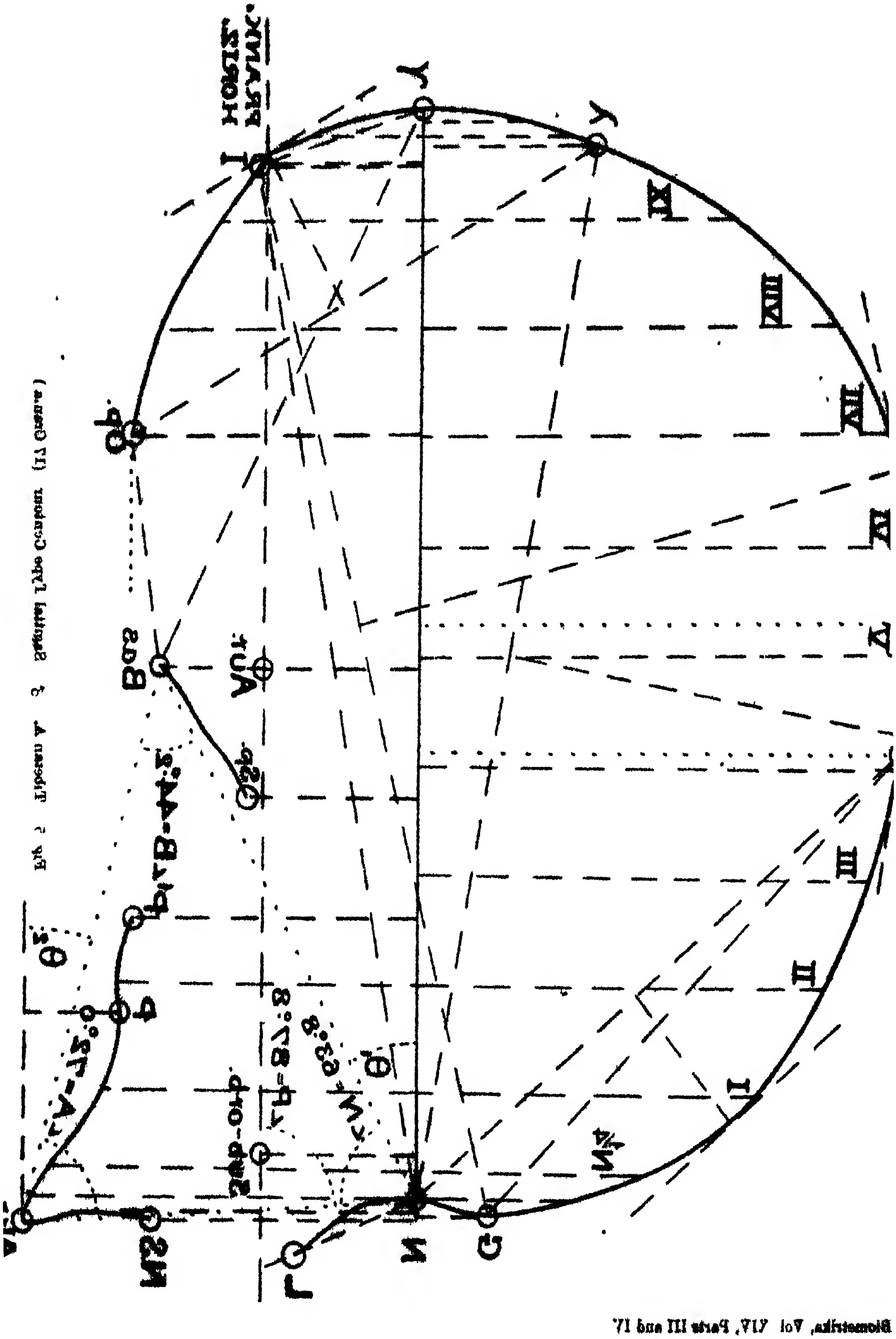

Fig. 5. Tibetan A. Sagittal Type Contour (17 Crania)

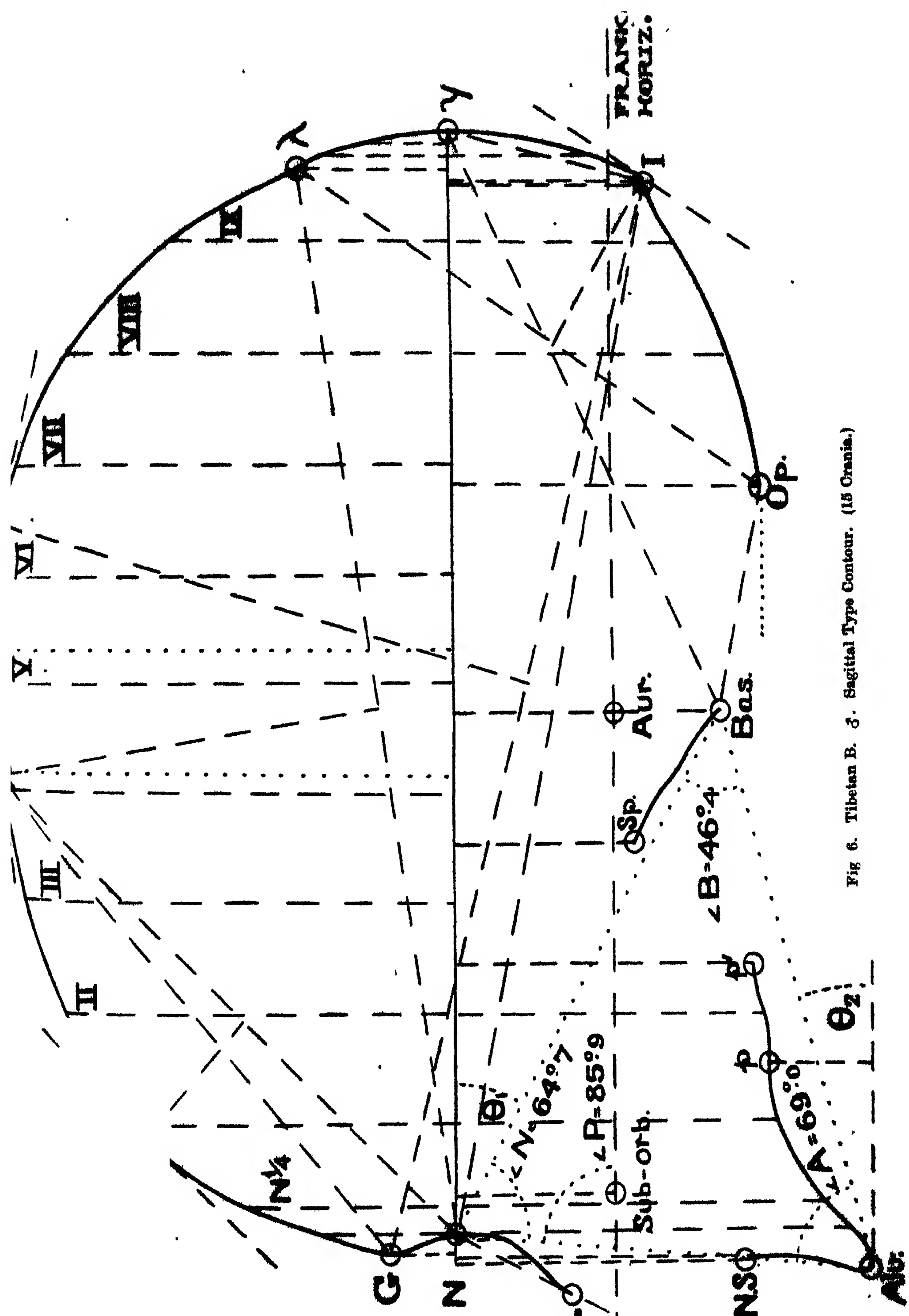

Fig 6. Tibetan B. ♂. Sagittal Type Contour. (15 Crania.)